

# ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

## Handout 20

### Problem Set 9

## Principles of Digital Communications

Apr. 30, 2025

PROBLEM 1. Prove the following properties of the self-similarity function. Recall that the self-similarity function of an  $\mathcal{L}_2$  pulse  $\xi(t)$  is

$$R_\xi(\tau) = \int_{-\infty}^{\infty} \xi(t + \tau) \xi^*(t) dt.$$

(a) *Value at zero:*

$$R_\xi(\tau) \leq R_\xi(0) = \|\xi\|^2, \quad \tau \in \mathbb{R}.$$

(b) *Conjugate symmetry:*

$$R_\xi(-\tau) = R_\xi^*(\tau), \quad \tau \in \mathbb{R}.$$

(c) *Convolution representation:*

$$R_\xi(\tau) = \xi(\tau) \star \xi^*(-\tau), \quad \tau \in \mathbb{R}.$$

(d) *Fourier relationship:*

$$R_\xi(\tau) \text{ is the inverse Fourier transform of } |\xi(f)|^2.$$

*Note:* The fact that  $\xi_{\mathcal{F}}(f)$  is in  $\mathcal{L}_2$  implies that  $|\xi_{\mathcal{F}}(f)|^2$  is in  $\mathcal{L}_1$ . The Fourier inverse of an  $\mathcal{L}_1$  function is continuous. Hence  $R_\xi(\tau)$  is continuous.

PROBLEM 2. Let

$$w(t) = \sum_{k=1}^K d_k \psi(t - kT)$$

be a transmitted signal where  $\psi(t)$  is a real-valued pulse that satisfies

$$\int_{-\infty}^{\infty} \psi(t) \psi(t - kT) dt = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and  $d_k \in \{-1, 1\}$ .

(a) Suppose that  $w(t)$  is filtered at the receiver by the matched filter with impulse response  $\psi(-t)$ . Show that the filter output  $y(t)$  sampled at  $mT$ ,  $m \in \mathbb{Z}$ , yields  $y(mT) = d_m$ , for  $1 \leq m \leq K$ .

(b) Now suppose that the (noiseless) channel outputs the input plus a delayed and scaled replica of the input. That is, the channel's impulse response takes the form  $h(t) = \delta(t) + \rho \delta(t - T)$  for some  $T$  and some  $\rho \in [-1, 1]$ . The transmitted signal  $w(t)$  is filtered by  $h(t)$ , then filtered at the receiver by  $\psi(-t)$ . The resulting waveform  $\tilde{y}(t)$  is again sampled at multiples of  $T$ . Determine the samples  $\tilde{y}(mT)$ , for  $1 \leq m \leq K$ .

- (c) Suppose that the  $k$ -th received sample is  $Y_k = d_k + \alpha d_{k-1} + Z_k$ , where  $Z_k \sim \mathcal{N}(0, \sigma^2)$  and  $0 \leq \alpha < 1$  is a constant.  $d_k$  and  $d_{k-1}$  are realizations of independent random variables that take on the values 1 and  $-1$  with equal probability. Suppose that the receiver decides  $\hat{d}_k = 1$  if  $Y_k > 0$ , and decides  $\hat{d}_k = -1$  otherwise. Find the probability of error for this receiver.

PROBLEM 3. For many years, telephone companies built their networks on *twisted pairs*. This is a twisted pair of copper wires invented by Alexander Graham Bell in 1881 as a means to mitigate the effect of electromagnetic interference. In essence, an alternating magnetic field induces an electric field in a loop. This applies also to the loop created by two parallel wires connected at both ends. If the wire is twisted, the electric field components that build up along the wire alternate polarity and tend to cancel out one another. If we swap the two contacts at one end of the cable, the signal's polarity at one end is the opposite of that on the other end. Differential encoding is a technique for encoding the information in such a way that it makes no difference when the polarity is inverted. The differential encoder takes the data sequence  $\{D_i\}_{i=1}^n$ , here assumed to have independent and uniformly distributed components taking value in  $\{0, 1\}$ , and produces the symbol sequence  $\{X_i\}_{i=1}^n$  according to the following encoding rule:

$$X_i = \begin{cases} X_{i-1}, & D_i = 0, \\ -X_{i-1}, & D_i = 1, \end{cases}$$

where  $X_0 = \sqrt{\mathcal{E}}$  by convention. Suppose that the symbol sequence is used to form

$$X(t) = \sum_{i=1}^n X_i \psi(t - iT),$$

where  $\psi(t)$  is normalized and orthogonal to its  $T$ -spaced time translates. The signal is sent over the AWGN channel of power spectral density  $\frac{N_0}{2}$  and at the receiver is passed through the matched filter of impulse response  $\psi^*(-t)$ . Let  $Y_i$  be the filter output at time  $iT$ .

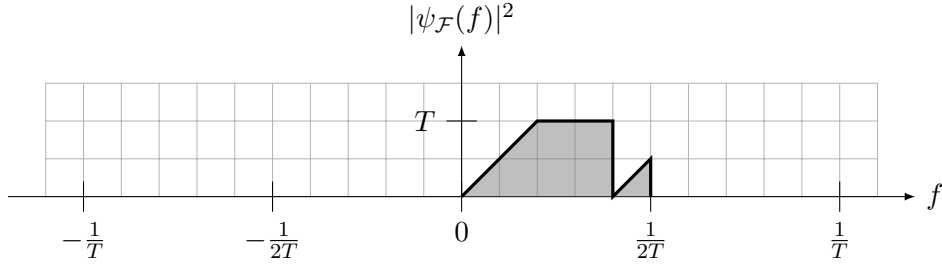
- (a) Determine  $K_X[k]$ ,  $k \in \mathbb{Z}$ , assuming an infinite sequence  $\{X_i\}_{i=-\infty}^{\infty}$ .
- (b) Describe a method to estimate  $D_i$  from  $Y_i$  and  $Y_{i-1}$ , such that the performance is the same if the polarity of  $Y_i$  is inverted for all  $i$ . We ask for a simple decoder, not necessarily ML.
- (c) Determine (or estimate) the error probability of your decoder.

PROBLEM 4. The following figure shows *part* of the plot of a function  $|\psi_{\mathcal{F}}(f)|^2$ , where  $\psi_{\mathcal{F}}(f)$  is the Fourier transform of some pulse  $\psi(t)$ . Complete the plot (for positive and negative frequencies) and label the ordinate, knowing that the following conditions are satisfied:

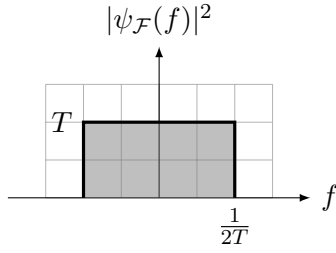
- For every pair of integers  $k$  and  $l$ ,

$$\int \psi(t - kT) \psi(t - lT) dt = \mathbb{1}\{k = l\}.$$

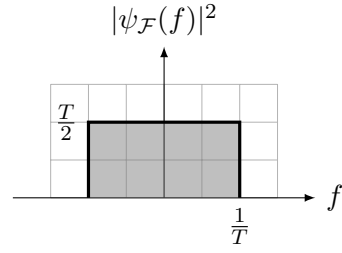
- $\psi(t)$  is real-valued.
- $\psi_{\mathcal{F}}(f) = 0$  for  $|f| > \frac{1}{T}$ .



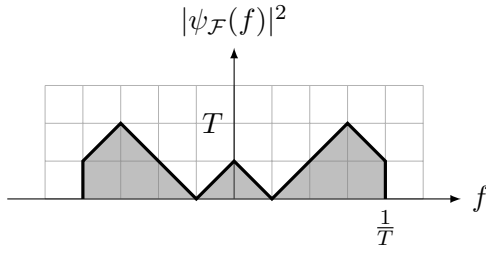
PROBLEM 5. For each of the following functions  $|\psi_{\mathcal{F}}(f)|^2$ , indicate whether the corresponding pulse  $\psi(t)$  has unit norm and/or is orthogonal to its time-translates by multiples of  $T$ . The function in (d) is  $\text{sinc}^2(fT)$ .



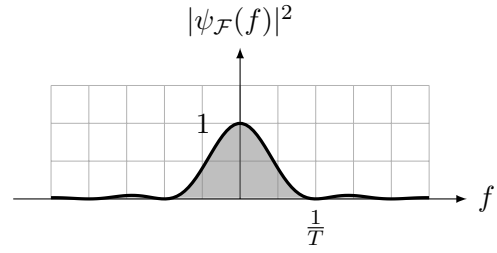
(a)



(b)



(c)



(d)

PROBLEM 6. Consider the following transmitter/receiver design problem for an additive non-white Gaussian noise channel.

- (a) Let the hypothesis  $H$  be uniformly distributed in  $\mathcal{H} = \{0, \dots, m-1\}$  and when  $H = i$ ,  $i \in \mathcal{H}$ , let  $w_i(t)$  be the channel input. The channel output is then

$$R(t) = w_i(t) + N(t),$$

where  $N(t)$  is Gaussian noise of power spectral density  $G(f)$ , where we assume that  $G(f) \neq 0$  for all  $f$ . Describe a receiver that, based on the channel output  $R(t)$ , decides on the value of  $H$  with least probability of error.

*Hint:* Find a way to transform this problem into one that you can solve.

- (b) Consider the setting as in part (a), except that now you get to design the signal set with the restrictions that  $m = 2$  and that the average energy cannot exceed  $\mathcal{E}$ . We also assume that  $|G(f)|^2$  is constant in the interval  $[a, b]$ ,  $a < b$ , where it also achieves its global minimum. Find two signals that achieve the smallest possible error probability under an ML decoding rule.