Principles of Digital Communication

May 28th, 2021

Question "How to design a convolutional code"?

- General question: How to design a code?

- Problem in the abstract:

$$\underbrace{b_1 \dots b_k}_{k \text{ information bits}} \longrightarrow \text{Enc} \xrightarrow{x_1 \dots x_n} \text{Channel} \longrightarrow Y_1 \dots Y_n \longrightarrow \text{Dec} \longrightarrow \widehat{b_1 \dots b_k}$$

$$\vec{Y} = \vec{x} + \vec{Z}$$

$\hookrightarrow Z \sim N(0,1)$

We need to design Enc/Dec

subject to constraints:

k/n large $= R$

$\frac{1}{n}\sum_{i=1}^{n} x_i^2$ small $= P$

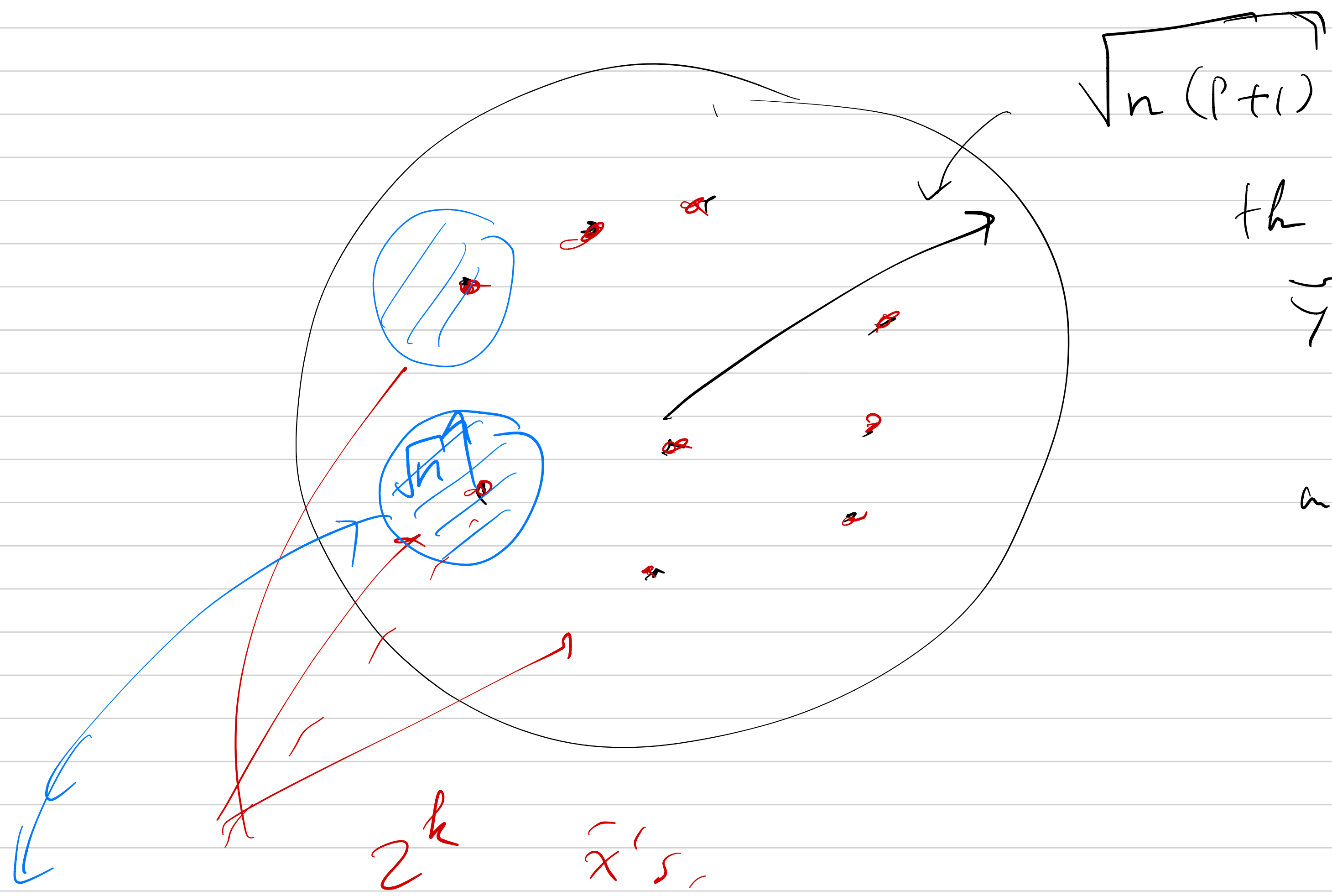
$\Pr(\widehat{b_1 \dots b_k} \neq b_1 \dots b_k)$ small P_{error}

Primary question:

given R, P, P_{error}, is there an enc/dec

with these parameters?

Recall from some weeks ago:



the vector $\bar{y}$ will belong to this sphere with high probability

If $\bullet$ is sent where $\bar{y}$ will be.

Reliable communication (P_{error} is small) requires that Blue spheres are $\sim$ disjoint.

$$\Rightarrow \quad 2^k \left(\sqrt{n}\right)^n \le \left(\sqrt{(1+P)n}\right)^n$$

$$\Rightarrow \quad 2^R \le \sqrt{1+P} \quad \Rightarrow \quad \boxed{R \le \frac{1}{2}\log_2(1+P)}$$

Ex: $P \le 1 \qquad \sigma^2 = 10$

$$R \le \frac{1}{2}\log_2\left(1+\frac{1}{10}\right) \approx \frac{1}{14} \approx 0.07$$

$n = 60000 \Rightarrow k \lessapprox 4000$

Question: how far is "$\frac{1}{2}\log(1+P)$" from the true value of best possible R.

Ans: for any $P_e > 0$ (no matter how small) any $R < \boxed{\frac{1}{2}\log(1+P)}$ (but arbitrarily close to $\frac{1}{2}\log(1+P)$)

for any P.

there exists an enc/dec s.t.

$\frac{k}{n} > R$ & $P_{error} < P_e$.

and $\frac{1}{n}\sum x_i^2 \le P$.

- More "finer" question: do there exist low-complexity (enc/dec)s that approach $\frac{1}{2}\log(1+P)$ limit?

- Ans: yes

- "Constructive" methods to design the enc/dec.

- Remarks: If we restrict ourselves to convolutional encoder & viterbi decoder, the "$\frac{1}{2}\log(1+P)$" needs to be replaced by $\frac{1}{2}\log(1+\frac{P}{2})$ (??)

- even with this it is not so clear how to choose