Principles of Digital Communication

April 28, 2021

Reading war 5.[1–4].

Motivation :

$$i : k \text{ bit} \longrightarrow \underbrace{c_i}_{\in \mathbb{C}^n} \longrightarrow w_i(t) = \sum_j c_{ij}\, \psi_j(t) \longrightarrow$$

TX

$\{\psi_i\}$ an ⊥-normal basis.

RCVR 1

$\xrightarrow{R(t)}$ $\langle R, w_1 \rangle$ ⋮ $\langle R, w_m \rangle$ requires $m = 2^k$ inner products

RCVR 2

$\xrightarrow{R(t)}$ $\langle R, \psi_1 \rangle$ ⋮ $\langle R, \psi_n \rangle$ requires n inner products.

if it is also the case that

$\{\psi_i\}$ are not only ⊥-normal but

time shifts of each other

$\xrightarrow{R(t)}$ $\boxed{\psi^*(-t)}$ ⟶

Sample at n time instants

$t_1, \dots t_n$ s.t

$\psi_j(t) = \psi(t - t_j)$. $j = 1 \dots n$.

Based on these considerations we are led to the following question: is there a simple test still tell us if the integer shifts of $\psi(\cdot)$ form an $\perp$-normal set?

$\equiv$ we want to test

$$\underline{\int \psi^*(t)\,\psi(t-n)\,dt} = \begin{cases} 1 & n=0 \\ 0 & n=\pm1,\pm2,\pm3,\dots \end{cases}$$

$$x(t) = \psi(t) \longleftrightarrow x_F(f) = \psi_F(f)$$

$$y(t) = \psi(t-n) \longleftrightarrow Y_F(f) = \psi_F(f)\cdot e^{-j2\pi n f}$$

Aside $\quad y_F(f) = \int y(t)\, e^{-j2\pi f t}\, dt$

$$= \int \psi(t-n)\, e^{-j2\pi f t}\, dt$$

$$= \int \psi(t-n)\, e^{-j2\pi f (t-n+n)}\, dt$$

$$= e^{-j2\pi f n} \int \psi(t-n)\, e^{-j2\pi f (t-n)}\, dt$$

$$= \quad '' \quad \int \psi(t)\, e^{-j2\pi f t}\, dt$$

$$= e^{-j2\pi f n} \cdot \psi_F(f).$$

$\perp$-normality test

$$\equiv \underline{\int \psi_F^*(f)\,\psi_F(f)\, e^{-j2\pi f n}}\, df = \begin{cases} 1 & n=0 \\ 0 & n=\pm1,\pm2,\pm3. \end{cases}$$

$$\equiv \int |\psi_F(f)|^2 e^{-j2\pi f n} \, df = \mathbb{1}\{n=0\}$$

$$\equiv \underline{\sum_{k \text{ integer}}' \int_{-\frac{1}{2}+k}^{\frac{1}{2}+k} |\psi_F(f)|^2 e^{-j2\pi f n} \, df} = \mathbb{1}\{n=0\}$$

$$\equiv \sum_k \int_{-\frac{1}{2}}^{\frac{1}{2}} |\psi_F(f+k)|^2 e^{-j2\pi (f+k) n} \, df = \text{"}$$

$$\equiv \sum_k \int_{-1/2}^{1/2} |\psi_F(f+k)|^2 e^{-j2\pi f n} \, df = \mathbb{1}\{n=0\}$$

$$\equiv \int_{-\frac{1}{2}}^{1/2} \underbrace{\left(\sum_k |\psi_F(f+k)|^2 \right)}_{g(f)} \cdot e^{-j2\pi f n} \, df = \underline{\underline{\mathbb{1}\{n=0\}}}$$

note that $f \mapsto g(f)$ is a periodic function with period 1.

so $g(f) = \sum_n' g_n e^{+j2\pi f n}$ (discrete Fourier series).

with $g_n = \int_{-1/2}^{1/2} g(f) e^{-j2\pi f n} \, df$

We now see that our condition for $\{\varphi(t-n)\}$ to be 1-normal family is

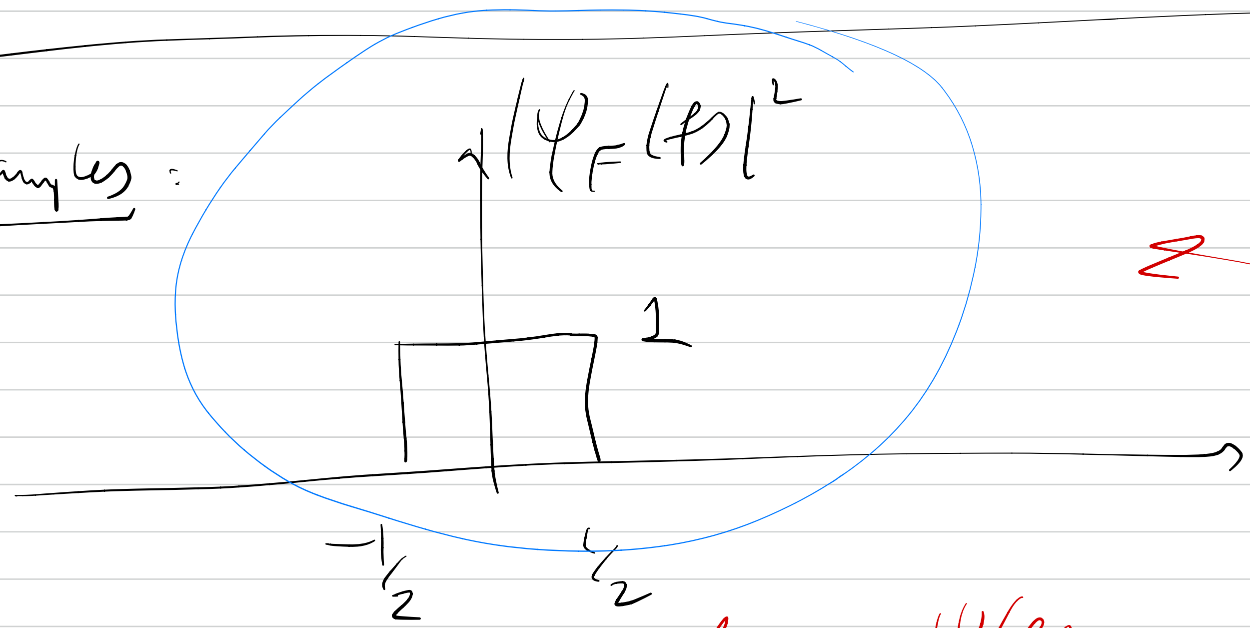
$$g_0 = 1 \quad g_1 = g_{-1} = g_2 = g_{-2} \cdots = 0,$$

i.e. $g(f) = 1$. Consequently we have the following test:

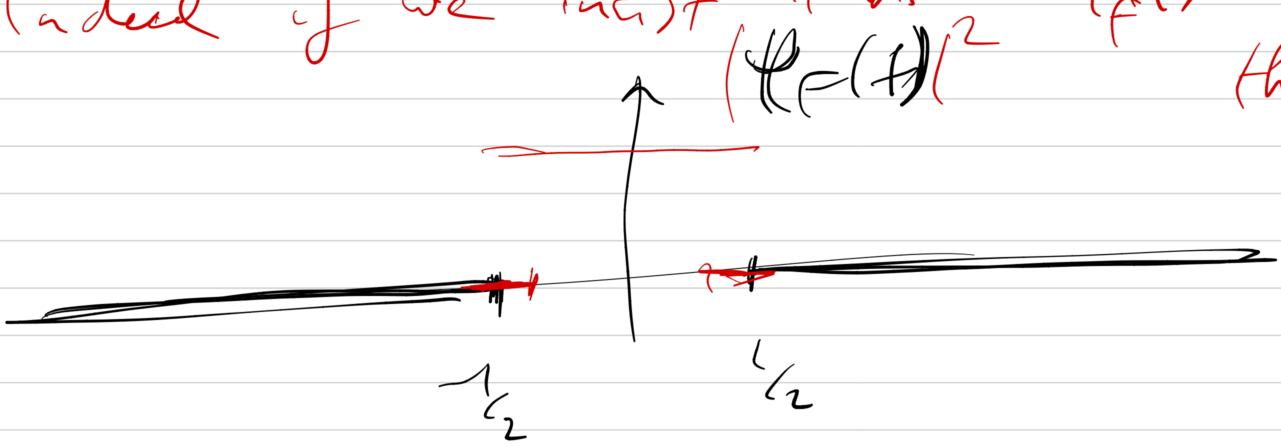
Thm (Nyquist Criterion): $\{\varphi_n = \varphi(t-n) : n \in \mathbb{Z}\}$ form an 1-normal collection if and only if

$$\sum_n{}' |\varphi_F(f-n)|^2 = 1.$$

Examples:



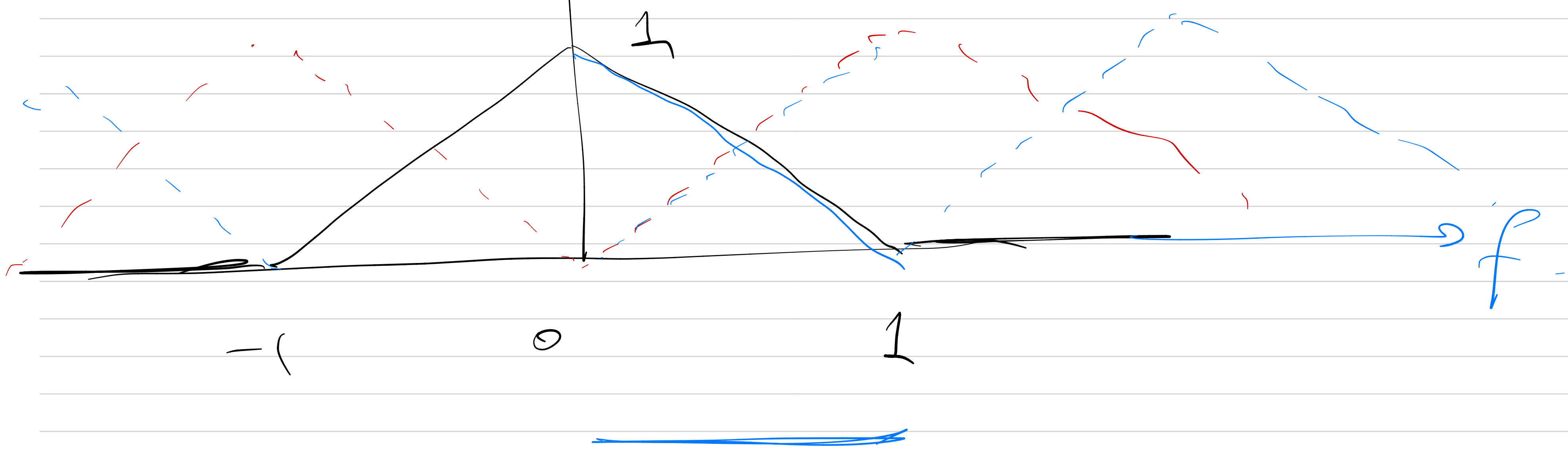
Indeed if we insist that $\varphi_F(f) = 0$ when $|f| > 1/2$ then φ_F needs to be equal to



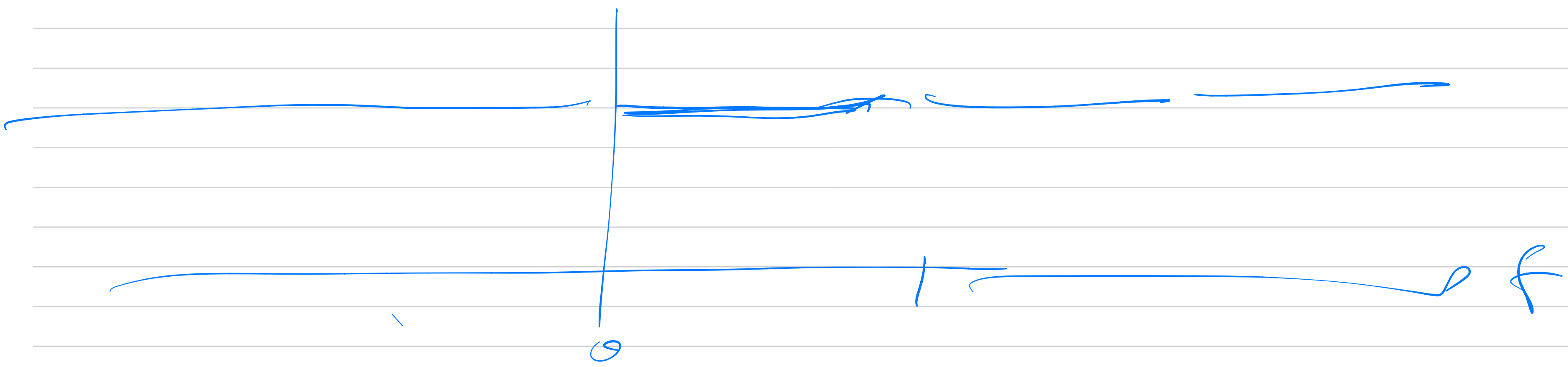
$\Rightarrow$ the example above gives the minimal bandwidth function that satisfies the Nyquist condition.

Ex.:

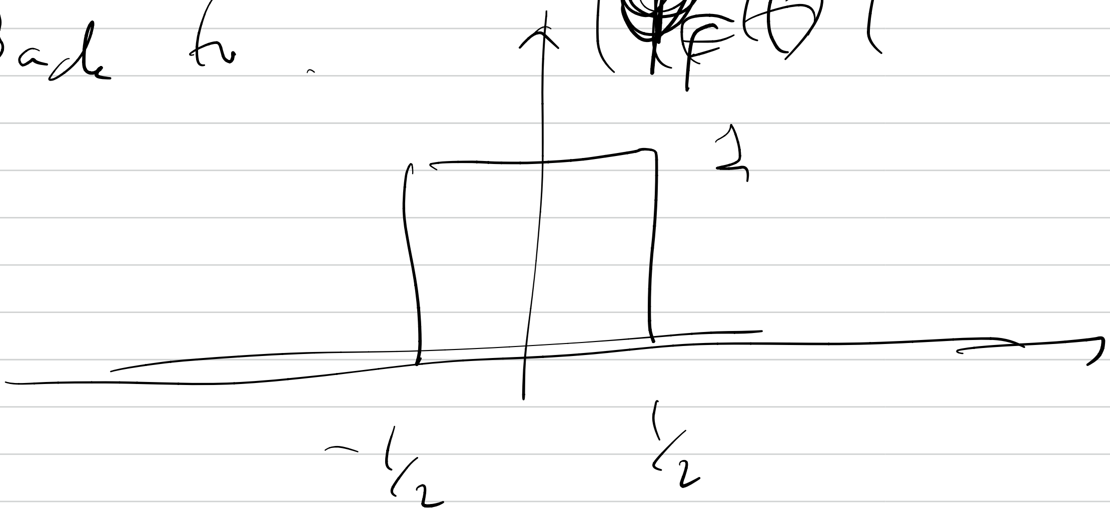
$|\psi_F(f)|^2$



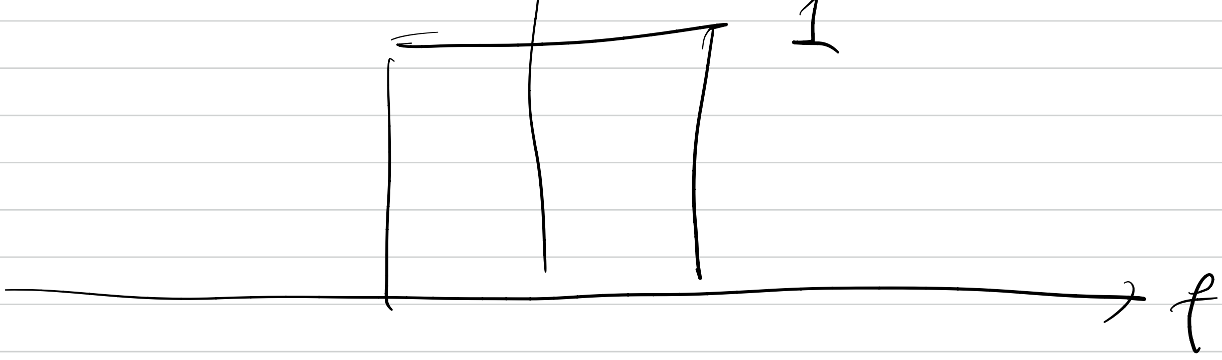
$$\sum_n |\psi_F(f-n)|^2$$



Back to: $|\psi_F(f)|^2$



We can choose $\psi_F(f)$



$\Rightarrow \quad \psi(t) = \text{sinc}(t).$

So, one can build a transmitter & RCVR as

$$i \longrightarrow c_i \longrightarrow w_i(t) = \sum_{j=1}^{n} c_{ij}\,\mathrm{sinc}(t-j)$$

$c_i \in \mathbb{C}^n$

$$R(t) \longrightarrow \boxed{\mathrm{sinc}(t)} \xrightarrow[1,2,3,\dots]{} Y_1, Y_2, \dots, Y_n$$

$$\hat{i} = \arg\min_{i} \|Y - c_i\|^2$$

~~S4.1~~ S_2

5.3 : Power Spectral Density.

We saw last Friday: if $\dots, X_{-2}, X_{-1}, X_0, X_1, \dots$ is

a stationary discrete-time stochastic process with zero-mean

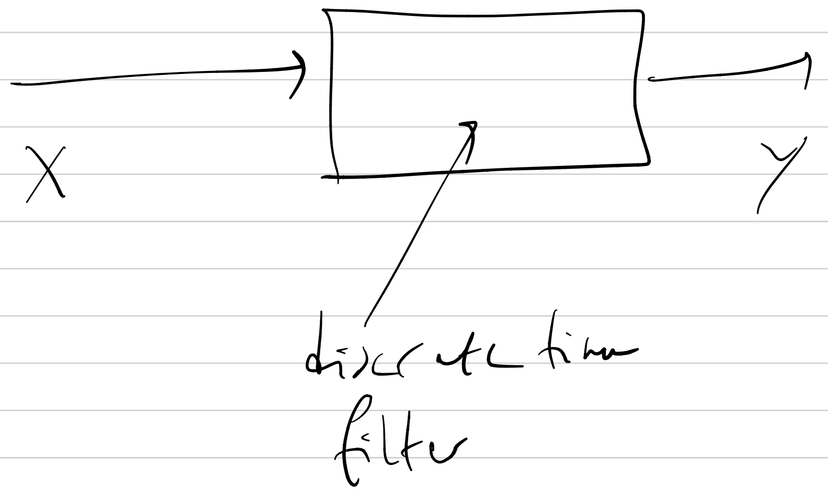
the quantity $S_X(f) \triangleq \sum_k R_X(k) e^{-j2\pi fk}$ ← DFT of R_X

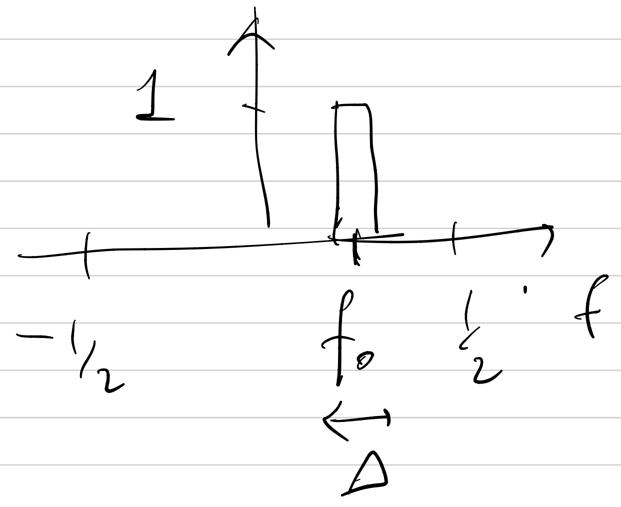
$$R_X(k) = E[X_0^* X_k] = E[X_i^* X_{i+k}].$$

↑ autocorrelation function

tells us how much power the process has at frequency f.

What I mean by this:





$$E\left[|Y_i|^2\right] = \int_{f_0 - \frac{\Delta}{2}}^{f_0 + \Delta/2} S_X(f)\, df$$

$$\approx \underline{\underline{S_X(f_0)\,\Delta}}.$$

This gives meaning to the name "Power Spectral Density" of the process X.

The continuous time analog is : $(X(t) : t \in \mathbb{R})$ a continuous time stochastic process. Assume $X(t)$ to be 0-mean. Define

$$R(\tau) = E\left[X(t)^* X(t+\tau)\right] \quad \left(\begin{array}{l}\text{assume } X(t)\\ \text{is stationary}\end{array}\right)$$

as the auto correlation function.

($R(0) = E(|X(t)|^2)$ is the power of X

Let us also define

$$S_X(f) = \text{F. Transform of } R_X(\tau)$$
$$= \int R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau.$$

One can show:

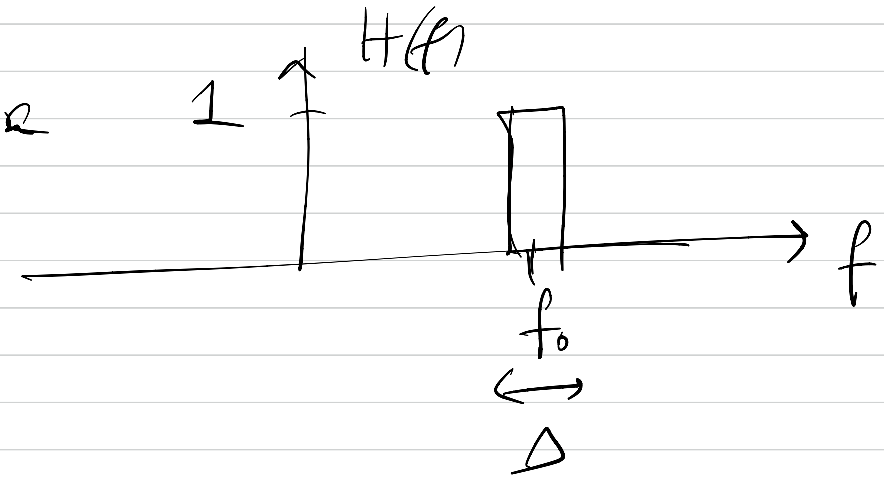
$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = (X*h)(t)$$

$$S_Y(f) = S_X(f)\,|H(f)|^2.$$

$$R_Y(0) = \int S_Y(f)\,df = \int S_X(f)\,|H(f)|^2\,df \qquad //$$

power in Y

choose



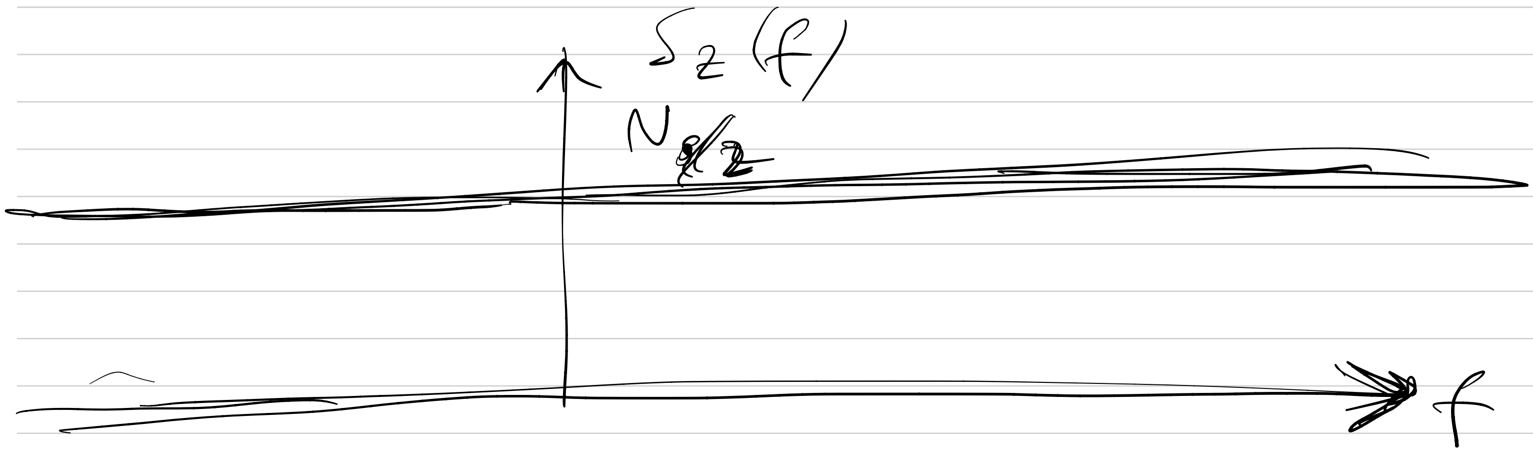
then Y(t) will be the part of X that lives in the freq. interval $(f_0 - \Delta/2,\ f_0 + \Delta/2)$

and will have power $\approx \underline{S_X(f_0)\,\Delta}$

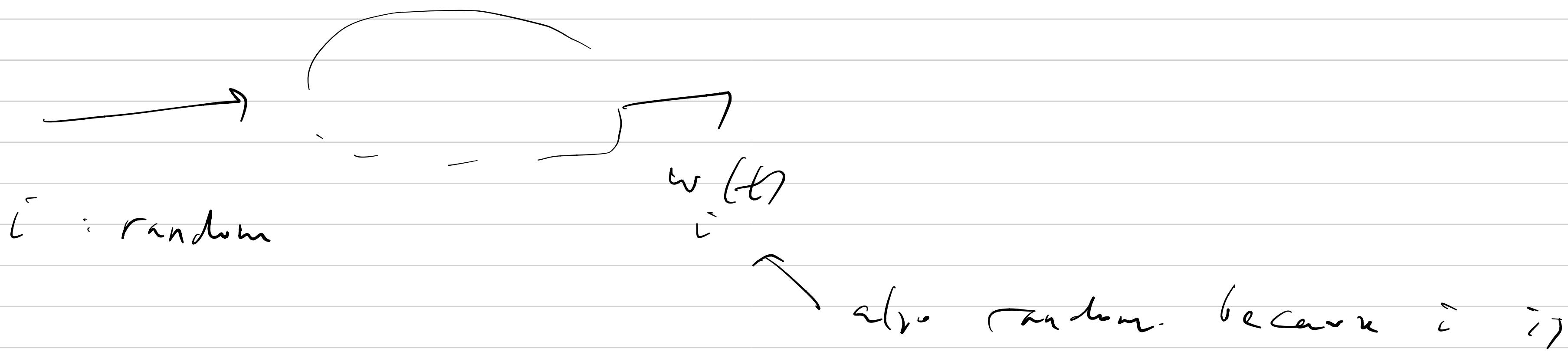
This justifies the name "power spectral density" for $S_X(f)$.

<u>Ex</u>: the "white noise" process $Z(t)$ has

$$S_Z(f) = \frac{N_0}{2} \quad \text{for every } f$$



Having told about what is Power-Spect-Dens. & its "Engineering meaning", its real-life use is



i = k bit integer = $b_1, \ldots, b_k$ $\qquad b_i \in \{0, 1\}$

$\downarrow$

c_i = vector of length n = $c_{i1}, \ldots, c_{in}$

random process

$$b_1, b_2, \ldots b_3, \ldots \longrightarrow \bigcirc \longrightarrow c_1\, c_2\, c_3\, c_4\, c_5\, c_6 \ldots$$

maintains $n = 2k$

random process, e.g.

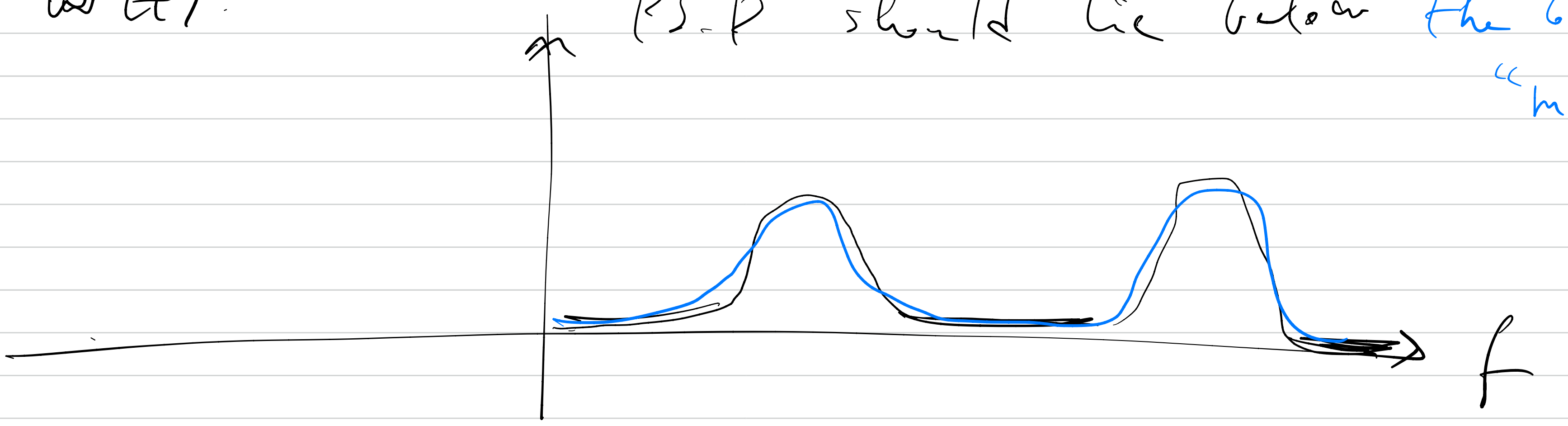
b_i iid $\Pr(b_i = 0) = \Pr(b_i = 1) = \frac{1}{2}$

$$\longrightarrow w(t) = \sum_{j=1}^{\infty} c_j\, \psi(t-j)$$

So the transmitter emits a random waveform with this structure

We will be subject to constraints on the P.S.D of $w(t)$:

PSD should lie below the blue "mask".



This suggests: try to compute the P.S.D of $w(\,)$ in terms of $\psi(\cdot)$ and the statistics of $\{c_j\}$.

We see in 5.3 that

$$\underline{X}(t) = \sum_i \underline{X}_i \, \psi(t-i) \quad \text{then}$$

$$S_X(f) = \underbrace{|\psi_F(f)|^2}\, \sum_k e^{-j2\pi k f} \cdot K_X(k)$$

$$K_X(k) \leftarrow E(X_i^* X_{i+k}).$$