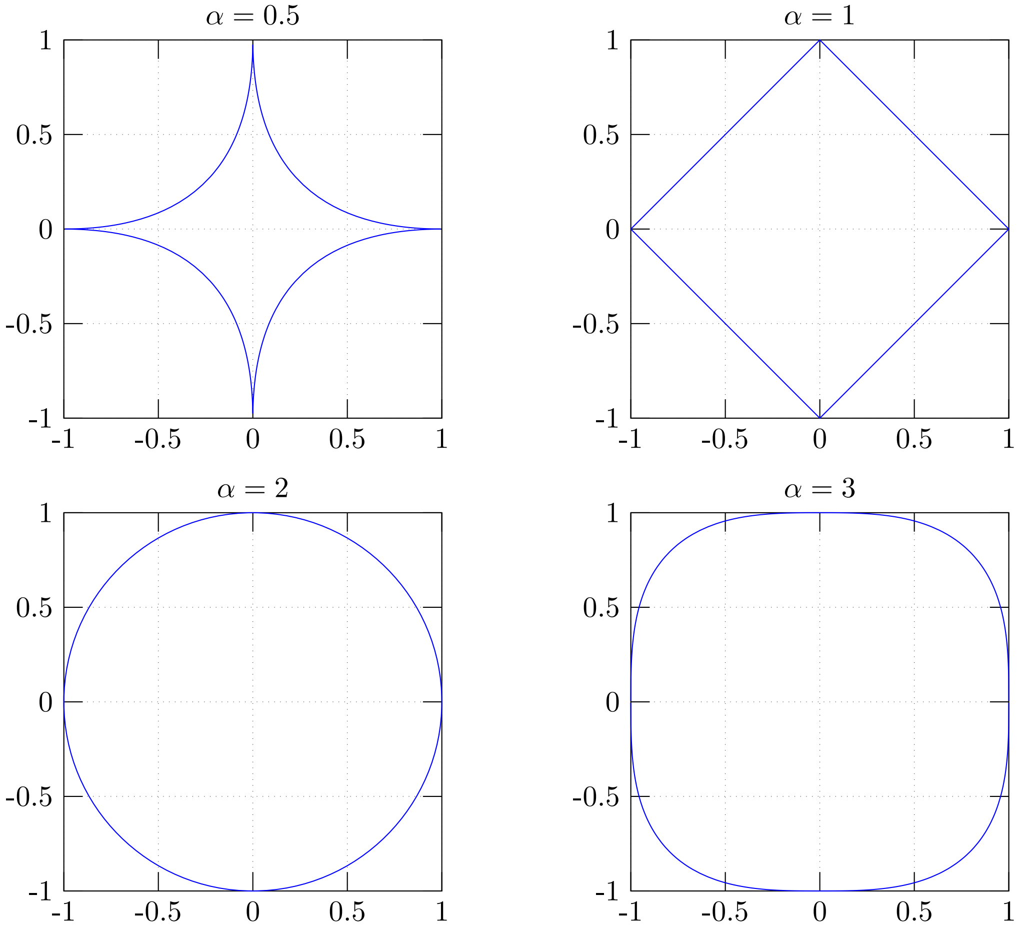
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 29 Principles of Digital Communications
Solutions to Problem Set 12 May 23, 2025

SOLUTION 1.

(a) (i) The plots are shown below:



(ii) The joint density function is invariant under rotation for $\alpha = 2$ only. For this value of α, we have $X, Y \sim \mathcal{N}\left(0, \frac{1}{2}\right)$.

(b) (i) We know that we can write (x, y) in polar coordinates (r, θ). Hence in general the joint distribution of X and Y is a function of r and θ. Because of circular symmetry the joint distribution should not depend on θ, which means that $f_{X,Y}(x, y)$ can be written as a function of r. Hence if we denote this function by ψ and use the independence of X and Y, we have $f_X(x)f_Y(y) = \psi(r)$.

(ii) Taking the partial derivative with respect to x and using the chain rule for differentiation, we have $f'_X(x)f_Y(y) = \psi'(r)\frac{\partial r}{\partial x} = \psi'(r)\frac{x}{r}$. If we divide both sides by $xf_X(x)f_Y(y)$ we have $\frac{f'_X(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)}$. Proceeding similarly for y, we obtain

$$\frac{f'_X(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{f'_Y(y)}{yf_Y(y)}.$$

(iii) $\frac{f'_X(x)}{xf_X(x)}$ is a function of x while $\frac{f'_Y(y)}{yf_Y(y)}$ is a function of y. Hence the only way for the equality to hold is that both of them equal a constant. If we denote this constant by $-\frac{1}{\sigma^2}$, we reach the final result.

(iv) We have $\frac{f'_X(x)}{f_X(x)} = -\frac{x}{\sigma^2}$. Integrating both sides we have $\log(\frac{f_X(x)}{C}) = -\frac{x^2}{2\sigma^2}$. Hence $f_X(x) = C\exp(-\frac{x^2}{2\sigma^2})$. $f_X(x)$ is a probability density function and so should integrate to 1, which gives $C = \frac{1}{\sqrt{2\pi\sigma^2}}$. Hence $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{x^2}{2\sigma^2})$ and by symmetry $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{y^2}{2\sigma^2})$, which shows that X and Y are Gaussian random variables.

Solution 2.

(a) Let $x_E(t) = x_R(t) + \mathrm{j}x_I(t)$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{[x_R(t) + \mathrm{j}x_I(t)]e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}[x_R(t)\cos(2\pi f_c t) - x_I(t)\sin(2\pi f_c t)]. \end{aligned}$$

Hence, we have

$$x_{EI}(t) = \sqrt{2}\Re\{x_E(t)\}$$

and

$$x_{EQ}(t) = \sqrt{2}\Im\{x_E(t)\}.$$

(b) Let $x_E(t) = \alpha(t)e^{\mathrm{j}\beta(t)}$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{\alpha(t)e^{\mathrm{j}\beta(t)}e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{\alpha(t)e^{\mathrm{j}(2\pi f_c t + \beta(t))}\} \\ &= \sqrt{2}\alpha(t)\cos[2\pi f_c t + \beta(t)]. \end{aligned}$$

We thus have

$$x_E(t) = \alpha(t)e^{\mathrm{j}\beta(t)} = \frac{a(t)}{\sqrt{2}}e^{\mathrm{j}\theta(t)}.$$

(c) From (b) we see that

$$x_E(t) = \frac{A(t)}{\sqrt{2}}e^{\mathrm{j}\varphi}.$$

This is consistent with Example 7.9 (DSB-SC) given in the text. We can also verify:

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\left\{\frac{A(t)}{\sqrt{2}}e^{\mathrm{j}\varphi}e^{\mathrm{j}2\pi f_c t}\right\} \\ &= \Re\{A(t)e^{\mathrm{j}(2\pi f_c t + \varphi)}\} \\ &= A(t)\cos(2\pi f_c t + \varphi). \end{aligned}$$

SOLUTION 3.

(a) The key observation is that while $e^{j2\pi f_1 t}$ and $e^{-j2\pi f_1 t}$ are two different signals if $f_1 \neq 0$, $\Re\{e^{j2\pi f_1 t}\}$ and $\Re\{e^{-j2\pi f_1 t}\}$ are identical.

Therefore, if we fix $f_1 \neq 0$ and choose $a_1(t)$ and $a_2(t)$ so that $a_1(t)e^{j2\pi f_c t} = e^{j2\pi f_1 t}$ and $a_2(t)e^{j2\pi f_c t} = e^{-j2\pi f_1 t}$, we get $a_1(t) \neq a_2(t)$ and $\Re\left\{a_1(t)e^{j2\pi f_c t}\right\} = \Re\left\{a_2(t)e^{j2\pi f_c t}\right\}$.

Let $a_1(t) = e^{-j2\pi(f_c - f_1)t}$ and $a_2(t) = e^{-j2\pi(f_c + f_1)t}$. Then $a_1(t) \neq a_2(t)$ and

$$\sqrt{2}\Re\left\{a_1(t)e^{j2\pi f_c t}\right\} = \sqrt{2}\Re\left\{a_2(t)e^{j2\pi f_c t}\right\}.$$

(b) Let $b(t) = a(t)e^{j2\pi f_c t}$, which represents a translation of $a(t)$ in the frequency domain. If $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, then $b_{\mathcal{F}}(f) = 0$ for $f < 0$. Because $\Re\{b(t)\} = \frac{1}{2}\left(a(t)e^{j2\pi f_c t} + a^*(t)e^{-j2\pi f_c t}\right)$, taking the real part has a scaling effect and adds a negative-frequency component. The negative spectrum is canceled by the $h_>$ filter, and the scaling is compensated by the $\sqrt{2}$ factors from the up-converter and down-converter. Multiplying by $e^{-j2\pi f_c t}$ translates the spectrum back to the initial position. In conclusion, we obtain $a(t)$.

(c) Take any baseband signal $u(t)$ with frequency domain support $[-f_c - \Delta, f_c + \Delta]$, $\Delta > 0$. The signal can be real-valued or complex-valued (for example $u_{\mathcal{F}}(f) = \mathbb{1}_{[-f_c-\Delta, f_c+\Delta]}(f)$, which is a sinc in time domain). After we up-convert, the support of $u_{\mathcal{F}}(f)$ will not extend beyond $2f_c + \Delta$. When we chop the negative frequencies we obtain a support contained in $[0, 2f_c + \Delta]$ and when we shift back to the left the support will be contained in $[-f_c, f_c + \Delta]$, which is too small to be the support of $u_{\mathcal{F}}(f)$.

(d) In time domain:

$$\begin{aligned} w(t) &= \sqrt{2}\Re\{a(t)e^{j2\pi f_c t}\} \\ &\overset{a \in \mathbb{R}}{=} \sqrt{2}a(t)\cos(2\pi f_c t). \end{aligned}$$

Therefore,

$$a(t) = \frac{w(t)}{\sqrt{2}\cos(2\pi f_c t)}.$$

In frequency domain: If $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, we obtain $a(t)$ as described in (b). In the following, we consider the case $a_{\mathcal{F}}(f) \neq 0$ for $f < -f_c$.

We have $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}\left[a_{\mathcal{F}}(f - f_c) + a_{\mathcal{F}}(f + f_c)\right] = a_{\mathcal{F}}^+(f) + a_{\mathcal{F}}^-(f)$, with $a_{\mathcal{F}}^+(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}(f - f_c)$ and $a_{\mathcal{F}}^-(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}(f + f_c)$, respectively. These two components have overlapping support in some interval centered at 0. However, there is no overlap for sufficiently large frequencies. This means that for sufficiently large frequencies f we have $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}^+(f)$, which implies that from $w_{\mathcal{F}}(f)$ we can observe the right tail of $a_{\mathcal{F}}^+(f)$ and use that information to remove the right tail of $a_{\mathcal{F}}^-(f)$ from $w_{\mathcal{F}}(f)$ (the right tails of $a_{\mathcal{F}}^+(f)$ and $a_{\mathcal{F}}^-(f)$ are the same because $a(t)$ is real). Hence, from $w_{\mathcal{F}}(f)$ we can read more of the right tail of $a_{\mathcal{F}}^+(f)$. The procedure can be repeated until we get to see $a_{\mathcal{F}}^+(f)$ for all frequencies above f_c. At this point, using $a_{\mathcal{F}}(f) = a_{\mathcal{F}}^+(f + f_c)\sqrt{2}$ and the fact that $a(t)$ is real-valued, we have $a_{\mathcal{F}}(f)$ for the positive frequencies, hence for all frequencies.

Solution 4.

$$\begin{aligned} x(t)\sqrt{2}\cos(2\pi f_c t) &= x(t)\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\ &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\}\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\ &= \left[\frac{x_E(t)e^{\mathrm{j}2\pi f_c t}+x_E^*(t)e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right]\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\ &= \frac{x_E(t)e^{\mathrm{j}4\pi f_c t}+x_E(t)+x_E^*(t)+x_E^*(t)e^{-\mathrm{j}4\pi f_c t}}{2}. \end{aligned}$$

At the lowpass filter output we have

$$\frac{x_E(t)+x_E^*(t)}{2}=\Re\{x_E(t)\}.$$

The calculation for the other path is similar.

Solution 5.

(a) Notice that the sinusoids of $w(t)$ have a period of $T_c = 4$ ms units of time, which implies that $f_c = \frac{1}{T_c} = \frac{1}{4\text{ ms}} = 250$ Hz.

(b) Notice that the phase of the sinusoidal signal changes every $T_s = 4$ ms. (Here we have $T_s = T_c$, but in general it is not the case. In practice we usually have $T_s \gg T_c$. See the note at the end.)

The expression of $w(t)$ as a function of t is:

$$\begin{aligned} w(t) &= \begin{cases} 4\cos(2\pi f_c t - \frac{\pi}{2}) & t\in]0,T_s[\\ 4\cos(2\pi f_c t) & t\in]T_s,2T_s[\\ 4\cos(2\pi f_c t+\pi) & t\in]2T_s,3T_s[\\ 4\cos(2\pi f_c t+\frac{\pi}{2}) & t\in]3T_s,4T_s[\end{cases} = \begin{cases} \Re\left\{4e^{\mathrm{j}(2\pi f_c t-\frac{\pi}{2})}\right\} & t\in]0,T_s[\\ \Re\left\{4e^{\mathrm{j}(2\pi f_c t)}\right\} & t\in]T_s,2T_s[\\ \Re\left\{4e^{\mathrm{j}(2\pi f_c t+\pi)}\right\} & t\in]2T_s,3T_s[\\ \Re\left\{4e^{\mathrm{j}(2\pi f_c t+\frac{\pi}{2})}\right\} & t\in]3T_s,4T_s[\end{cases} \\ &= \begin{cases} \Re\left\{-4\mathrm{j}e^{\mathrm{j}2\pi f_c t}\right\} & t\in]0,T_s[\\ \Re\left\{4e^{\mathrm{j}2\pi f_c t}\right\} & t\in]T_s,2T_s[\\ \Re\left\{-4e^{\mathrm{j}2\pi f_c t}\right\} & t\in]2T_s,3T_s[\\ \Re\left\{4\mathrm{j}e^{\mathrm{j}2\pi f_c t}\right\} & t\in]3T_s,4T_s[\end{cases} = \sqrt{2}\Re\left\{w_E(t)e^{\mathrm{j}2\pi f_c t}\right\}, \end{aligned}$$

where

$$\begin{aligned} w_E(t) &= -\frac{4\mathrm{j}}{\sqrt{2}}\mathbb{1}\{t\in]0,T_s[\} + \frac{4}{\sqrt{2}}\mathbb{1}\{t\in]T_s,2T_s[\} \\ &\quad - \frac{4}{\sqrt{2}}\mathbb{1}\{t\in]2T_s,3T_s[\} + \frac{4\mathrm{j}}{\sqrt{2}}\mathbb{1}\{t\in]3T_s,4T_s[\} \\ &= -\mathrm{j}\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]0,T_s[\} + \sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]T_s,2T_s[\} \\ &\quad - \sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]2T_s,3T_s[\} + \mathrm{j}\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]3T_s,4T_s[\}. \end{aligned}$$

If we define $\psi(t) = \frac{1}{\sqrt{T_s}}\mathbb{1}\{t \in]0, T_s[\}$, $c_0 = -\mathtt{j}\sqrt{8T_s}$, $c_1 = \sqrt{8T_s}$, $c_2 = -\sqrt{8T_s}$ and $c_3 = \mathtt{j}\sqrt{8T_s}$, we get

$$w_E(t) = \sum_{i=0}^{3} c_i \psi(t - iT_s). \tag{1}$$

Therefore, the pulse used in the waveform former is $\psi(t) = \frac{1}{\sqrt{T_s}}\mathbb{1}\{t \in]0, T_s[\}$, and the waveform former output signal is given by (1). The orthonormal basis that is used is $\{\psi(t - iT_s)\}_{i=0}^{3}$.

(c) The symbol sequence is $\{c_0,\ c_1,\ c_2,\ c_3\} = \left\{-\mathtt{j}\sqrt{\mathcal{E}_s},\ \sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ \mathtt{j}\sqrt{\mathcal{E}_s}\right\}$, where $\mathcal{E}_s = 8T_s$. We can see that the symbol alphabet is $\left\{\sqrt{\mathcal{E}_s},\ \mathtt{j}\sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ -\mathtt{j}\sqrt{\mathcal{E}_s}\right\}$.

(d) We have:

- The output sequence of the encoder is the symbol sequence, which is

$$\{c_0,\ c_1,\ c_2,\ c_3\} = \left\{-\mathtt{j}\sqrt{\mathcal{E}_s},\ \sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ \mathtt{j}\sqrt{\mathcal{E}_s}\right\}.$$

- The symbol alphabet contains 4 symbols. This means that each symbol represents two bits. Since the symbol rate is $f_s = \frac{1}{T_s} = 250$ symbols/s, the bit rate is $2 \times 250 = 500$ bits/s.
- The input/output mapping can be obtained by assigning two bits for each symbol in the symbol alphabet. Keeping in mind that it is better to minimize the number of bit-differences between close symbols, we obtain the following input/output mapping (which is not unique, i.e., we can obtain other mappings that satisfy the mentioned criterion): $\sqrt{\mathcal{E}_s} \longleftrightarrow 00$, $\mathtt{j}\sqrt{\mathcal{E}_s} \longleftrightarrow 01$, $-\sqrt{\mathcal{E}_s} \longleftrightarrow 11$ and $-\mathtt{j}\sqrt{\mathcal{E}_s} \longleftrightarrow 10$.
- Assuming that the above input/output mapping was used, we can obtain the input sequence of the encoder: 10001101.

Note that in this example, we have $T_s = T_c$, so $f_c = f_s$. This is very unusual. In practice we almost always have $f_c \gg f_s$, especially if we are using electromagnetic waves.