ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

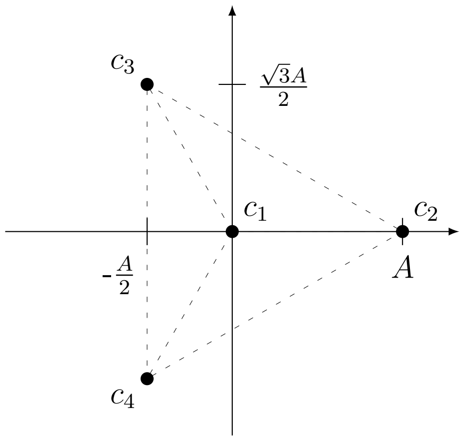
School of Computer and Communication Sciences

Handout 32 — Principles of Digital Communications

Solutions to Final exam — June 19, 2025

PROBLEM 1. (8 points)
Answer the following questions with proper justification.

(a) (2 pts) Assuming equally likely messages, is it true that the energy of the constellation $\{c_1, c_2, c_3, c_4\}$ given in the figure below cannot be reduced by isometric transformations? If not, give a minimum energy version.



Solution: It is true. The given configuration already has mean 0.

(b) (2 pts) Re-do part (a) with $P_H(1) = 1/2$, $P_H(2) = P_H(3) = P_H(4) = 1/6$ instead.

Solution: Still true, as the mean is still zero.

(c) (2 pts) In a binary hypothesis problem with observation Y, suppose $f_{Y|H}(y|0) = \exp(-y)\,\mathbb{1}\{y > 0\}$, and $f_{Y|H}(y|1) = (1/\alpha)\exp(-y/\alpha)\mathbb{1}\{y > 0\}$, with $\alpha > 1$. Is it true that if T is a sufficient statistic, then T determines Y?

Solution: True. We know that in a binary hypothesis test any sufficient statistic determines the log-likelihood ratio $\log \frac{f_{Y|H}(y|1)}{f_{Y|H}(y|0)}$. In this case the LLR is equal to $y\left(1 - \frac{1}{\alpha}\right) - \log\alpha$, which is a one-to-one function of y.

(d) (2 pts) Continuing with (c), is it true that no matter what the a priori probabilities of the two hypotheses are, the error probability of the MAP decision rule is upper bounded by $2\sqrt{\alpha}/(1+\alpha)$?
Hint: Bhattacharyya bound.

Solution: True. The Bhattacharyya bound tells us that

$$P_e(0) \le \sqrt{\frac{P_H(1)}{P_H(0)}} \int_{\mathbb{R}} \sqrt{f_{Y|H}(y|1) f_{Y|H}(y|0)}\,\mathrm{d}y$$

$$P_e(1) \le \sqrt{\frac{P_H(0)}{P_H(1)}} \int_{\mathbb{R}} \sqrt{f_{Y|H}(y|1) f_{Y|H}(y|0)}\,\mathrm{d}y.$$

Hence, the average probability of error is upper bounded by

$$2\sqrt{P_H(0)P_H(1)}\int_{\mathbb{R}}\sqrt{f_{Y|H}(y|1)f_{Y|H}(y|0)}\,\mathrm{d}y.$$

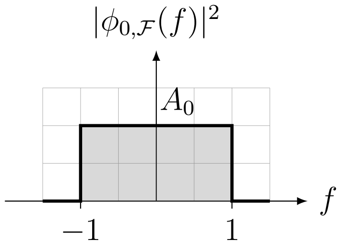
For any value of $P_H(0)$ and $P_H(1)$, $\sqrt{P_H(0)P_H(1)}$ is at most $1/2$ (A.M. $\geq$ G.M.), and the integral evaluates to exactly $2\sqrt{\alpha}/(1+\alpha)$.

Remarks: Parts (a) and (b) look at "minimum energy constellation has zero mean" — this is true even if the hypotheses are not equally likely, the mean should be weighted according to the prior probabilities. Part (c) is a direct application of the principle "sufficient statistic determines LLR" — in this case the LLR also happens to determine the observation itself, i.e., the LLR is a one-to-one function of the observation, so the sufficient statistic also detrmines the observation. In part (d), using the tighter version of the Bhattacharyya bound as described in the book gives an improvement by a factor of $1/2$, i.e., the statement would be true even if the bound was given to be $\sqrt{\alpha}/(1+\alpha)$.

PROBLEM 2. (9 points)

(a) (2 pts) Suppose $\phi_0(t)$ is a waveform with Fourier transform $\phi_{0,\mathcal{F}}(f)$ satisfying

$$|\phi_{0,\mathcal{F}}(f)|^2 = A_0\,\mathbb{1}\{|f|<1\} \qquad \equiv$$

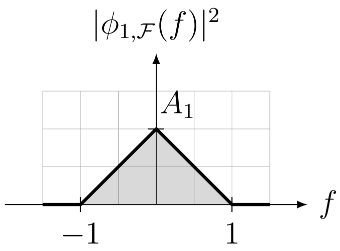


What is A_0 if ϕ_0 is a unit energy waveform (i.e., $\int_{-\infty}^{\infty}|\phi_0(t)|^2\,\mathrm{d}t = 1$)? Find the smallest T such that $\{\phi_0(t-jT) : j\in\mathbb{Z}\}$ is an orthogonal set of waveforms, and call it T_0.

Solution: $A_0 = \frac{1}{2}$ via Parseval's relation, i.e., $\int_{-\infty}^{\infty}|\phi_0(t)|^2dt = \int_{-\infty}^{\infty}|\phi_{0,\mathcal{F}}(f)|^2\,\mathrm{d}f$. The smallest T that satisfies Nyquist criterion is $T_0 = 1/2$, as the sum of shifts by multiples of $1/T_0 = 2$ is the constant $1/2 = T_0$.

(b) (2 pts) Repeat (a) for $\phi_1(t)$ whose Fourier transform's square magnitude is given by

$$|\phi_{1,\mathcal{F}}(f)|^2 = A_1(1-|f|)\,\mathbb{1}\{|f|<1\} \qquad \equiv$$



That is, find A_1 such that ϕ_1 has unit energy and T_1 such that it is the smallest T that makes $\{\phi_1(t-jT) : j\in\mathbb{Z}\}$ an orthogonal set.

Solution: $A_1 = 1$ via Parseval's relation. The smallest T that satisfies Nyquist criterion is $T_1 = 1$, as the sum of shifts by multiples of $1/T_0 = 1$ is the constant $1 = T_0$.

(c) (3 pts) Suppose we have two systems using the same codebook set $\mathcal{C} = \{c_1, \dots, c_m\}$. One system uses as orthonormal basis the shifts of ϕ_0 by T_0 obtained in (a), and the other uses the shifts of ϕ_1 by T_1 obtained in (b), to generate the waveforms $w_1, \dots, w_m$, which are then transmitted over an AWGN channel with noise intensity $\frac{N_0}{2}$. Compare the two systems in terms of their

(i) energy per bit.

(ii) error probability.

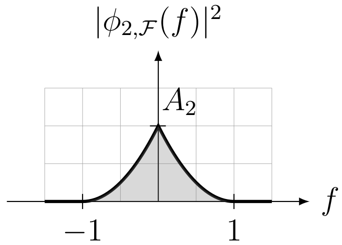
(iii) bandwidth occupied (i.e., the smallest B such that all the w_i's have Fourier transform zero outside of $[-B, B]$).

(iv) rate (i.e., number of bits transmitted per second).

Solution: Since the codebooks are the same, the energy per bit and error probability are the same. The bandwidth is also the same, as ϕ_0 and ϕ_1 both have bandwidth 1 (and hence, so does $w_i = \sum_j c_{ij}\phi_k(t - jT_k)$ for both $k = 0, 1$). Since both systems send $\log_2 m$ bits, the system using ϕ_0 transmits twice as many bits per second as the one using ϕ_1, as $T_0 = T_1/2$.

(d) (2 pts) Repeat (a) for $\phi_2(t)$ whose Fourier transform's square magnitude is given by

$$|\phi_{2,\mathcal{F}}(f)|^2 = A_2(1 - |f|)^2\,\mathbb{1}\{|f| < 1\} \qquad \equiv$$



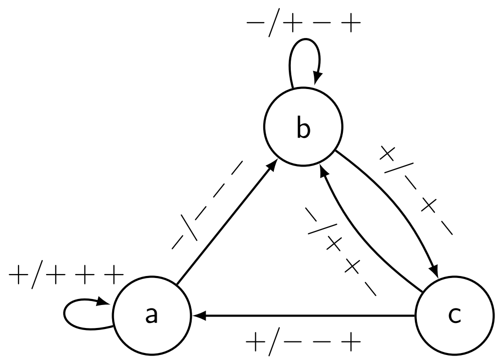
That is, find A_2 such that ϕ_2 has unit energy and T_2 such that it is the smallest T that makes $\{\phi_2(t - jT) : j \in \mathbb{Z}\}$ an orthogonal set (if it exists).
Hint: Look for band-edge symmetry in $|\phi_{2,\mathcal{F}}(f)|^2$ or argue in time-domain.

Solution: $A_2 = \frac{3}{2}$, via Parseval's relation again. There is no T that makes $\{\phi_2(t-jT) : j \in \mathbb{Z}\}$ an orthogonal set. From the frequency domain, this can be seen as there is no point with band-edge symmetry and Nyquist's criterion cannot be satisfied. From the time domain, observe that $|\phi_{2,\mathcal{F}}(f)| = \sqrt{A_2}(1 - |f|)\,\mathbb{1}\{|f| < 1\}$, which is the convolution of $A_2^{\frac{1}{4}}\mathbb{1}\{|f| < \frac{1}{2}\}$ with itself. Hence, the time domain signal is the square of a sinc function, which is always non-negative, and hence, the inner product with any shift of itself can never be zero.

Remarks: The waveforms ϕ_0 and ϕ_1 have the same bandwidth, but we need to shift ϕ_1 by twice as much as ϕ_0 to get an orthonormal waveform. Hence, though they occupy the same bandwidth, the data rate of the system using ϕ_0 is twice as much (meaning that we can send twice as much information in the same time). In previous exercises (and previous exams) we have seen that among all waveforms that give the same data rate (i.e., same shift to obtain an orthonormal waveform), the one occupying the minimum bandwidth is the rect (in frequency domain). Here, we conversely see that among all waveforms occupying the same bandwidth, the rect has the highest data rate. Part (d) shows an example where it is NOT possible to obtain an orthonormal basis by shifts (unlike usual examples you have seen earlier) — the time domain argument is elegant and makes it intuitively clear why it is impossible to get its inner product with a shift to be zero — the function is simply non-negative everywhere.

Problem 3. (11 points)
Consider the following 3-state machine that transforms a binary data sequence $b_1, b_2, \ldots$ (with $b_i \in \{+, -\}$) to a coded sequence $x_1, x_2, x_3, x_4, x_5, x_6, \ldots$ as follows, starting from state a.

current state	data bit	next state	output
a	$+$	a	$+++$
a	$-$	b	$---$
b	$+$	c	$-+-$
b	$-$	b	$+-+$
c	$+$	a	$--+$
c	$-$	b	$++-$

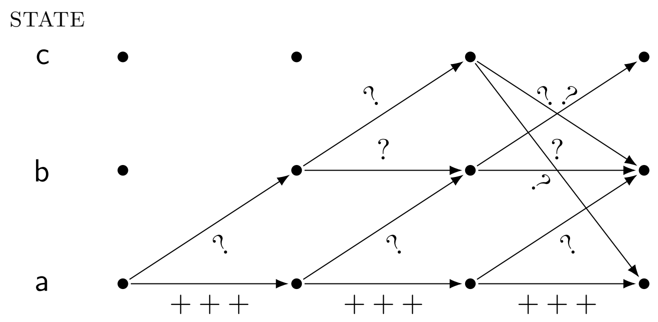


(a) (2 pts) Show that there is a "sentinel sequence of bits" $t_1, t_2, \ldots, t_L$ such that no matter the current state of the machine, after the input $t_1, \ldots, t_L$ the machine will be found in state a. Among all such sequences, find the shortest one.

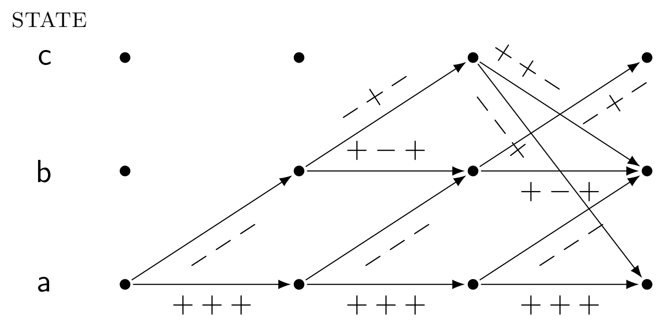
Solution: $t_1, t_2 = ++$ is clearly such a sequence. Further, no length one sequence works, as from b one cannot reach a in a single step, so $++$ is the shortest such sequence.

From now on, assume that the shortest such $t_1, \ldots, t_L$ (as found in (a)) will be appended to the data sequence before transmission, i.e., to send $b_1, \ldots, b_k$, we encode $b_1, \ldots, b_k, t_1, \ldots, t_L$.

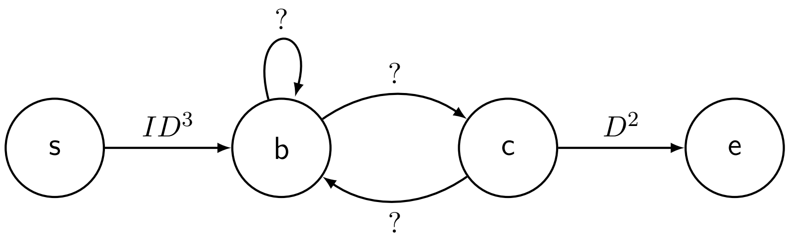
(b) (2 pts) Complete the following trellis diagram by filling in the ?'s:



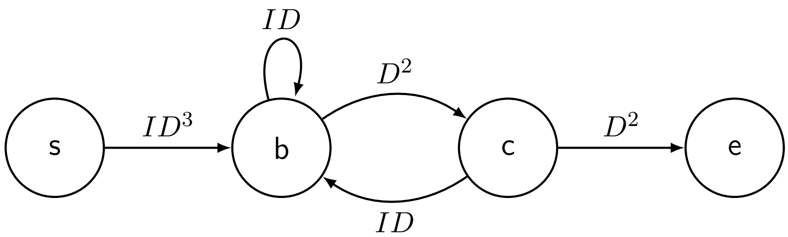
textitSolution: The completed figure is:



(c) (2 pts) Fill the ?'s in the following detour flow graph formed by splitting state a:



Solution: The completed figure is:



(d) (3 pts) Find the transfer function $T(I, D)$ from s to e.

Solution: The equations are:

$$\begin{aligned} T_{\mathsf{e}}(I, D) &= D^2 T_{\mathsf{c}}(I, D) \\ T_{\mathsf{c}}(I, D) &= D^2 T_{\mathsf{b}}(I, D) \\ T_{\mathsf{b}}(I, D) &= IDT_{\mathsf{b}}(I, D) + IDT_{\mathsf{c}}(I, D) + ID^3 T_{\mathsf{s}}(I, D). \end{aligned}$$

Substituting the first and second into the third, we get $T_{\mathsf{b}} = (ID + ID^3)T_{\mathsf{b}} + ID^3 T_{\mathsf{s}}$, hence $T_{\mathsf{b}} = \frac{ID^3}{1-ID-ID^3} T_{\mathsf{s}}$, and $T_{\mathsf{e}} = D^4 T_{\mathsf{b}}$, so we have the transfer function from s to e is

$$T(I, D) = \frac{ID^7}{1 - ID - ID^3}.$$

(e) (2 pts) Consider the communication system described as follows: the bit sequence $b_1, \ldots$, together with the sentinel bits from (a), are encoded using this 3-state machine to obtain $x_1, x_2, x_3, \ldots$. The receiver observes $Y_1, Y_2, Y_3, \ldots$ where $Y_i = \sqrt{\mathcal{E}_s} x_i + Z_i$, with $Z_i \sim \mathcal{N}(0, \sigma^2)$ i.i.d. A MAP decoder converts the received sequence to decoded bits $\hat{b}_1, \ldots$.

Find an upper bound to the bit error probability of this communication system, when the "all +" data sequence is transmitted.

Solution: The bit error probability is upper bounded by

$$\frac{\partial}{\partial I} T(I, D)\Big|_{I=1, D=z} = \frac{z^7}{(1 - z - z^3)^2},$$

where $z = \exp(-\frac{\mathcal{E}_s}{2\sigma^2})$.

Remarks: As seen in the lectures, the encoding from bits to codewords is done using a finite-state machine. This is not a "convolutional" code as the output bits cannot be written as the convolution of the input bits with some filter. In fact, the transformation from the input to the output is not linear. Nonetheless, the same principles apply — we can upper bound the bit error probability by counting detours using the transfer function. The upper bound that we obtain is nearly $\frac{1}{3}z^7$ if z is small, which means that the error probability decays approximately as $\exp(-\frac{7\mathcal{E}_s}{2\sigma^2})$.

PROBLEM 4. (16 points)

Hint: For this problem, you may find the following background on Fourier transforms useful:

(i) By Parseval's relation, $\langle a, b\rangle = \langle A_{\mathcal{F}}, B_{\mathcal{F}}\rangle$, where $A_{\mathcal{F}}(f)$ and $B_{\mathcal{F}}(f)$ are the Fourier transforms of $a(t)$ and $b(t)$.

(ii) $a(t)\exp(\mathrm{j}2\pi\Delta t)$ has Fourier transform $A_{\mathcal{F}}(f-\Delta)$.

(iii) $\phi(t) = \mathrm{sinc}(t)$ has Fourier transform $\mathrm{rect}(f) := \mathbb{1}\{|f| < 1/2\}$.

Consider a transmitter that operates with $f_0 > 0$ and $\phi(t) = \mathrm{sinc}(t)$ as follows:

$$\big[i\big] \to \big[c_i \in \mathbb{C}\big] \to \big[w_{i,E}(t) = c_i\phi(t)\big] \to \big[w_i(t) = \sqrt{2}\Re\{w_{i,E}(t)\exp(\mathrm{j}2\pi f_0 t)\}\big].$$

The waveform $w_i(t)$ is then input to an AWGN channel with noise intensity $N_0/2$; the channel's output is $R(t)$.

(a) (3 pts) Consider the following receiver:

$$\big[R(t)\big] \to \big[R_E(t) = R(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_0 t)\big] \to \big[Y = \langle R_E, \phi\rangle\big] \to \big[\hat{\imath} = \arg\min_i |Y - c_i|\big].$$

What are the conditions on f_0 and the probability distribution of the message i so that this receiver is optimal?

Solution: We need that the inner product of R_E with ϕ also removes the high frequency terms in $R_E(t)$ that occupy frequencies in $[-2f_0 - 1/2, -2f_0 + 1/2]$, i.e., we require $-2f_0 + 1/2 \leq -1/2$. Hence, we need $f_0 \geq 1/2$. (This also ensures that there is no interference between the positive and negative frequencies in the passband signal.) The given receiver based on Y is the ML receiver, which is optimal (equal to MAP) if the message i is uniformly distributed (i.e., the hypothesis is a priori equlaly likely).

For the rest of the problem suppose that there is an error made in the receiver design: while forming $R_E(t)$, an oscillator frequency $f_1 > 0$ instead of f_0 is used, i.e., $R_E(t) = R(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_1 t)$. The rest of the receiver is identical to the design in (a).

(b) (2 pts) Noting that for any complex number z, $2\Re\{z\} = z + z^*$, find f_2 and f_3 such that

$$R_E(t) = c_i\phi(t)\exp(\mathrm{j}2\pi f_2 t) + c_i^*\phi(t)\exp(\mathrm{j}2\pi f_3 t) + N(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_1 t).$$

Solution: By direct computation, we get $f_2 = f_0 - f_1, f_3 = -(f_0 + f_1)$.

Suppose we write $Y = d_i + Z$, where $d_i \in \mathbb{C}$ represents the part of Y which depends on the message and Z represents noise.

(c) (3 pts) Find α and β so that

$$d_i = \alpha c_i + \beta c_i^*.$$

Solution: We have $\alpha = \langle\phi(t)\exp(\mathrm{j}2\pi f_2 t), \phi(t)\rangle = \langle\mathrm{rect}(f - f_2), \mathrm{rect}(f)\rangle = (1 - |f_2|)\mathbb{1}\{|f_2| < 1\}$. Similarly, $\beta = \langle\phi(t)\exp(\mathrm{j}2\pi f_3 t), \phi(t)\rangle = \langle\mathrm{rect}(f - f_3), \mathrm{rect}(f)\rangle = (1 - |f_3|)\mathbb{1}\{|f_3| < 1\}$.

(d) (2 pts) Find necessary and sufficient conditions on f_0 and f_1 so that $\beta = 0$. Find necessary and sufficient conditions on f_0 and f_1 so that $\alpha \neq 0$.

Solution: From the expressions in (d), we see that $\beta = 0$ if and only if $f_2 = f_0 + f_1 \geq 1$. Similarly $\alpha \neq 0$ if and only if $|f_3| = |f_0 - f_1| < 1$.

(e) (2 pts) Find waveforms $a(t)$ and $b(t)$ so that $\Re\{Z\} = \langle N, a\rangle$ and $\Im\{Z\} = \langle N, b\rangle$. Under what conditions (on f_0 and f_1) will $\Re\{Z\}$ and $\Im\{Z\}$ be independent?

Solution: $a(t) = \sqrt{2}\phi(t)\cos(2\pi f_1 t)$, $b(t) = \sqrt{2}\phi(t)\sin(2\pi f_1 t)$. The orthogonality of $a(t)$ and $b(t)$ requires $\langle a, b\rangle = \langle \phi(t), \phi(t)\sin(2\pi(2f_1)t)\rangle = \Im\{\langle \phi(t), \phi(t)\exp(\mathrm{j}2\pi(2f_1)t)\rangle\}$ to be zero. By Parseval's relation the last is always equal to zero since $\langle \mathrm{rect}(f), \mathrm{rect}(f - \Delta)\rangle$ is real.

(f) (2 pts) Suppose $f_0 = 3$, $f_1 = 3.2$, and the codewords are $c_0 = \sqrt{\mathcal{E}}$, $c_1 = \mathrm{j}c_0$, $c_2 = -c_0$, $c_3 = -c_1$. What is the error probability?

Solution: By the parts above we have $\alpha = 0.8$ and $\beta = 0$. Hence $Y = 0.8c_i + Z_i$, and is equivalent to a QAM with $0.64\mathcal{E}$. Thus the error probability is $2Q(a) - Q(a)^2$ where $a = \sqrt{0.64\mathcal{E}/N_0}$

(g) (2 pts) Suppose $f_0 = 3$, $f_1 = 4$, and the codewords are the same as in part (f). What is the error probability?

Solution: Here both α and β are equal to zero, so Y is independent of the message. The error probability is the same as that of a random guess between the four messages, which is $3/4$.

Remarks: We see from part (c) that $\beta = 0$ if $f_0 + f_1 \geq 1$. Note that the bandwidth of the baseband signal is $1/2$. Typically, the carrier frequency is much larger than the bandwidth of the baseband signal, so we usually have $\beta = 0$ in practice, even with the mismatch. (If the mismatch is zero, then $f_0 + f_1 = 2f_0 \geq 1$ anyway, so $\beta = 0$ again.) However, the quantity α depends on the difference between f_0 and f_1. If there is any mismatch at all (i.e. $f_0 \neq f_1$) then $\alpha < 1$, and we will suffer in terms of error probability. Nonetheless, if this difference is smaller than twice the bandwidth of the signal, we can still make a non-trivial decision (as in part (f)). If this difference happens to be larger than twice the bandwidth, we can do no better than a random guess (as in part (g)).