Principles of Digital Communication

Feb 26th, 2021

Binary Hypothesis Testing

Setup:

Object of Interest		Observation	Guess
H	$\rightsquigarrow$	Y	$\hat{H} = \text{funct}(Y)$
i		y	$\hat{i} = \phi(y)$

Reading for next week:

Chapter 1: 1.2 – 1.6

" 2: 2.1 – 2.2.1

Ex: Y is nothing

$H \in \{0, 1\}$ $\qquad \hat{H} = ?$

prob 0.7 $\qquad$ prob 0.3

Proposal 1: set $\hat{H} = 1$

proposal 2 " $\hat{H} = 0$

proposal 3 " $\hat{H} = \begin{cases} 0 & \text{wp } 0.7 \\ 1 & \text{wp } 0.3 \end{cases}$ $\quad$ wp = with probability

Q: $\Pr(\text{correct}) = ?$ with proposal $\cancel{2}$

$= 0.7$.

Copy from prev:

$$H = \begin{cases} 0 & \text{wp } 0.7 \\ 1 & \text{wp } 0.3 \end{cases}$$

Y = nothing

proposal 1 $\hat{H} = 1 \longrightarrow$ Prob(correct) $= 0.3$

2 $\hat{H} = 0 \longrightarrow$ Pr(correct) $= 0.7$

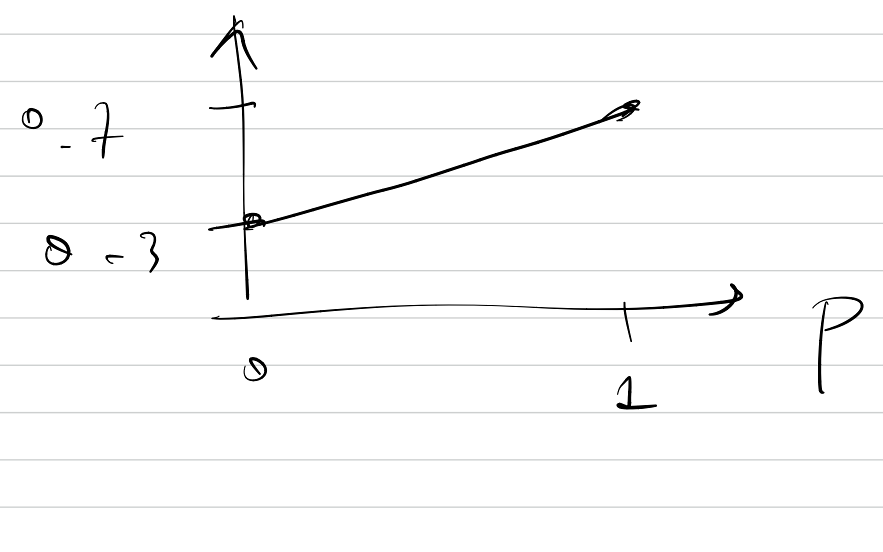
3 $\hat{H} = \begin{cases} 0 & \text{wp } 0.7 \\ 1 & \text{wp } 0.3 \end{cases} \longrightarrow$ Pr( ) $= (0.7)^2 + (0.3)^2 = 0.58.$

H & $\hat{H}$ are indep.

Can we do better? Suppose

$$\hat{H} = \begin{cases} 0 & \text{wp } p \leftarrow \\ 1 & \text{wp } 1-p \end{cases}$$

$$\Pr(H = \hat{H}) = P((H,\hat{H}) = 00) + \Pr((H\hat{H}) = (11))$$
$$= (0.7)p + (0.3)(1-p)$$



General (Binary) Case:

$H \in \{0,1\} \qquad Y \in \mathcal{Y}$

A-priori probabilities of $H = \begin{cases} 0 & \text{w.p. } 1-p = P_H(0) \\ 1 & \text{w.p. } p = P_H(1) \end{cases}$

The observation has two possible statistics

$$P_{Y|H}(y|0), \quad P_{Y|H}(y|1) = \Pr(Y=y \mid H=1)$$

We want a rule: $\hat{H} = \phi(Y)$ so as to maximize the probability that $H = \hat{H}$.

$$\Pr(H=\hat{H}) = \cancel{\sum_y \Pr(Y=y) \Pr(H=\hat{H} \mid Y=y)}$$

$$= \Pr(H=\hat{H}, Y \in \mathcal{Y})$$

$$= \sum_{y \in \mathcal{Y}} \Pr(\hat{H}=H, Y=y)$$

$$= \sum_{y \in \mathcal{Y}} \Pr(Y=y) \Pr(H=\hat{H} \mid Y=y)$$

$$= \sum_{y \in \mathcal{Y}} \Pr(Y=y) \Pr(H=\phi(y) \mid Y=y).$$

To maximize $\Pr(H=\hat{H})$ we should maximize for each y

$\Pr(H=\phi(y) \mid Y=y)$.

Note that conditioned on $Y=y$,

$$\longrightarrow \Pr(H=0 \mid Y=y) = \Pr(Y=y \mid H=0)\,\frac{P(H=0)}{P(Y=y)}$$

$$\Pr(H=1 \mid Y=y) = \Pr(Y=y \mid H=1)\,\frac{P(H=1)}{P(Y=y)}$$

Consequently we should choose

$$\phi(y) = \begin{cases} 0 & \text{if } \Pr(H=0 \mid Y=y) > \Pr(H=1 \mid Y=y) \\ 1 & \text{"} \qquad < \\ \text{don't care between 0 \& 1} & \text{"} \qquad = \end{cases}$$

This rule

① it maximizes the $\Pr(H=\hat{H})$

② it is know as the "MAP" rule

maximum a posteriori probability

Example:

$$H = \begin{cases} 0 & 1-p \\ 1 & p \end{cases} \qquad Y = \begin{cases} \alpha \\ \beta \end{cases}$$

$$\Pr(Y=\alpha \mid H=0) = \underline{0.9}, \quad \Pr(Y=\beta \mid H=0) = \underline{0.1}$$

$$\Pr(Y=\alpha \mid H=1) = 0.2, \quad \Pr(Y=\beta \mid H=1) = 0.8$$

Conditional on $Y=\alpha$, what is

$$\Pr(H=0|Y=\alpha) = (1-p)\overset{0.9}{\Pr(Y=\alpha|H=0)}\Big/\Pr(Y=\alpha)$$

$$\left(\begin{aligned}\text{also note } \Pr(Y=\alpha) &= \Pr(H=0)\Pr(Y=\alpha|H=0) \\ &\quad + \Pr(H=1)\Pr(Y=\alpha|H=1) \\ &= (1-p)(0.9) + p\cdot(0.2)\end{aligned}\right)$$

$$\Pr(H=1|Y=\alpha) = p\cdot\underbrace{\Pr(H=\alpha|H=0)}_{0.2}\Big/\Pr(Y=\alpha)$$

Decision when $Y=\alpha$:

$$\Pr(H=0|Y=\alpha) \quad \text{vs} \quad \Pr(H=1|Y=\alpha)$$

$$\equiv \quad (1-p)\cdot(0.9) \quad \text{vs} \quad p\cdot(0.2)$$

$$\equiv \quad \frac{0.9}{0.2} \underset{\text{decide } \hat{H}=1}{\overset{\text{decide } \hat{H}=0}{\gtrless}} \frac{p}{1-p}$$

$$\boxed{\equiv \quad \frac{\Pr(Y=y|H=0)}{\Pr(Y=y|H=1)} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{\Pr(H=1)}{\Pr(H=0)}.}$$

"ML" rule: $\quad \dfrac{\Pr(Y=y|H=0)}{\Pr(Y=y|H=1)} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 1$

maximum likelihood

in general sub optimal but is $\equiv$ ML if $\Pr(H=0)=\Pr(H=1)$.