

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE  
School of Computer and Communication Sciences

## Handout 2

### Problem Set 1

# Principles of Digital Communications

## Feb. 19, 2025

PROBLEM 1. Assume that  $X_1$  and  $X_2$  are independent random variables that are uniformly distributed in the interval  $[0, 1]$ . Compute the probability of the following events.

(a)  $0 \leq X_1 - X_2 \leq \frac{1}{3}$ .      (c)  $X_2 - X_1 = \frac{1}{2}$ .

(b)  $X_1^3 \leq X_2 \leq X_1^2$ .      (d)  $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq (\frac{1}{2})^2$ .

(e) Given that  $X_1 \geq \frac{1}{4}$ , compute the probability that  $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq (\frac{1}{2})^2$ .

*Hint:* For each event, identify the corresponding region inside the unit square.

PROBLEM 2. Find the following probabilities.

- (a) A box contains  $m$  white and  $n$  black balls. Suppose  $k$  balls are drawn. Find the probability of drawing at least one white ball.
- (b) We have two coins; the first is fair and the second is two-headed. We pick one of the coins at random, toss it twice, and obtain heads both times. Find the probability that the coin is fair.

PROBLEM 3. Assume  $X$  and  $Y$  are random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} A, & 0 \leq x < y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Are  $X$  and  $Y$  independent?
- (b) Compute the value of  $A$ .
- (c) Find the density function of  $Y$ . Do this first by arguing geometrically, then compute it analytically.
- (d) Find  $\mathbb{E}[X|Y = y]$ . *Hint:* Argue geometrically.
- (e) The  $\mathbb{E}[X|Y = y]$  found in (d) is a function of  $y$ , call it  $f(y)$ . Find  $\mathbb{E}[f(Y)]$ . This is  $\mathbb{E}[\mathbb{E}[X|Y]]$ .
- (f) Find  $\mathbb{E}[X]$  from the definition. Verify that  $\mathbb{E}[X]$  is equal to  $\mathbb{E}[\mathbb{E}[X|Y]]$  computed in (e). Is this a coincidence?

PROBLEM 4. Let  $Z_1$  and  $Z_2$  be i.i.d. zero-mean Gaussian random variables, i.e., the pdf of  $Z_i$ ,  $i = 1, 2$  is

$$f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{z^2}{2\sigma^2}}$$

for some  $\sigma > 0$ . Define

$$X := \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2}} \quad \text{and} \quad Y := \frac{Z_2}{\sqrt{Z_1^2 + Z_2^2}}.$$

Prove that  $(X, Y)$  is a uniformly chosen point on the unit circle.

PROBLEM 5.

- (a) Let  $X$  and  $Y$  be two continuous real-valued random variables with joint probability density function  $f_{X,Y}$ . Show that if  $X$  and  $Y$  are independent, they are also *uncorrelated*.
- (b) Consider two independent and uniformly distributed random variables  $U \in \{0, 1\}$  and  $V \in \{0, 1\}$ . Assume that  $X$  and  $Y$  are defined as follows:  $X = U + V$  and  $Y = |U - V|$ . Are  $X$  and  $Y$  independent? Compute the covariance of  $X$  and  $Y$ . What do you conclude?

PROBLEM 6. Assume you pick a point on the surface of the unit sphere (i.e. the sphere centered at the origin with radius 1) uniformly at random and  $(X, Y, Z)$  denotes its coordinates (in 3D space). Compute  $\mathbb{E}[X^2]$ .

PROBLEM 7. Assume the random variable  $X$  has an exponential distribution given by  $f_X(x) = e^{-x}$  when  $x \geq 0$ . Similarly,  $\hat{X}$  is exponentially distributed with  $f_{\hat{X}}(\hat{x}) = 2e^{-2\hat{x}}$  for  $\hat{x} \geq 0$ .

- (a) For what values of  $x$  do we have  $f_X(x) \leq f_{\hat{X}}(x)$ ?
- (b) Calculate  $\mathbb{P}\left(f_X(X) \leq f_{\hat{X}}(X)\right)$ .
- (c) Calculate  $\mathbb{P}\left(f_X(\hat{X}) \geq f_{\hat{X}}(\hat{X})\right)$ .