Principles of Digital Communication

March 26, 2021

Reading Assignment: 3.[5, 6, 7, 9].

Where are we in the course

- Hypothesis testing

  $H \in \{0, \dots, m-1\}$ $\quad P_H(i)$

  $Y \in \mathcal{Y}$ $\quad P_{Y|H}(y|i)$

- MAP rule $\hat{H}(Y)$, $\hat{H}(y) = \arg\max_i \underline{P_{H|Y}(i|y)}$

  $= \arg\max_i \underline{P_{Y|H}(y|i)\, P_H(i)}.$

- Suff. stats & irrelevance.

  $Y = (t, u)$ $\quad$ if $P_{H|Y}(i|(t,u))$

  $= \text{funct}(i, t)$

  $\Rightarrow$ u is irrelevant.

  t is a suff stat.

- Special Case: $Y \in \mathbb{R}^n$ $(\mathbb{C}^n)$.

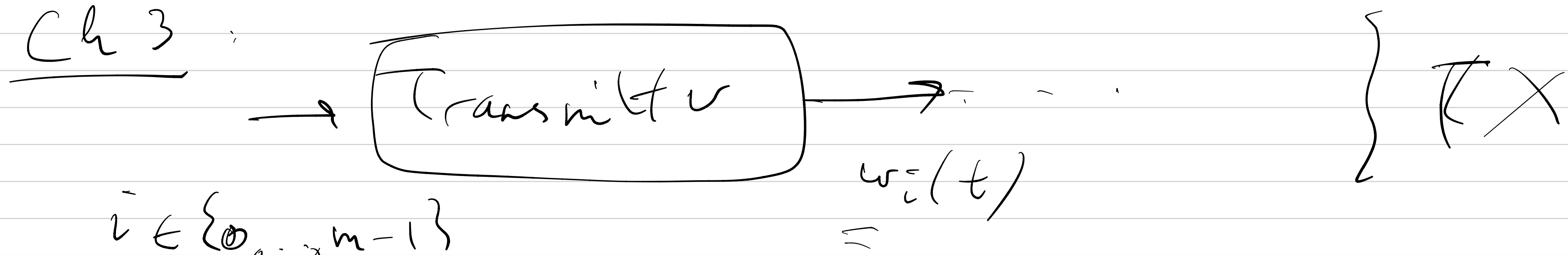
  when $H = i$ $\quad Y = c_i + Z$

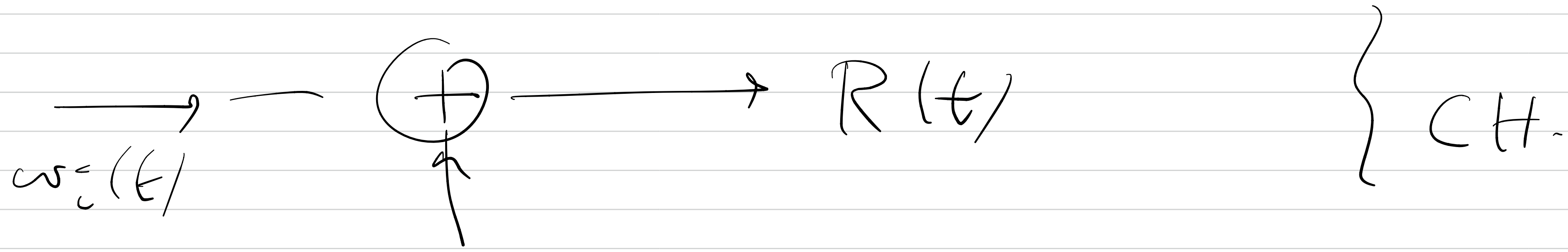
  $c_i \in \mathbb{R}^n$, $Z \sim N(0, I_n \sigma^2)$.

$$\hat{H}_{MAP}(y) = \arg\max_{i} \; p_H(i) \frac{1}{\dots} \exp\left(-\frac{1}{2\sigma^2}\|y-c_i\|^2\right)$$

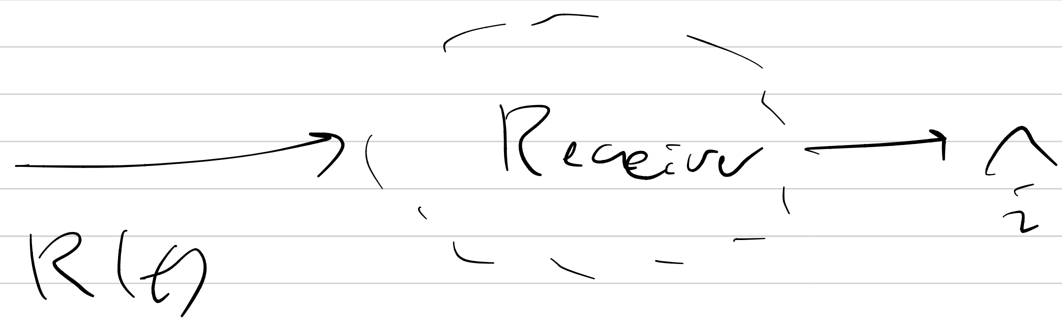
$$\equiv \arg\min_{i} \; \|y-c_i\|^2 - 2\sigma^2 \ln p_H(i).$$

Ch 3:





$Z(t)$: additive white Gaussian noise



To understand how to design the Receiver:

Suppose $\psi_1(t), \dots, \psi_n(t)$ are $\perp$-normal signals

such that $\text{span}\{\psi_1, \dots, \psi_n\} = \text{span}\{w_0, \dots, w_{m-1}\}$

At the receiver form

$$Y_j = \langle R, \psi_j \rangle = \int R(t)\,\psi_j(t)\,dt$$

Claim $(Y_1 \dots Y_n)$ is a sufficient statistic

for any measurements we could have performed that include the Y's.

$\equiv$ given $\left(\langle R, \psi_j\rangle = Y_j \; : \; j = 1 \dots n\right)$ additional

measurements $\left(U_j = \langle R, g_j \rangle \quad j = 1 \dots \ell\right)$

are irrelevant.

Recall $\text{span}(\psi) = \text{span}(w)$ so

$$w_0(t) = c_{01}\,\psi_1(t) + \dots + c_{0n}\,\psi_n(t)$$

$$w_i(t) = c_{i1}\,\psi_1(t) \qquad + c_{in}\,\psi_n(t)$$

So we have m vectors $c_0, \dots, c_{m-1}$

$$R(t) = \sum_{j=1}^{n} c_{ij}\,\psi_j(t) + Z(t) \qquad H = i$$

The receiver now forms

$$Y_j = \langle R, \psi_j \rangle \qquad j = 1 \dots n$$

$$= \left\langle \sum_{\ell=1}^{n} c_{i\ell} \psi_\ell + Z, \psi_j \right\rangle$$

when

$H = i$

$$= \sum_{\ell=1}^{n} c_{i\ell} \langle \psi_\ell, \psi_j \rangle + \underbrace{\langle Z, \psi_j \rangle}_{Z_j}$$

$$= c_{ij} + Z_j$$

So when $H = i$

$$\underbrace{\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} c_{i1} \\ \vdots \\ c_{in} \end{bmatrix}}_{c_i} + \underbrace{\begin{bmatrix} z_1 \\ \\ z_n \end{bmatrix}}$$

$$Z \sim N(0, \tfrac{N_0}{2} I_n)$$

MAP decision of $\hat{i}$ from $(Y_1 \dots Y_n)$.
need to compute

$$\underbrace{\|y - c_i\|^2} - N_0 \ln p_H(i) \quad \text{for } i = 0, \dots, m-1$$

choose smallest.

$$\cancel{\|y\|^2} - 2\langle y, c_i \rangle + \|c_i\|^2 - N_0 \ln p_H(i)$$

$$\|w_i\|^2$$

observe:

$$w_i(t) = c_{i1} \varphi_1(t) + \dots + c_{in} \varphi_n(t)$$

$$w_i(t)^2 = c_{i1}^2 \varphi_1(t)^2 + \dots + c_{in}^2 \varphi_n(t)^2$$

$$+ \dots \varphi_1(t) \varphi_2(t) + \dots + \varphi_{n-1}(t) \varphi_n(t)$$

$$\int dt = c_{i1}^2 + c_{i2}^2 + \dots + c_{in}^2 = \|c_i\|^2$$

what about

$$\langle Y, c_i \rangle = \sum_j c_{ij} \int R(t) \varphi_j(t) dt$$

$$= \int R(t) \left(\sum_j c_{ij} \varphi_j(t) \right) dt$$

So $\langle y, c_i \rangle = \langle R, w_i \rangle$,

So the map rule: compute for $i = 0, \ldots, m-1$

$$-2 \int R(t) w_i(t)\, dt + \int w_i(t)^2\, dt - N_0 \ln p_H(i)$$

and find which is smallest.

$$\equiv \arg\max_i \underline{\int R(t) w_i(t)} \underbrace{- \frac{\|w_i\|^2}{2} + \frac{N_0}{2} \ln p_H(i)}_{q_i}$$

A note on computing inner products of functions. $\int R(t)\, w(t)\, dt$

$$R(t) \longrightarrow \boxed{h(t)} \longrightarrow s(t)$$

$s(t)$ output.

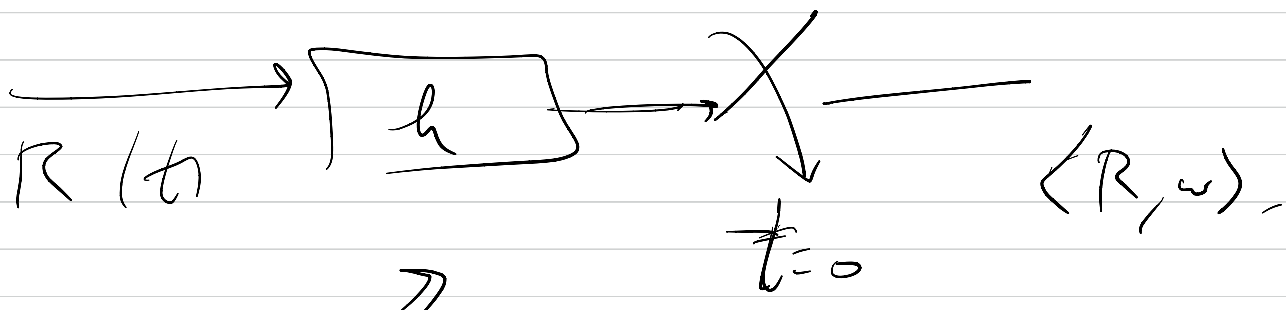
$$s(t) = \int R(\tau)\, h(t-\tau)\, d\tau$$

$$s(0) = \int R(\tau)\, h(-\tau)\, d\tau$$

$$= \langle R, g \rangle \qquad g(t) = h(-t)$$

$$s(t_0) = \langle R, \tilde{g} \rangle \qquad \tilde{g}(\tau) = h(t_0 - \tau)$$

So to compute $\langle R, w \rangle$ we could design a filter with impulse response $h(t) = w(-t)$ and



matched filter implementation of $\langle R, w \rangle$.