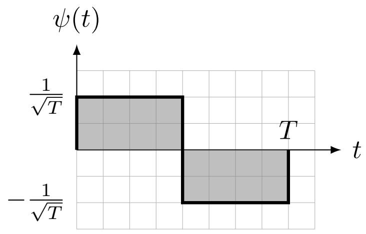
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 24 Principles of Digital Communications
Problem Set 10 May 7, 2025

PROBLEM 1. Derive the power spectral density of the random process

$$X(t) = \sum_{i \in \mathbb{Z}} X_i \psi(t - iT - \Theta),$$



where $\{X_i\}_{i \in \mathbb{Z}}$ is an i.i.d. sequence of uniformly distributed random variables taking values in $\{\pm\sqrt{\mathcal{E}}\}$, Θ is uniformly distributed in the interval $[0, T]$, and $\psi(t)$ is as shown in the plot (called Manchester pulse). The Manchester pulse guarantees that $X(t)$ has at least one transition per symbol, which facilitates the clock recovery at the receiver.

PROBLEM 2. Consider the random process

$$X(t) = \sum_{i \in \mathbb{Z}} X_i \sqrt{\mathcal{E}_s} \psi(t - iT_s - T_0),$$

where T_s and $\mathcal{E}_s$ are fixed positive numbers, $\psi(t)$ is some unit-energy function, T_0 is a uniformly distributed random variable taking values in $[0, T_s)$, and $\{X_i\}_{i \in \mathbb{Z}}$ is the output of the convolutional encoder described by

$$X_{2n} = B_n B_{n-2}, \qquad X_{2n+1} = B_n B_{n-1} B_{n-2},$$

with i.i.d. input sequence $\{B_i\}_{i \in \mathbb{Z}}$ taking values in $\{\pm 1\}$.

(a) Express the power spectral density of $X(t)$ for a general $\psi(t)$.

(b) Plot the power spectral density of $X(t)$ assuming that $\psi(t)$ is a unit-norm rectangular pulse of width T_s.

PROBLEM 3. From the decoder's point of view, inter-symbol interference (ISI) can be modeled as follows:

$$\begin{aligned} Y_i &= X_i + Z_i \\ X_i &= \sum_{j=0}^{L} B_{i-j} h_j, \quad i \in \mathbb{N} \end{aligned} \tag{$*$}$$

where B_i is the ith information bit, $h_0, \dots, h_L$ are coefficients that describe the inter-symbol interference, and Z_i is zero-mean, Gaussian, of variance σ^2, and statistically independent of everything else. Relationship $(*)$ can be described by a trellis, and the ML decision rule can be implemented by the Viterbi algorithm.

(a) Draw the trellis that describes all sequences of the form $X_1, \dots, X_6$ resulting from information sequences of the form $B_1, \dots, B_5, 0$, $B_i \in \{0, 1\}$, assuming

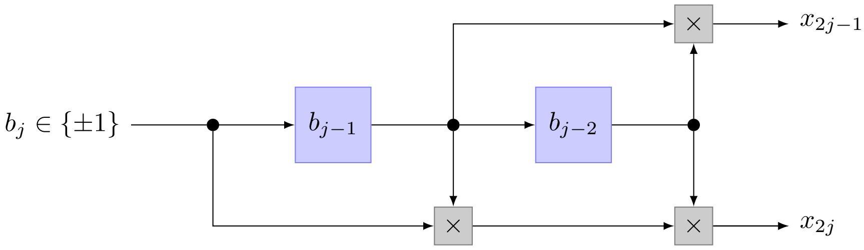
$$h_i = \begin{cases} 1, & i = 0 \\ -2, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

To determine the initial state, you may assume that the preceding information sequence terminated with 0. Label the trellis edges with the input/output symbols.

(b) Specify a metric $f(x_1,\ldots,x_6)=\sum_{i=1}^{6} f(x_i,y_i)$ whose minimization or maximization with respect to the valid $x_1,\ldots,x_6$ leads to a maximum likelihood decision. Specify if your metric needs to be minimized or maximized.

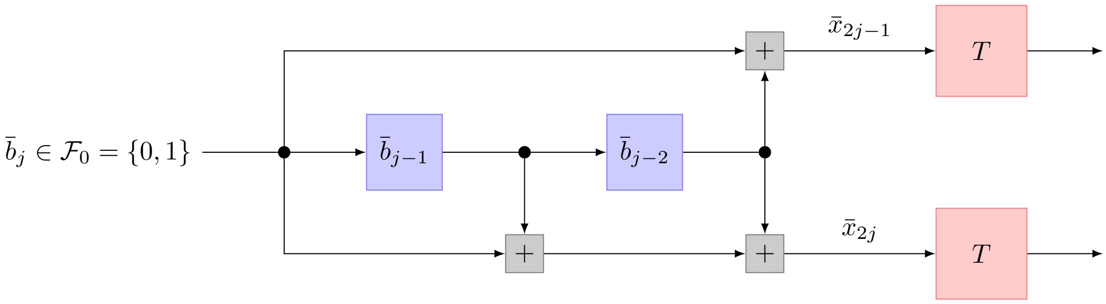
(c) Assume $y_1,\ldots,y_6=\{2,0,-1,1,0,-1\}$. Find the maximum likelihood estimate of the information sequence $B_1,\ldots,B_5$.

PROBLEM 4. An output sequence $x_1,\ldots,x_{10}$ from the convolutional encoder shown below is transmitted over the discrete-time AWGN channel. The initial and final state of the encoder is $(1,1)$. Using the Viterbi algorithm, find the maximum likelihood information sequence $\hat{b}_1,\ldots,\hat{b}_3,1,1$, knowing that $b_1,\ldots,b_3$ are drawn independently and uniformly from $\{\pm 1\}$ and that the channel output $y_1,\ldots,y_{10}=\{1,2,-1,4,-2,1,1,-3,-1,-2\}$. (It is for convenience that we are choosing integers rather than real numbers.)

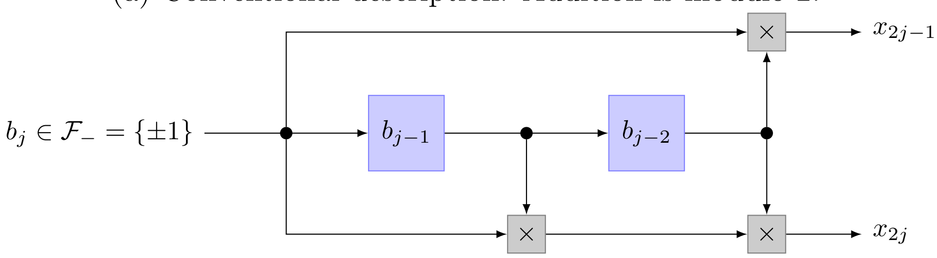


PROBLEM 5. Consider the following two encoders where the map $T:\mathcal{F}_0\to\mathcal{F}_-$ sends 0 to 1 and 1 to -1. Show that the two encoders produce the same output when their inputs are related by $b_j=T(\bar{b}_j)$.

Hint: For $a,b\in\mathcal{F}_0$, $T(a+b)=T(a)\times T(b)$, where addition is modulo 2 and multiplication is over $\mathbb{R}$.



(a) Conventional description. Addition is modulo 2.



(b) Description used in the book. Multiplication is over $\mathbb{R}$.

Comment: The encoder of (b) is linear over the field $\mathcal{F}_-$, whereas the encoder of (a) is linear over $\mathcal{F}_0$ only if we omit the output map T. The comparison of the two figures should explain why in this chapter we have opted for the description of (b) even though the standard description of a convolutional encoder is as in (a).