ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 36 Principles of Digital Communications
Solution to Final exam August 10, 2020

3 problems, 60 points
165 minutes
2 double-sided A4 sheets of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. (20 points) Consider the following additive noise channel,

$$Y[i] = X[i] + \tilde{Z}[i]$$

where $X[i] \in \mathbb{R}$ is the transmitted signal and $Y[i] \in \mathbb{R}$ is the received signal. The noise $\tilde{Z}[i]$ is a colored gaussian noise, that can also be written as,

$$\tilde{Z}[i] = \sum_{j=0}^{\infty} \frac{Z[i-2j]}{2^j} = Z[i] + \frac{1}{2} \cdot Z[i-2] + \frac{1}{4} \cdot Z[i-4] + \dots$$

where $Z[i]$ are i.i.d. $\mathcal{N}(0,1)$ random variables. The transmitter uses antipodal signalling without any coding, i.e., an n bit message $b_0, \dots, b_{n-1}$ in $\{+1, -1\}^n$ is sent as $X[0], \dots, X[n-1]$ with $X[i] = b_i$, with $X[i] = 0$ if $i < 0$. We also assume that all messages are equiprobable.

(a) (4 pts) Assume that we are only given the value of $Y[0]$, determine the error probability of estimating $X[0]$ under the MAP rule.

As $\tilde{Z}[i]$'s is a linear combination of Gaussians, hence it is also Gaussian. It is easy to see that $E\left[\tilde{Z}[0]\right] = 0$. For the variance, we have

$$\begin{aligned}
\mathrm{Var}\left(\tilde{Z}[0]\right) &= \mathrm{Var}\left(\sum_{j=0}^{\infty} \frac{Z[i-2j]}{2^j}\right) \\
&= \sum_{j=0}^{\infty} \mathrm{Var}\left(\frac{Z[i-2j]}{2^j}\right) \\
&= \sum_{j=0}^{\infty} \frac{1}{4^j} \mathrm{Var}\left(Z[i-2j]\right) \\
&= \frac{4}{3}.
\end{aligned}$$

It is easy to see by symmetry that the decision region boundary for estimating $X[0] \in \{+1, -1\}$ is at $Y[0] = 0$. Hence the error probability is equal to $Q\left(\sqrt{\frac{3}{4}}\right)$.

(b) (4 pts) Determine the whitening filter for this noise, i.e., find c_j's such that, for all i

$$\sum_{j=0}^{\infty} c_j \tilde{Z}[i-j] = Z[i].$$

Notice that

$$\tilde{Z}[i] = Z[i] + \left(\frac{1}{2} \cdot Z[i-2] + \frac{1}{4} \cdot Z[i-4] + \dots\right) = Z[i] + \frac{1}{2}\tilde{Z}[i-2].$$

This implies that $Z[i] = \tilde{Z}[i] - \frac{1}{2}\tilde{Z}[i-2]$. Hence

$$c_j = \begin{cases} 1 & j = 0 \\ -\frac{1}{2} & j = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

For the rest of the problem, define

$$\tilde{Y}[i] \triangleq \sum_{j=0}^{\infty} c_j Y[i-j].$$

(c) (4 pts) If we are only given the values of $\tilde{Y}[0], \tilde{Y}[1]$, and $\tilde{Y}[2]$, determine the MAP rule for estimating $X[2]$.

From the c_j's that we found in (b), we have

$$\begin{aligned}
\tilde{Y}[2] &= X[2] - \frac{1}{2}X[0] + Z[2] \\
\tilde{Y}[1] &= X[1] + Z[1] \\
\tilde{Y}[0] &= X[0] + Z[0].
\end{aligned}$$

As the noises are independent, $\tilde{Y}[2], \tilde{Y}[0]$ is a sufficient statistics for estimating $X[0]$. As we assumes that the messages are equiprobable, ML rule is equal to the MAP rule. Let $f(x)$ be the PDF of $N(0,1)$, the ML rule is equal to

$$\begin{aligned}
\hat{X}[2] &= \underset{x_2' \in \{+1,-1\}}{\arg\max} \max_{x_0' \in \{+1,-1\}} \log f\left(\tilde{Y}[2] - x_2' + \frac{1}{2}x_0'\right) + \log f\left(\tilde{Y}[0] - x_0'\right) \\
&= \underset{x_2' \in \{+1,-1\}}{\arg\min} \min_{x_0 \in \{+1,-1\}} \left(\tilde{Y}[2] - x_2' + \frac{1}{2}x_0'\right)^2 + \left(\tilde{Y}[0] - x_0'\right)^2
\end{aligned}$$

(d) (4 pts) Explain how to use the Viterbi decoder to implement the MAP decoding of the sequence $X[0], X[1], \ldots$ from the observation $\tilde{Y}[0], \tilde{Y}[1], \ldots$. I.e., you should find the constraint length d and a function $f(y, x_0, x_{-1}, \ldots, x_{-d})$ such that performing MAP decoding is the same as finding $\{X[i]\}_{i=0}^{n-1}$ which minimizes $\sum_{i=0}^{n-1} f(\tilde{Y}[i], X[i], \ldots, X[i-d])$.

By the same argument as in (c), we have the MAP rule given as follows,

$$\underset{x_0, x_1, \ldots, x_{n-1} \in \{+1,-1\}^n}{\arg\min} \sum_{j=0}^{n-1} \left(\tilde{Y}[j] - x_j - \frac{1}{2}x_{j-2}\right)^2$$

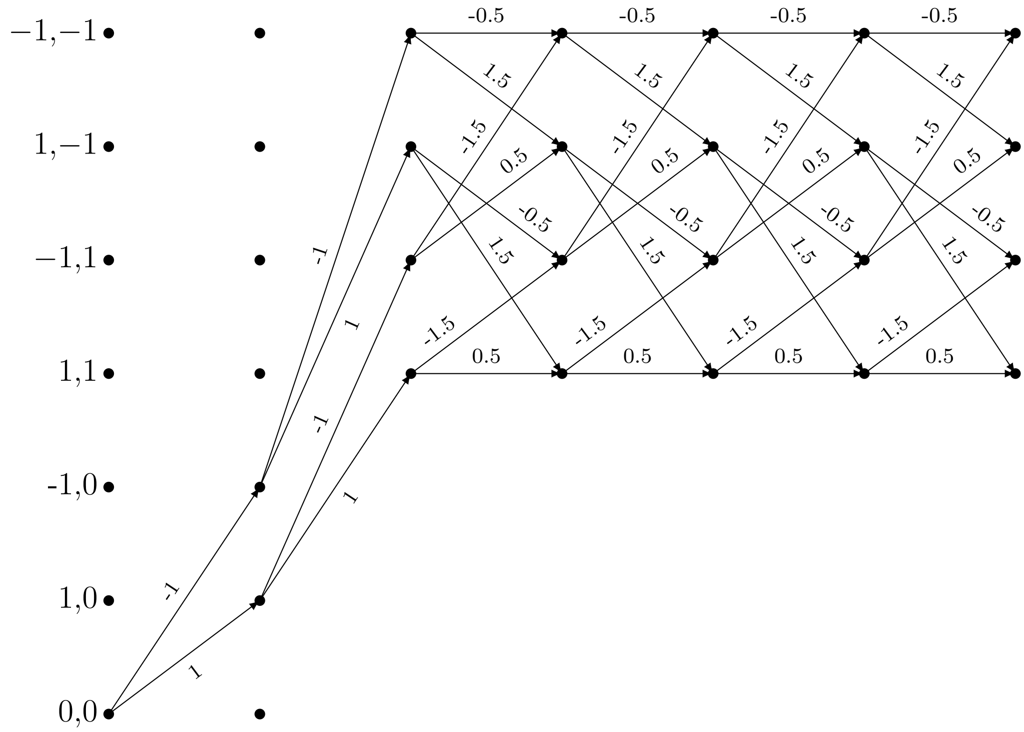
with $x_{-1}, x_{-2} = 0$ as the initial state. Hence, we have $d = 2$ and $f(Y[j], x_j, x_{j-1}, x_{j-2}) = \left(\tilde{Y}[j] - x_j - \frac{1}{2}x_{j-2}\right)^2$.

(e) (4 pts) Given whitening filter output

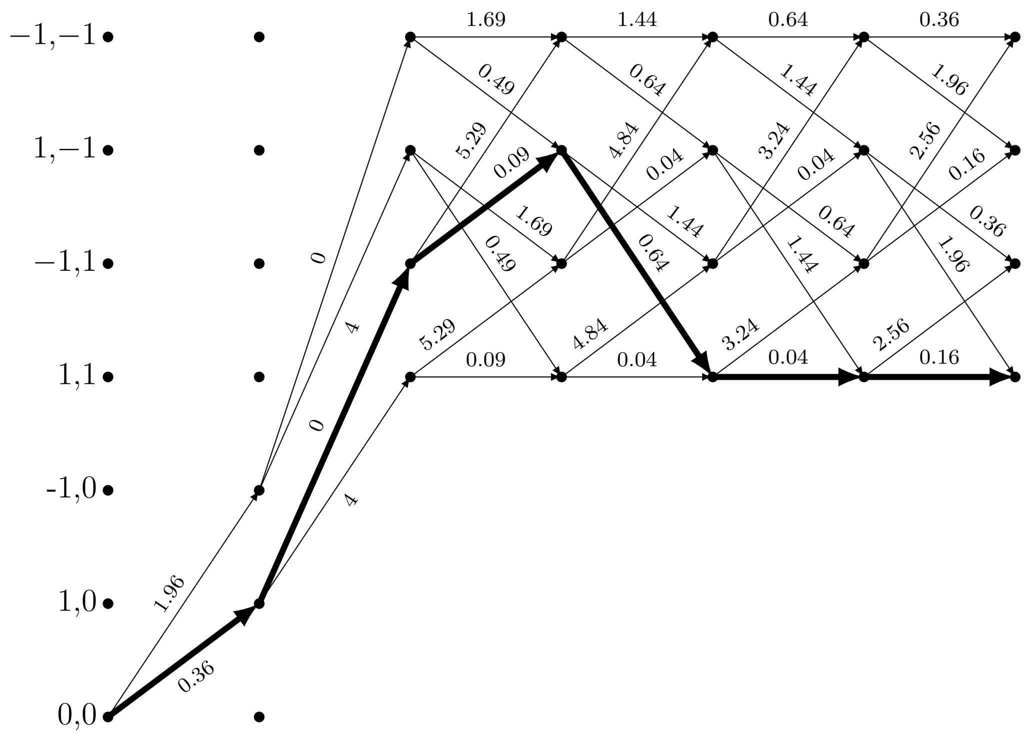
$\tilde{Y}[0]$	$\tilde{Y}[1]$	$\tilde{Y}[2]$	$\tilde{Y}[3]$	$\tilde{Y}[4]$	$\tilde{Y}[5]$
0.4	−1	0.8	0.7	0.3	0.1,

draw the trellis diagram and find the MAP sequence $X[0], \ldots, X[5]$.

The trellis labeled by its output,



and the trellis labeled by the decoding metric,



Hence the optimal sequence is equal to $X[0] = 1, X[1] = -1, X[2] = 1, X[3] = 1, X[4] = 1, X[5] = 1.$

PROBLEM 2. (12 points) Suppose we have a binary hypothesis testing problem where the observable Y may take k distinct values $y_1, \dots, y_k$ distributed according to a conditional probability law $p_{Y|H}$. Suppose we process Y to form a new observable

$$Z = \begin{cases} Y & \text{if } Y \in \{y_1, \dots, y_{k-1}\}, \\ y_{k-1} & \text{if } Y = y_k. \end{cases}$$

In other words the processing removes the distinction between y_{k-1} and y_k.

Let B_Y denote the Bhattacharrya bound on the error probabililty when the observation is Y and B_Z denote the Bhattacharrya bound when the observation is Z.

(a) (4 pts) Show that $B_Z \geq B_Y$.

Hint: You may first want to show $\sqrt{(a+b)(c+d)} \geq \sqrt{ac} + \sqrt{bd}$ for non-negative a, b, c, d.

Let us first shows the hint, both sides of the inequality are guaranteed to be nonnegative so we have

$$\begin{aligned} (\sqrt{(a+b)(c+d)})^2 &\geq (\sqrt{ac} + \sqrt{bd})^2 \\ ac + bd + bc + ad &\geq ac + bd + 2\sqrt{abcd} \\ \frac{bc + ad}{2} &\geq \sqrt{abcd} \end{aligned}$$

The last line is true by the AM-GM inequality.

Now we have

$$\begin{aligned} B_Z - B_Y &= \sum_{y \in \{y_1, \dots, y_{k-1}\}} \sqrt{P_{Z|H}(y|0) P_{Z|H}(y|1)} - \sum_{y \in \{y_1, \dots, y_k\}} \sqrt{P_{Y|H}(y|0) P_{Y|H}(y|1)} \\ &= \sqrt{P_{Z|H}(y_{k-1}|0) P_{Z|H}(y_{k-1}|1)} - \sqrt{P_{Y|H}(y_{k-1}|0) P_{Y|H}(y_{k-1}|1)} - \sqrt{P_{Y|H}(y_k|0) P_{Y|H}(y_k|1)} \end{aligned}$$

Take $a = P_{Y|H}(y_{k-1}|0), b = P_{Y|H}(y_k|0), c = P_{Y|H}(y_{k-1}|1)$ *and* $d = P_{Y|H}(y_k|1)$*, we have*

$$\begin{aligned} B_Z - B_Y &= \sqrt{(a+b)(c+d)} - \sqrt{ac} - \sqrt{bd} \\ &\geq 0 \end{aligned}$$

where the last inequality is from the hint.

(b) (4 pts) Find the condition for equality in (a), express the condition in terms of $\Lambda(y_{k-1})$ and $\Lambda(y_k)$ where $\Lambda(y) = p_{Y|H}(y|0)/p_{Y|H}(y|1)$ is the likelihood ratio.

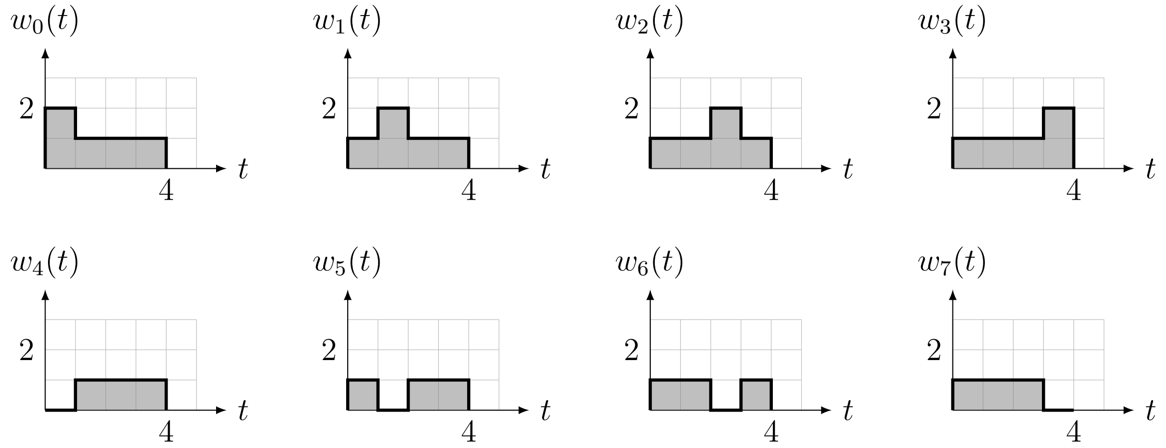
The AM-GM Inequaly is an equality if and only if both terms are equivalent, i.e., $\frac{p+q}{2} = \sqrt{pq}$ *iff* $p = q$*. In our case, this translates to*

$$\begin{aligned} P_{Y|H}(y_{k-1}|0) P_{Y|H}(y_k|0) &= P_{Y|H}(y_{k-1}|1) P_{Y|H}(y_k|1) \\ \frac{P_{Y|H}(y_{k-1}|0)}{P_{Y|H}(y_{k-1}|1)} &= \frac{P_{Y|H}(y_k|0)}{P_{Y|H}(y_k|1)} \\ \Lambda(y_{k-1}) &= \Lambda(y_k). \end{aligned}$$

(c) (4 pts) Show that when the observation Y is replaced with $\Lambda(Y)$ the Bhattacharrya bound is not changed.

The Bhattacharrya bound is not changed because the Bhattacarya value of this hypothesis testing problem is not changed. Notice that from (b) we know that combining two observations symbols with similar LLR does not change the Bhattacharrya value. Hence, by the applying the result inductively, we can group the observations symbols, such that all observation symbols with the same LLR is mapped to the same symbol. If we label each of this groups by its LLR value, then we can assert that replacing Y *by* $\Lambda(Y)$ *does not change the Bhattacharrya value.*

PROBLEM 3. (28 points) Consider the following waveforms.



(a) (4 pts) Find an orthonormal basis for these waveforms such that the waveform elements are time shifts of a single basis function $\psi(t)$.

We can use $\psi(t) = \mathbb{1}(t \in [0,1))$*, then the basis is for* $i = 0,\ldots,3$ *the function* $\psi_i(t) = \psi(t-i)$.

(b) (4 pts) Assuming equally likely messages, translate the above waveforms to form a minimum energy signal set $\tilde{\mathcal{W}} = \{\tilde{w}_i : i = 0,\ldots,7\}$. What are codewords $c_0,\ldots,c_7$ that describe $\tilde{\mathcal{W}}$ in the orthonormal basis you found in (a). What is the energy per bit of $\tilde{\mathcal{W}}$?

We need to remove the mean of the codewords which is exactly $\mathbb{1}(t \in [0,4))$ *so that the codewords are respectively* $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(-1,0,0,0)$, $(0,-1,0,0)$, $(0,0,-1,0)$ *and* $(0,0,0,-1)$*. The energy per bit is* $1/3$.

(c) (4 pts) What can you say about the magnitude squared Fourier transforms of the waveforms $|\tilde{w}_{i,F}(f)|^2$ and $|\psi_F(f)|^2$?

The $|\tilde{w}_{i,F}(f)|^2$ *are all equal since they corespond to a time shift and scaling by* ± 1 *and in the fourier domain this corespond to changing the phase but not the amplitude. They are actually are all equal to* $|\psi_F(f)|^2$ *for the same reason. By Nyquist's criterion we have* $\sum_{i\in\mathbb{Z}} |\psi_F(f-i)|^2 = 1$.

(d) (4 pts) $\tilde{\mathcal{W}}$ is used to communicate over an AGWN channel with noise spectral density $N_0/2$. At the receiver, the received signal is first passed through a filter $h(t) = \mathbb{1}\{t \in [0,1/3]\}$. How should the output of the filter be processed to ensure MAP detection?

Hint: The receiver can sample the output of filter h multiple times.

In order to fall back on the case where we would have filtered with $\tilde{h}(t) = \mathbb{1}(t \in [0,1))$ *we observe the following* $f * \tilde{h}(T) = f * h(T) + f * h(T+1/3) + f * h(T+2/3)$*. We can now do as usual, form the statistic* $Y_i = f * \tilde{h}(i)$ *for* $i = 0,\ldots,3$ *and it will be, under hypothesis* $H = j$*, be distributed as* $\mathcal{N}(c_{j,i}, N_0/2)$*. To make the decision it is enough to get* $i = \arg\max_j |Y_j|$ *and sgn*(Y_i) *such that the decision is* i *if the sign is* 1 *and* $i+4$ *otherwise.*

(e) (4 pts) Suppose the communication channel multiplies the transmitted waveform with either $+1$ or -1 with equal probability. We want to use the transmitter and receiver in (b) and (c) to communicate. You will notice that because of the "multiplicative" defect in the channel, certain waveforms cannot be distinguished. Describe a four element subset of $\tilde{\mathcal{W}}$ that can be distinguished despite this imperfection and how to interpret the decisions of the receiver in (c) to yield a MAP decoder.

We can take $\tilde{w}_i$ for $i = 1, \ldots, 4$, those can be distinguished. For the MAP decoder, it is easy to see that the procedure in (d) will work except that the sign is random with probability $1/2$, the decoder of (d) is still optimal since we cannot dinstinguish between c_i and c_{i+4}, it is as if the random permutation forced $c_{i+4} = c_i$ for $i = 0, \ldots, 3$.

(f) (4 pts) Find the error probability of the system in (e).

The probability of being correct is the probability of having $H \equiv \arg\max_i |Y_i|$ [mod 4] multiplied by $1/2$, the probability of the random shift. That is the probability of error is

$$\begin{aligned}
&1 - 1/2 \cdot \mathbb{P}[|Y_0| \geq |Y_i| \text{ for } i = 1, \ldots, 3 | H = 0] \\
=&1 - 1/2 \cdot \mathbb{P}\left[\bigcap_{i=1}^{3} \{|Z_i| \leq |Z_0 + 1|\} \middle| H = 0\right] \\
=&1 - 1/2 \int_{-\infty}^{\infty} (1 - 2 \cdot Q(2|z+1|/N_0))^3 \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{z^2}{N_0}\right) dz
\end{aligned}$$

(g) (4 pts) We would like to keep the codewords as in (b) but change the function $\psi(t)$ so that the transmitted signals occupy a smaller region in the frequency domain. The partial figure below plot the magnitude squared Fourier transform of a candidate function $\tilde{\psi}(t)$ to replace $\psi(t)$. Complete the figure such that the time shifts $\{\tilde{\psi}(t-i) : i \in \mathbb{Z}\}$ forms an orthogonal basis.