ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 14 Principles of Digital Communications
Problem Set 7 Apr. 02, 2025

PROBLEM 1. In this problem, we develop further intuition about matched filters. You may assume that all waveforms are real-valued. Let $R(t) = \pm w(t) + N(t)$ be the channel output, where $N(t)$ is additive white Gaussian noise of power spectral density $\frac{N_0}{2}$ and $w(t)$ is an arbitrary but fixed pulse. Let $\phi(t)$ be a unit-norm but otherwise arbitrary pulse, and consider the receiver operation

$$Y = \langle R, \phi \rangle = \langle w, \phi \rangle + \langle N, \phi \rangle$$

The signal-to-noise ratio (SNR) is defined as

$$\text{SNR} \triangleq \frac{|\langle w, \phi \rangle|^2}{\mathbb{E}[|\langle N, \phi \rangle|^2]}$$

Notice that the SNR remains the same if we scale $\phi(t)$ by a constant factor. Notice also that

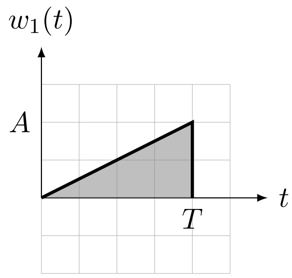
$$\mathbb{E}[|\langle N, \phi \rangle|^2] = \frac{N_0}{2}$$

(a) Use the Cauchy–Schwarz inequality to give an upper bound on the SNR. What is the condition for equality in the Cauchy–Schwarz inequality? Find the $\phi(t)$ that maximizes the SNR. What is the relationship between the maximizing $\phi(t)$ and the signal $w(t)$?

(b) Let us verify that we would get the same result using a pedestrian approach. Instead of waveforms we consider tuples. So let $c = (c_1, c_2)^\mathsf{T} \in \mathbb{R}^2$ and use calculus (instead of the Cauchy–Schwarz inequality) to find the $\phi = (\phi_1, \phi_2)^\mathsf{T} \in \mathbb{R}^2$ that maximizes $\langle c, \phi \rangle$ subject to the constraint that ϕ has unit norm.

(c) Verify with a picture (convolution) that the output at time T of a filter with input $w(t)$ and impulse response $h(t) = w(T - t)$ is indeed $\langle w, w \rangle = \int_{-\infty}^{\infty} w^2(t)dt$.

PROBLEM 2. Let $w_1(t)$ be as shown below and let $w_2(t) = w_1(t - T_d)$, where $T_d \geq T$ is a fixed number known to the receiver. One of the two pulses is selected at random and transmitted across the AWGN channel of noise power spectral density $\frac{N_0}{2}$.



(a) Describe an ML receiver that decides which pulse was transmitted. The n-tuple former must contain a single causal matched filter. Finally, draw the matched filter impulse response.

(b) Express the error probability of the receiver in (a) in terms of A, T, T_d, N_0. Consider both cases $T_d \geq T$ and $T_d < T$.

PROBLEM 3. In this problem, we consider the implementation of matched filter receivers. In particular, we consider frequency-shift keying (FSK) with the following signals:

$$w_j(t) = \begin{cases} \sqrt{\frac{2}{T}} \cos 2\pi \frac{n_j}{T} t & 0 \leq t \leq T \\ 0 & \text{otherwise,} \end{cases}$$

where $n_j \in \mathbb{Z}$ and $0 \leq j \leq m-1$. Thus, the communication scheme consists of m signals $w_j(t)$ of different frequencies $\frac{n_j}{T}$.

(a) Determine the impulse response $h_j(t)$ of a causal matched filter for the signal $w_j(t)$. Plot $h_j(t)$ and specify the sampling time.

(b) Sketch the matched filter receiver. How many matched filters are needed?

(c) Sketch the output of the matched filter with impulse response $h_j(t)$ when the input is $w_j(t)$.

PROBLEM 4. Let the message $H \in \{1, \ldots, m\}$ be uniformly distributed and consider the communication problem described by

$$H = i: \quad Y = c_i + Z, \quad Z \sim \mathcal{N}(0, \sigma^2 I_m),$$

where $Y = (Y_1, \ldots, Y_m)^\mathsf{T} \in \mathbb{R}^m$ is the received vector and $\{c_1, \ldots, c_m\} \subset \mathbb{R}^m$ is the codebook consisting of constant-energy codewords that are orthogonal to each other. Without loss of essential generality, we can assume

$$c_i = \sqrt{\mathcal{E}} e_i,$$

where e_i is the ith unit vector in $\mathbb{R}^m$, i.e. the vector that contains 1 at position i and 0 elsewhere, and $\mathcal{E}$ is some positive constant.

(a) Describe the statistics of Y_j for $j = 1, \ldots, m$ given that $H = 1$.

(b) Consider a suboptimal receiver that uses a threshold $t = \alpha\sqrt{\mathcal{E}}$ where $0 < \alpha < 1$. The receiver declares $\hat{H} = i$ if i is the only integer such that $Y_i \geq t$. If there is no such i or there is more than one index i for which $Y_i \geq t$, the receiver declares that it cannot decide. This will be viewed as an error. Let $E_i = \{Y_i \geq t\}$ and describe, in words, the meaning of the event

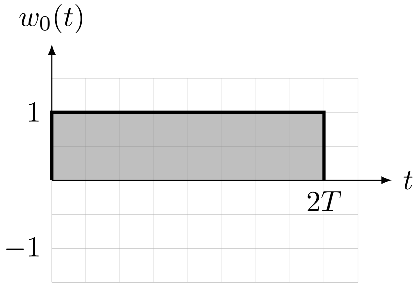
$$E_1 \cap E_2^c \cap E_3^c \cap \cdots \cap E_m^c$$

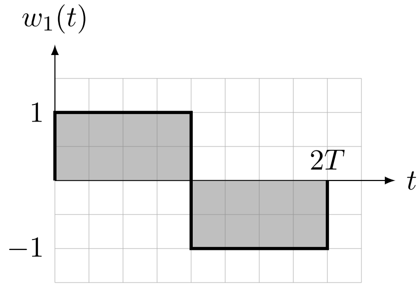
(c) Find an upper bound to the probability that the above event *does not* occur when $H = 1$. Express your result using the Q function.

(d) Now let $m = 2^k$ and let $\mathcal{E} = k\mathcal{E}_b$ for some fixed energy per bit $\mathcal{E}_b$. Prove that the error probability goes to 0 as $k \to \infty$, provided that $\frac{\mathcal{E}_b}{\sigma^2} > \frac{2 \ln 2}{\alpha^2}$.
Hint: Use $m - 1 < m = e^{\ln m}$ and $Q(x) < \frac{1}{2} e^{-\frac{x^2}{2}}$.

PROBLEM 5. *(Signal translation)*

Consider the signals $w_0(t)$ and $w_1(t)$ shown below, used to communicate 1 bit across the AWGN channel of power spectral density $\frac{N_0}{2}$.





(a) Determine an orthonormal basis $\{\psi_0(t), \psi_1(t)\}$ for the space spanned by $\{w_0(t), w_1(t)\}$ and find the corresponding codewords c_0 and c_1. Work out two solutions, one obtained via Gram–Schmidt and one in which $\psi_1(t)$ is a delayed version of $\psi_0(t)$. Which of the two solutions would you choose if you had to implement the system?

(b) Let X be a uniformly distributed binary random variable that takes values in $\{0, 1\}$. We want to communicate the value of X over an additive white Gaussian noise channel. When $X = 0$, we send $w_0(t)$, and when $X = 1$, we send $w_1(t)$. Draw the block diagram of an ML receiver based on a single matched filter.

(c) Determine the error probability P_e of your receiver as a function of T and N_0.

(d) Find a suitable waveform $v(t)$ such that the signals $\tilde{w}_0(t) = w_0(t) - v(t)$ and $\tilde{w}_1(t) = w_1(t) - v(t)$ have minimum energy. Plot the resulting waveforms.

(e) What is the name of the signaling scheme that uses signals such as $\tilde{w}_0(t)$ and $\tilde{w}_1(t)$? Argue that one obtains this kind of signaling scheme independently of the initial choice of $w_0(t)$ and $w_1(t)$.

Problem 6. *(Orthogonal signal sets)*

Consider a set $\mathcal{W} = \{w_0(t), \ldots, w_{m-1}(t)\}$ of mutually orthogonal signals with squared norm $\mathcal{E}$, each used with equal probability.

(a) Find the minimum-energy signal set $\tilde{\mathcal{W}} = \{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ obtained by translating the original set.

(b) Let $\tilde{\mathcal{E}}$ be the average energy of a signal picked at random within $\tilde{\mathcal{W}}$. Determine $\tilde{\mathcal{E}}$ and the energy saving $\mathcal{E} - \tilde{\mathcal{E}}$.

(c) Determine the dimension of the inner product space spanned by $\tilde{\mathcal{W}}$.