**Handout 23** Principles of Digital Communications
Solutions to Problem Set 9 May 2, 2025

SOLUTION 1.

(a)

$$\begin{aligned} R_\xi(\tau) &= \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t)\ dt = \langle \xi(t+\tau), \xi(t)\rangle \\ &\overset{(1)}{\leq} \|\xi(t+\tau)\| \cdot \|\xi(t)\| = \|\xi\| \cdot \|\xi\| = \|\xi\|^2 \overset{(2)}{=} R_\xi(0), \end{aligned}$$

where (1) follows from the Cauchy–Schwarz inequality and (2) from the fact that $R_\xi(0) = \int_{-\infty}^{\infty} \xi(t)\xi^*(t)\ dt = ||\xi||^2$.

(b)

$$\begin{aligned} R_\xi(-\tau) &= \int_{-\infty}^{\infty} \xi(t-\tau)\xi^*(t)\ dt \\ &= \left(\int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau) dt\right)^* \\ &\overset{t\to t+\tau}{=} R_\xi^*(\tau). \end{aligned}$$

(c)

$$\begin{aligned} R_\xi(\tau) &= \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t)\ dt \\ &\overset{t\to t-\tau}{=} \int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau)\ dt \\ &= \xi(\tau) \star \xi^*(-\tau). \end{aligned}$$

(d) By Parseval's identity, we have

$$\begin{aligned} R_\xi(\tau) &= \langle \xi(t+\tau), \xi(t)\rangle \\ &= \langle \xi_{\mathcal{F}}(f) e^{\mathrm{j}2\pi f\tau}, \xi_{\mathcal{F}}(f)\rangle \\ &= \int_{-\infty}^{\infty} \xi_{\mathcal{F}}(f)\xi_{\mathcal{F}}^*(f) e^{\mathrm{j}2\pi f\tau}\ df \\ &= \int_{-\infty}^{\infty} |\xi_{\mathcal{F}}(f)|^2 e^{\mathrm{j}2\pi f\tau}\ df, \end{aligned}$$

which is the inverse Fourier transform of $|\xi_{\mathcal{F}}(f)|^2$.

Solution 2.

(a) We have

$$y(t) = \int_{-\infty}^{\infty} w(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of $T$ are

$$\begin{aligned}
y(mT) &= \int_{-\infty}^{\infty} w(\tau)\psi(\tau - mT)d\tau \\
&= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - kT)\right] \psi(\tau - mT)d\tau \\
&= \sum_{k=1}^{K} d_k \int_{-\infty}^{\infty} \psi(\tau - kT)\psi(\tau - mT)d\tau \\
&= \sum_{k=1}^{K} d_k \mathbb{1}\{k = m\} \\
&= d_m.
\end{aligned}$$

(b) Let $\tilde{w}(t)$ be the channel output. Then, $\tilde{y}(t)$ is $\tilde{w}(t)$ filtered by $\psi(-t)$. We have

$$\tilde{w}(t) = w(t) + \rho w(t - T)$$

and

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of $T$ are

$$\begin{aligned}
\tilde{y}(mT) &= \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - mT)d\tau \\
&= \int_{-\infty}^{\infty} [w(\tau) + \rho w(\tau - T)]\psi(\tau - mT)d\tau \\
&= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - kT)\right] \psi(\tau - mT)d\tau + \\
&\quad \rho \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - T - kT)\right] \psi(\tau - mT)d\tau \\
&= \sum_{k=1}^{K} d_k \mathbb{1}\{k = m\} + \rho \sum_{k=1}^{K} d_k \mathbb{1}\{k = m - 1\} \\
&= d_m + \rho d_{m-1}.
\end{aligned}$$

(c) From the symmetry of the problem, we have

$$P_e = P_e(1) = P_e(-1).$$

$$
\begin{aligned}
P_e(1) &= \Pr\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = -1\} \Pr\{D_{k-1} = -1\} + \\
&\quad \Pr\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = 1\} \Pr\{D_{k-1} = 1\} \\
&= \frac{1}{2} \left(\Pr\{Y_k < 0 | D_k = 1, D_{k-1} = -1\} + \Pr\{Y_k < 0 | D_k = 1, D_{k-1} = 1\}\right) \\
&= \frac{1}{2} \left(\Pr\{1 - \alpha + Z_k < 0\} + \Pr\{1 + \alpha + Z_k < 0\}\right) \\
&= \frac{1}{2} \left(\Pr\{Z_k < -1 + \alpha\} + \Pr\{Z_k < -1 - \alpha\}\right) \\
&= \frac{1}{2} \left[Q\left(\frac{1-\alpha}{\sigma}\right) + Q\left(\frac{1+\alpha}{\sigma}\right)\right].
\end{aligned}
$$

SOLUTION 3.

(a) We can easily see that

$$\mathbb{E}[X_i | X_{i-1}] = \frac{1}{2} X_{i-1} + \frac{1}{2}(-X_{i-1}) = 0.$$

Consequently (using the law of total expectation)

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i | X_{i-1}]] = 0.$$

Therefore,

$$K_X[k] = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_{i-k} - \mathbb{E}[X_{i-k}])^*] = \mathbb{E}[X_i X_{i-k}^*]$$

Moreover, using the fact that $X_i = X_{i-1} \times (-1)^{D_i}$ repeatedly, we can write

$$X_i = X_{i-k} \times \prod_{j=i-k+1}^{i} (-1)^{D_j}$$

Thus,

$$
\begin{aligned}
K_X[k] &= \mathbb{E}[X_i X_{i-k}^*] \\
&= \mathbb{E}\left[X_{i-k} \prod_{j=i-k+1}^{i} (-1)^{D_j} X_{i-k}^*\right] \\
&\overset{(a)}{=} \mathbb{E}[X_{i-k} X_{i-k}^*] \prod_{j=i-k+1}^{i} \mathbb{E}[(-1)^{D_j}] \\
&= \mathcal{E} \prod_{j=i-k+1}^{i} \mathbb{E}[(-1)^{D_j}] \\
&\overset{(b)}{=} \begin{cases} \mathcal{E} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

where $(a)$ follows from the independence of data bits $\{D_i\}$ and $(b)$ since $\mathbb{E}[(-1)^{D_i}] = 0$.

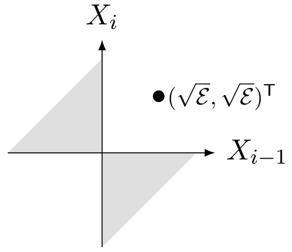
(b) By sampling the signal at the output of the matched filter, $Y(t)$, at multiples of $T$, we obtain

$$Y(iT) = X_i + Z_i,$$

where $Z_i$ is normally distributed with zero mean and variance $N_0/2$. By looking at the definition of $X_i$, we see that it is equal to $X_{i-1}$ if $D_i = 0$ and equal to $-X_{i-1}$ if $D_i = 1$. Therefore a simple decoder estimates that $\hat{D}_i = 0$ if $Y_i$ and $Y_{i-1}$ have the same sign, and $\hat{D}_i = 1$ otherwise. This is equivalent to

$$Y_i Y_{i-1} \underset{\hat{D}_i=1}{\overset{\hat{D}_i=0}{\gtrless}} 0.$$

(c) We first compute the error probability when $D_i = 0$. If $X_{i-1} = \sqrt{\mathcal{E}}$, then $X_i = \sqrt{\mathcal{E}}$. When we decode, we will make an error if the signal $(Y_{i-1}, Y_i)^\mathsf{T}$ is in the second or fourth quadrants (shaded regions in the following figure).




Due to the symmetry of the problem, the probability for this to happen is two times the probability for $(Y_{i-1}, Y_i)^\mathsf{T}$ to be in the second quadrant:

$$\Pr\{Z_{i-1} < -\sqrt{\mathcal{E}} \cap Z_i > -\sqrt{\mathcal{E}}\} = Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right),$$

so,

$$P_e(D_i = 0 | D_{i-1} = 0) = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

Again, due to the symmetry of the problem,

$$P_e(D_i = 0 | D_{i-1} = 1) = P_e(D_i = 0 | D_{i-1} = 0) = P_e(D_i = 0),$$

and

$$P_e(D_i = 1) = P_e(D_i = 0);$$

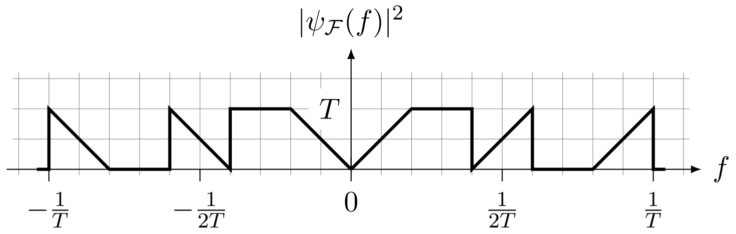
hence

$$P_e = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

Solution 4. Because $\psi(t)$ is real, its Fourier transform is conjugate symmetric ($\psi_{\mathcal{F}}(f) = \psi_{\mathcal{F}}^*(-f)$).

From the condition $\int \psi(t-kT)\psi(t-lT)dt = \mathbb{1}\{k = l\}$ for every pair $k$, $l$, it follows that $|\psi_{\mathcal{F}}(f)|^2$ satisfies Nyquist's criterion with parameter $T$, $\sum_{k\in\mathbb{Z}} |\psi_{\mathcal{F}}(f - k/T)|^2 = T$. On the other hand, since $\psi_{\mathcal{F}}(f) = 0$ for $|f| > \frac{1}{T}$, $|\psi_{\mathcal{F}}(f)|^2$ must have band-edge symmetry.

Putting everything together, we obtain the complete plot of $|\psi_{\mathcal{F}}(f)|^2$.




Solution 5. From Theorem 5.6, we know that $\{\psi(t-jT)\}_{j=-\infty}^{\infty}$ is an orthonormal set if and only if

$$\sum_{k\in\mathbb{Z}} |\psi_{\mathcal{F}}(f-\frac{k}{T})|^2 = T.$$

(a)

$$\sum_{k\in\mathbb{Z}} T\mathbb{1}_{\left[\frac{k}{T}-\frac{1}{2T},\frac{k}{T}+\frac{1}{2T}\right]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

$\Rightarrow$ $\psi(t)$ is orthonormal to its time-translates by multiples of $T$.

(b)

$$\sum_{k\in\mathbb{Z}} \frac{T}{2}\mathbb{1}_{\left[\frac{k-1}{T},\frac{k+1}{T}\right]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

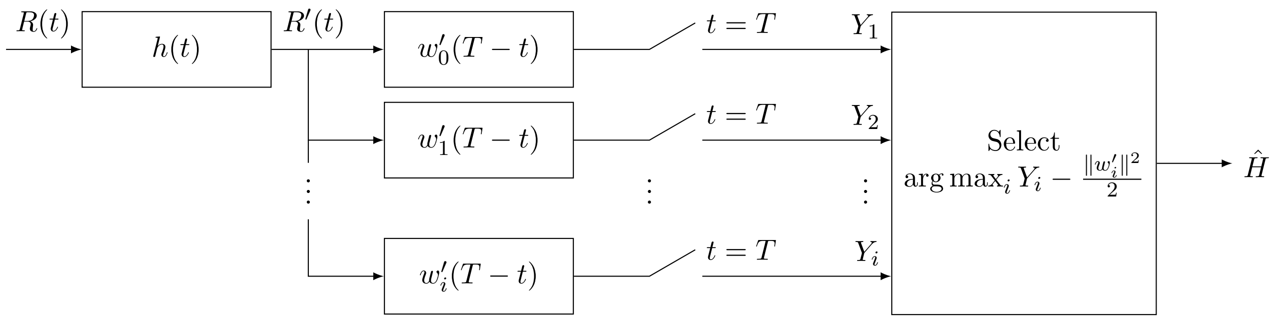
$\Rightarrow$ $\psi(t)$ is orthonormal to its time-translates by multiples of $T$.

(c) Because $|\psi_{\mathcal{F}}(f)|^2$ vanishes outside $\left[-\frac{1}{T},\frac{1}{T}\right]$, we verify whether the band-edge symmetry is fulfilled, which is the case. Hence, the Nyquist criterion is satisfied and $\psi(t)$ is orthonormal to its time-translates by multiples of $T$. Note: the same reasoning can be applied to (b).

(d) $\psi_{\mathcal{F}}(f)$ is a sinc function, therefore $\psi(t)$ is a box function, equal to $\frac{1}{T}\mathbb{1}_{\left[-\frac{T}{2},\frac{T}{2}\right]}(t)$. This is orthogonal to its time-translates by multiples of $T$, but does not have unit norm (unless $T=1$): $\int_{-\infty}^{\infty}|\psi(t)|^2\,dt = \frac{1}{T}$.

Solution 6.

(a) We pass $R(t)$ through a whitening filter $h(t)$ such that the output $R'(t)$ looks like the output of an AWGN channel. After this step we are facing a familiar situation and can implement a matched filter receiver. The receiver architecture is shown below:



Let $N'(t) = \int N(\alpha)h(t-\alpha)\ d\alpha$ be the noise at the output of the whitening filter. We want to select the filter $h(t)$ such that $\frac{N_0}{2} = G(f)|h_{\mathcal{F}}(f)|^2$, i.e.,

$$|h_{\mathcal{F}}(f)|^2 = \frac{N_0}{2G(f)}.$$

The output of the filter is

$$\begin{aligned} R'(t) &= \int R(\alpha)h(t-\alpha)\ d\alpha = \int w_i(\alpha)h(t-\alpha)\ d\alpha + \int N(\alpha)h(t-\alpha)\ d\alpha \\ &= w_i'(t) + N'(t), \end{aligned}$$

where $N'(t)$ is white Gaussian noise and $w_i'(t) = \int w_i(\alpha)h(t-\alpha)\ d\alpha$. We need to design the matched filter for the signals $w_i'(t)$.

(b) To minimize both the noise and the energy of the signal, we need to select an antipodal signal pair that is frequency-limited to $[a, b]$ and has energy $\mathcal{E}$.