Principles of Digital Communication

March 24, 2021

Reading

① "Vector & Inner Product Spaces". [2.12].
(Gram-Schmidt procedure to find orthonormal
basis from a given basis) $\perp$-normal

② Chapter 3 : "White Gaussian Noise"
"What is a measurement"
"Sufficient statistics" when specialized
to Additive WGN channel

Recall : Vector space over $\mathbb{C}$

a set $\mathcal{V}$ of objects (vectors) that obey the rules of vector algebra.

- vector addition $+$

$$\alpha, \beta \in \mathcal{V} \Rightarrow \alpha + \beta \in \mathcal{V}$$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

$$(\alpha + \beta) = (\beta + \alpha)$$

0 vector

$$0 + \alpha = \alpha$$

-ve vector

$$\alpha + (-\alpha) = 0.$$

- scalar-vector multiplication : $a \in \mathbb{C}, \alpha \in \mathcal{V}$

$$a\alpha \in \mathcal{V}. \quad 1.\alpha = \alpha$$

$$(ab)\alpha = a(b\alpha)$$

$$(a+b)\alpha = a\alpha + b\alpha$$

$$a(\alpha+\beta) = a\alpha + a\beta.$$

Ex of vectors: $\mathbb{C}^3$, $\mathbb{C}^4$, etc.

$\frac{d^2y}{dx^2} + y = 0$. all solutions $y(x)$ to this diff equation.

- space of all $\mathbb{C}$ valued RVs.
- space of all $\mathbb{C}$ " RVs with $\mathbb{E}[\cdot] = 0$

~~Inner products. $\langle \alpha, \beta \rangle \in \mathbb{C}$ or two vectors.~~

Linear forms: a map $f: V \to \mathbb{C}$ is a linear form if $f(a\alpha + b\beta) = af(\alpha) + bf(\beta)$. ←

Bi-linear forms: $\varphi: V \times V \to \mathbb{C}$

$\varphi(\alpha, \beta)$ is a linear form in α

is conjugate linear in β

an inner product $\langle \cdot, \cdot \rangle$ is a bilinear form $V \times V \to \mathbb{C}$ with the additional properties:

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle^*$$

$$\langle \alpha, \alpha \rangle \geq 0, \quad \langle \alpha, \alpha \rangle = 0 \implies \alpha = 0.$$

$$\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$$

Ex: in $\mathbb{C}^n$, $\langle \alpha, \beta \rangle = \beta^* \alpha = \sum \alpha_i \beta_i^*$

for the space of 0-mean $\mathbb{C}$-valued RVs

$$\langle X, Y \rangle \equiv E[Y^* X].$$

α, β are $\perp$ $\equiv$ $\langle \alpha, \beta \rangle = 0$.

$\{\psi_1, \dots, \psi_m\}$ is a $\perp$-normal collection if

$$\langle \psi_i, \psi_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

G-S procedure. Suppose we are given vectors $v_1, \dots, v_m$ in an inner product space

we want: $\psi_1, \dots, \psi_m$ s.t. $\{\psi_1, \dots, \psi_m\}$ $\perp$-normal

$$\underbrace{\text{span}\{v_1, \dots, v_n\}}_{\substack{\text{all linear comb of} \\ v_1, \dots, v_n}} = \text{span}\{\psi_1, \dots, \psi_m\}.$$

How can we do this? G-S procedure:

$\psi_1 = v_1 / \|v_1\|$ $\qquad$ $\text{span}\{\psi_1\} = \text{span}\{v_1\}$

$\psi_2 = \dfrac{v_2 - \langle v_2, \psi_1 \rangle \psi_1}{\| \quad " \quad \|}$ $\qquad$ $\text{span}\{\psi_2, \psi_1\} = \text{span}\{v_2, v_1\}$

$$\langle \psi_2, \psi_1 \rangle = \langle v_2, \psi_1 \rangle - \langle \langle v_2, \psi_1 \rangle \psi_1, \psi_1 \rangle$$

$$= \quad '' \quad - \langle v_2, \psi_1 \rangle \underbrace{\langle \psi_1, \psi_1 \rangle}_{1}$$

$$= 0$$

$$\psi_3 = v_3 - \langle v_3, \psi_1 \rangle \psi_1 - \langle v_3, \psi_2 \rangle \psi_2$$

$$\| \quad '' \quad \|$$

⋮

At the end of the process we will have

$(\psi_1 \ldots \psi_m)$ are O-normal

Span$\{ \quad \}$ = span$\{v_1 \ldots v_n\}$.

WGN (3.2)

In the discrete world: $Z \sim N(0, \sigma^2 I_n)$

Has the following property: if we form

$$X_1 = \langle a_1, z \rangle$$

$$X_2 = \langle a_2, z \rangle$$

$$\vdots$$

$$X_m = \langle a_m, z \rangle$$

then $(X_1, \ldots, X_m)$ is Gaussian RVector

with $E(X_i) = 0 \quad \forall i$

$E(X_i X_j) = \langle a_i, a_j \rangle \sigma^2$.

Observe: not only is this a consequence of $Z \sim N(0, \sigma^2 I_n)$ but is equivalent to it.

We could have defined "Z is the white Gaussian vector of intensity σ^2 if for every set of vectors $a_1, \ldots, a_m$ the collection $X_1 = \langle a_1, Z \rangle, X_2 = \langle a_2, Z \rangle, \ldots,$ is $N(0, \sigma^2 [\langle a_i, a_j \rangle])$"

In complete analogy we will define

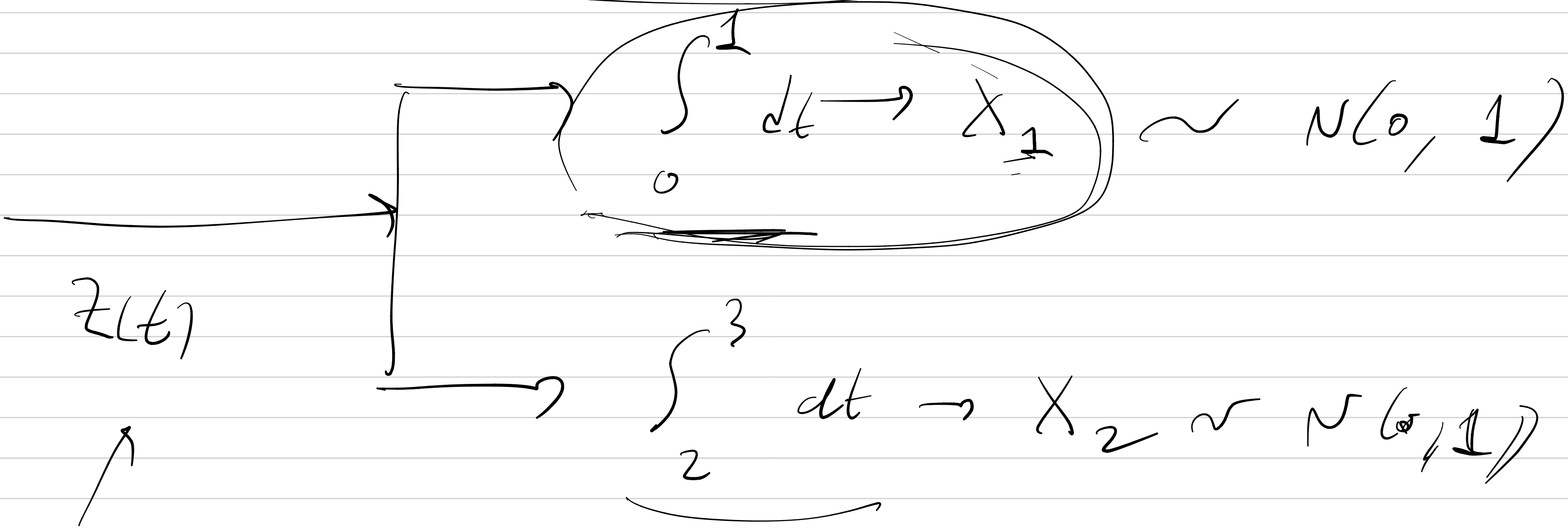
"$Z(t)$ is white Gaussian noise if for every set of functions $a_1(t), a_2(t), \ldots, a_m(t)$, The collection $X_1 = \int a_1(t) Z(t)\, dt$

$X_2 = \int a_2(t) Z(t)\, dt$

$\vdots$

$X_m = \int \qquad dt$

is $N(0, \sigma^2[\langle a_i, a_j\rangle])$.

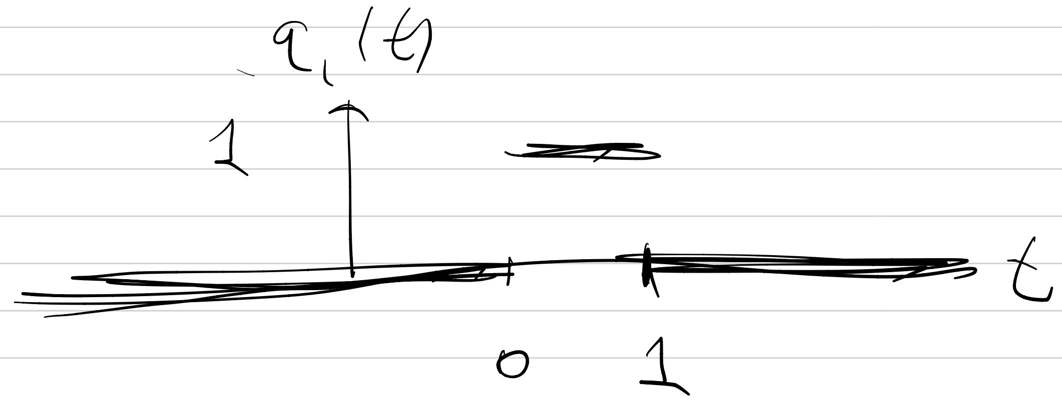


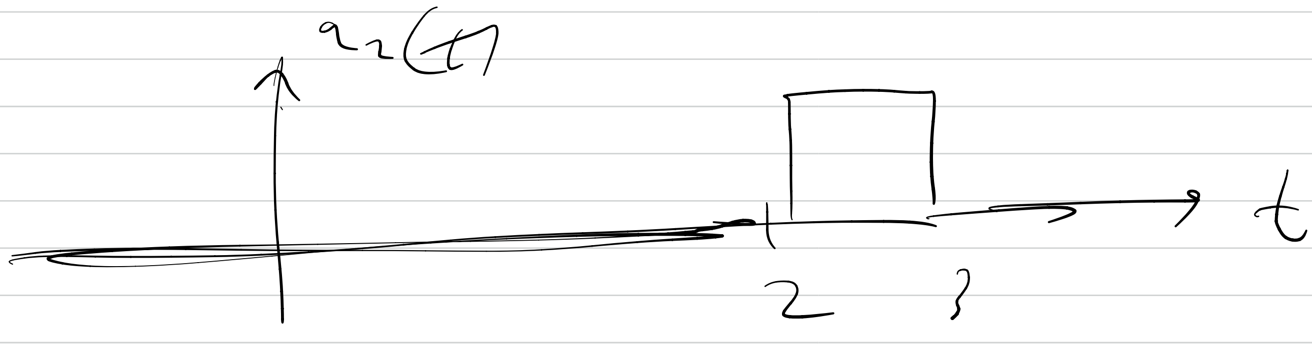
WGN intensity = 1

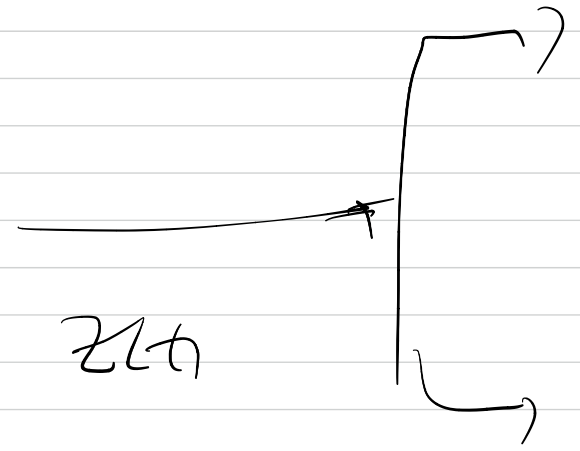
X_1, X_2 indep ?? ✓

$E[X_1 X_2] = \sigma^2 \int a_1(t) a_2(t)\, dt$

$= 0.$







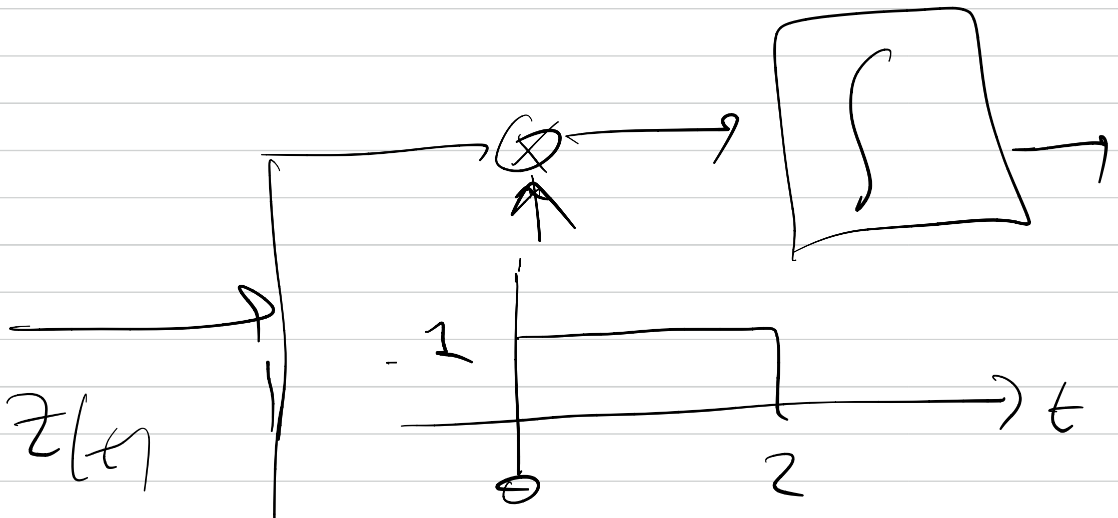
$\int_0^1 dt \Rightarrow X_1 \quad \sim (0, 1)$

$\int_0^2 dt \Rightarrow X_2 \quad \sim (0, 2)$

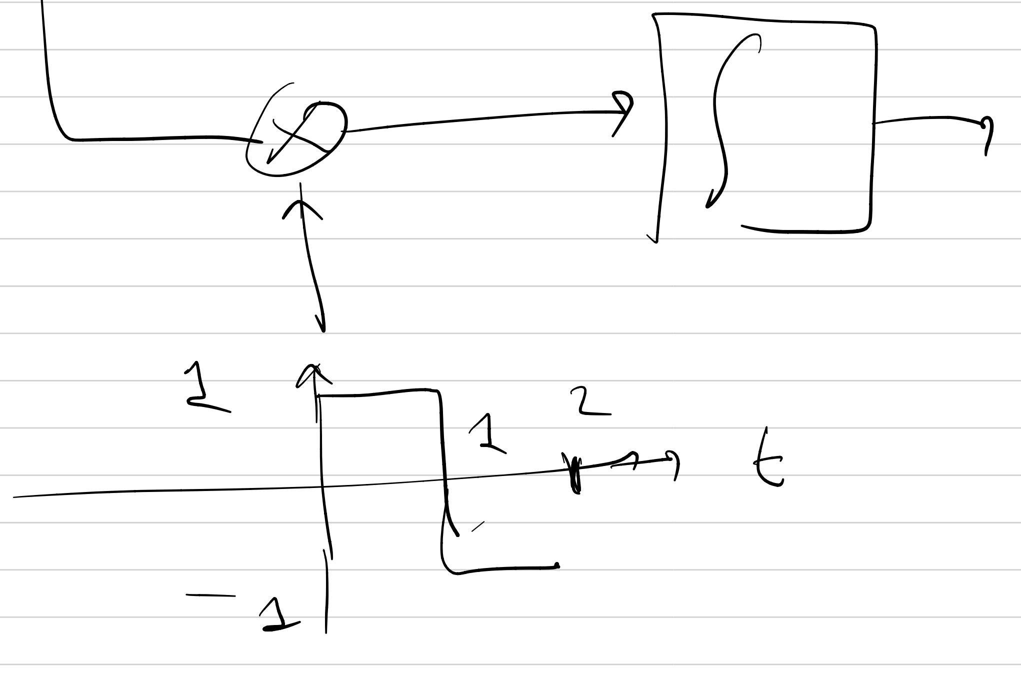
$\int a_2(t)^2 dt = \int_0^2 1\, dt = 2$

X_1, X_2 ~~indep~~ ?

$E(X_1 X_2) = 1 = \int a_1(t) a_2(t) = \int_0^1 1\, dt = 1$



$X_1 \sim N(0, 2)$



$X_2 \sim N(0, 2)$

$E(X_1 X_2) = 0$

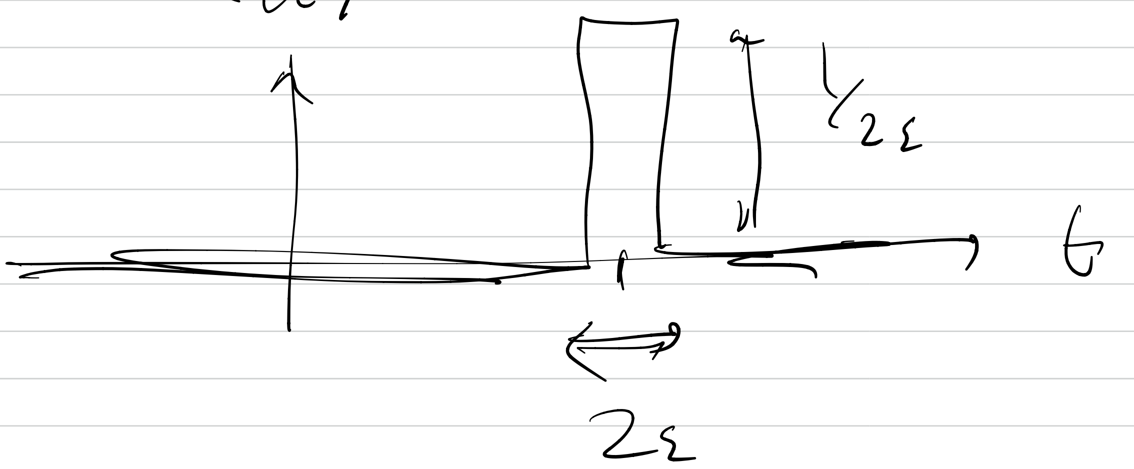
Usual function $f(t)$, continuous

$$f(t_0) \approx \frac{1}{2\varepsilon}\int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\,dt$$

$$\underline{z(t_0)} = \frac{1}{2\varepsilon}\int_{t_0-\varepsilon}^{t_0+\varepsilon} z(t)\,dt = \underline{\int a(t)\, z(t)\,dt}$$

$$N\left(0,\ 2\varepsilon\left(\frac{1}{2\varepsilon}\right)^2\right) = \underline{N\left(0,\ \frac{1}{2\varepsilon}\right)}$$

$a(t) =$



Consequently $z(t_0)$ is not well defined,

thus we only consider objects of the

form $$X = \int_{-\infty}^{+\infty} z(t)\, a(t)\,dt$$

with $\int a^2(t)\,dt < \infty$.

3.3: System with $H \in \{0, \dots, m-1\}$,

received signal $R(t) = Z(t) + w_i(t)$ if $H = i$.

we are allowed to perform the following type of measurements:

- pick any n and any n L_2 functions $g_1, \dots, g_n$ and form

$$Y_1 = \int R(t) g_1(t)\, dt, \dots, Y_n = \int R(t) g_n(t)\, dt.$$

Q: how do we choose $g_1 \dots g_n$?

Ans: Do G-S on $w_0, \dots, w_{m-1}$ to obtain 1-normal functions $\psi_1(t), \dots, \psi_n(t)$, use these as g's.

Reason: Suppose $Y_i = \langle R, \psi_i \rangle$ $i = 1, \dots, n$ are given. Suppose we augment this collection with $U_1 = \langle R, a_1 \rangle, \dots, U_k = \langle R, a_k \rangle$

we can show by Fisher-Neiman factorization that

$$H - Y - (Y, U)$$

So $t(U, Y) = Y$ is a suff. stat.