Principles of Digital Communication

April 23, 2021

Reading for next week: 5.(1-4)

Recall: we can design a comm. system for the AWGN channel

$$i \longrightarrow c_i \in \mathbb{C}^n \longrightarrow w_i(t) = \sum_j c_{ij}\,\psi_j(t) \longrightarrow$$

$i \in \{0,1,\dots,M-1\}$

$\perp$-normal basis waveforms $\psi_1,\dots,\psi_n$

TX

$$R(t) \longrightarrow \begin{array}{l} \langle R,\psi_1\rangle =: Y_1 \\ \vdots \\ \langle R,\psi_n\rangle =: Y_n \end{array} \quad Y$$

$$\hat{i} = \arg\min_i \|Y - c_i\| \longrightarrow \hat{i}$$

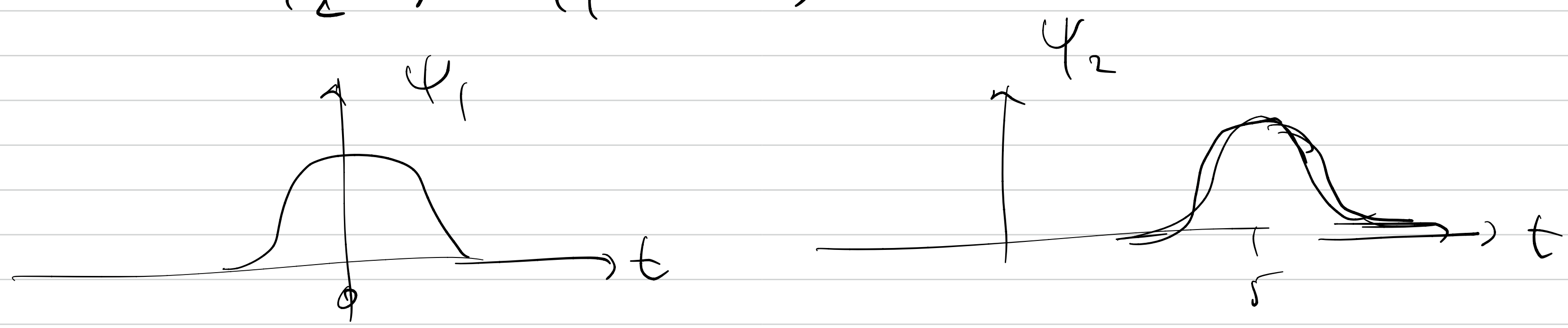
(assuming equally likely messages)

$$\langle R,\psi_j\rangle = \underbrace{\int R(t)\,\psi_j^*(t)\,dt}$$

$$= (R * h_j)(0) \quad \text{if } h_j(t) = \psi_j^*(-t)$$

if the h_j's are time shifts of each other, then all the inner products can be computed by sampling the output of one filter:

Ex: $\psi_2(t) = \psi_1(t-5)$.



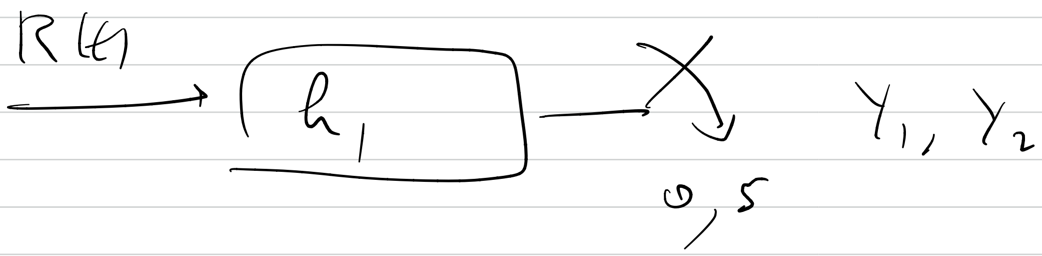
$$Y_1 = \int R(t)\, \psi_1^*(t)\, dt$$

$$Y_2 = \int R(t)\, \psi_2^*(t)\, dt = \int R(t)\, \psi_1^*(t-5)\, dt$$

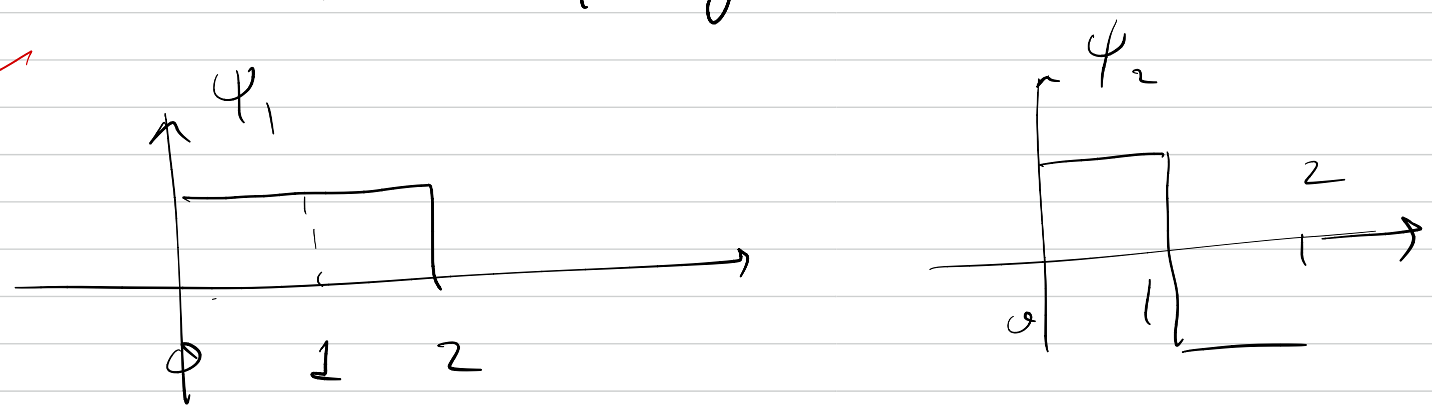
$$= \int R(t+5)\, \psi_1^*(t)\, dt$$

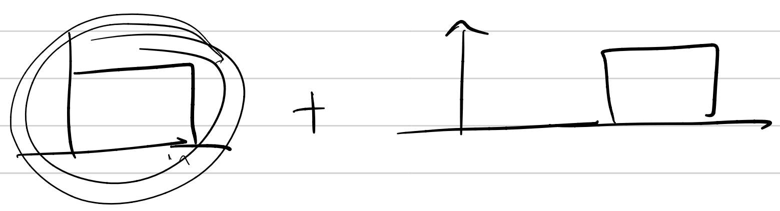
$$Y_1 = (R * h_1)(0) = \int R(t)\, h_1(-t)\, dt$$

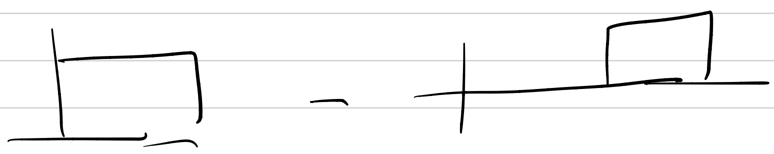
$$Y_2 = (R * h_1)(5) = \int R(t)\, h_1(5-t)\, dt$$



⇒ The design will be easy if all these $\psi_1, \psi_2, \ldots, \psi_n$ are time-shifts of each other.



$\psi_1(t) =$ 

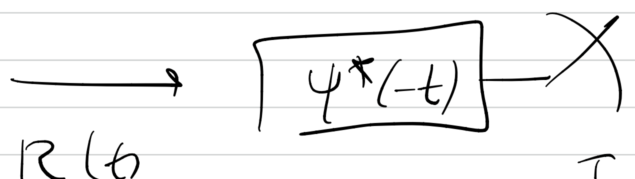
$\psi_2(t) =$ 

So we want

(1) $\psi_1, \psi_2, \ldots, \psi_n$ $\perp$-normal

(2) ~~$\psi_1, \psi_2 \ldots \psi_n$ time-shifts of each other~~

$$\psi_j(t) = \psi(t - jT).$$

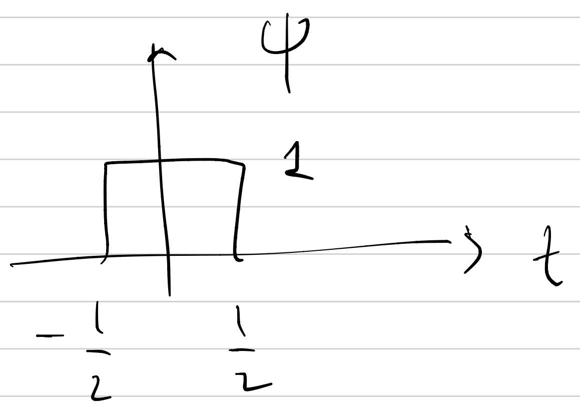


$Y_1, Y_2 \ldots Y_n$,

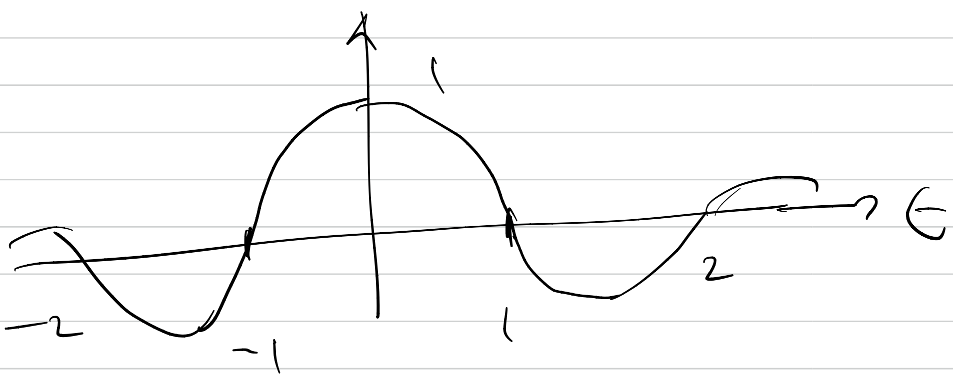
$T, 2T, \ldots, nT$

Q: what sort of ψ's are possible?

Ex..

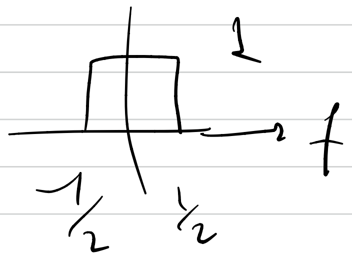


Ex. $\psi(t) = \text{sinc}(t) = \dfrac{\sin \pi t}{\pi t}$



$$\int \psi(t)\,\psi(t-1)\,dt = \int \underbrace{\psi(t)}_{x(t)}\, \underbrace{\psi^*(t-1)}_{y^*(t)}\, dt$$

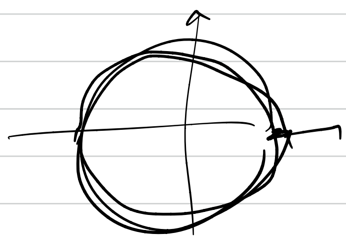
$$= \int x_F(f)\, y_F^*(f)\, df \quad : \text{Parseval.}$$

Recall that $\mathrm{sinc}(t) \xleftrightarrow{FT}$ 

$$x_F(f) = \mathbb{1}\{|f| \le \tfrac{1}{2}\}$$

$$y_F(f) = \mathbb{1}\{|f| < \tfrac{1}{2}\} \cdot e^{-j2\pi f}$$

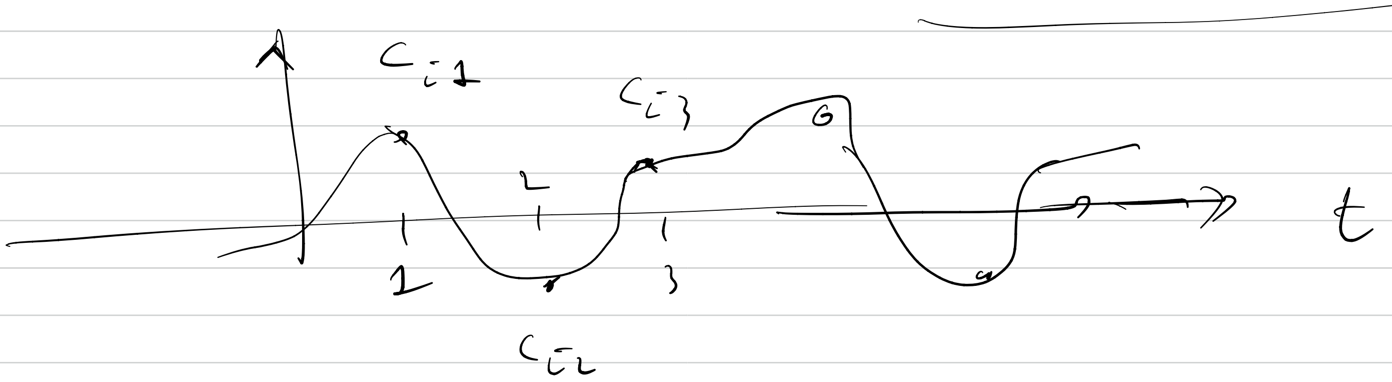
$$\int x_F(f)\, y_F(f)^* \, df = \int_{-1/2}^{1/2} e^{j2\pi f} \, df = 0.$$



$$\Rightarrow \int \psi(t)\, \psi(t-j)\, dt = 0.$$

in this "sinc" example

$$i \longrightarrow \underline{c_i} \longrightarrow w_i(t) = \sum_{j=1}^{n} \underline{c_{ij}}\, \underline{\mathrm{sinc}(t-j)}$$



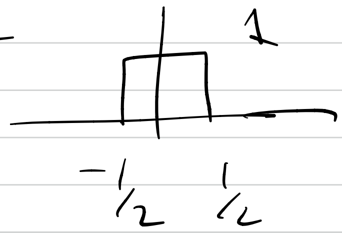
"Ideal Bandlimited case".

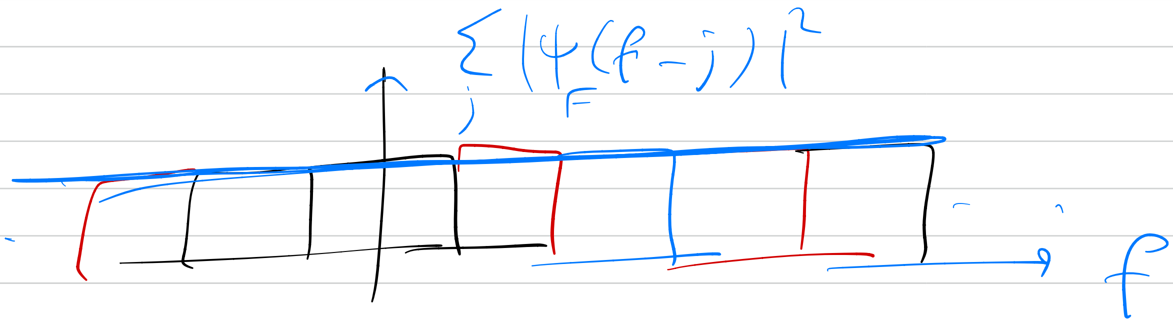
In S_4 we will see at criteria to test if ψ has the property that its integer shifts are $\perp$-normal. (Nyquist criterion).

Thm: $\{\psi(t-j) : j \in \mathbb{Z}\}$ form an ortho-normal collection

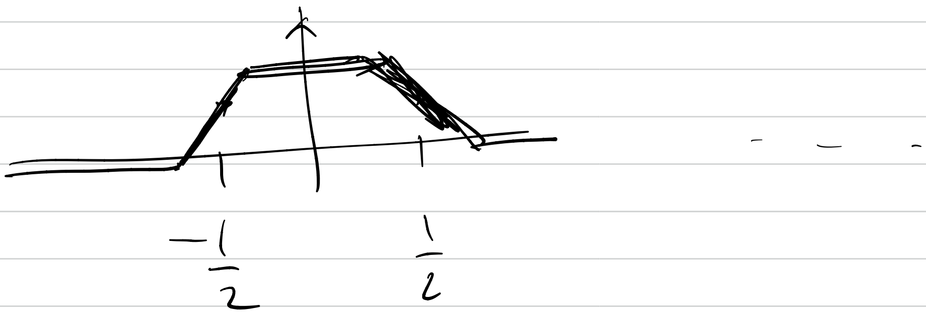
if and only if

$$\sum_j |\psi_F(f-j)|^2 = 1 \quad \forall f.$$

Ex: $\psi(t) = \operatorname{sinc}(t)$ $\quad |\psi_F(f)|^2 =$ 



Ex: $|\psi_F(f)|^2$



So far I have covered S.(1, 2, 4).

Let me say a few words about S.3 (Power Spectral Density.)

Discrete-time Stoch Processes $\quad \mathbb{C}$-valued

$(X_i : i \in \mathbb{Z})$ a collection of RVs.

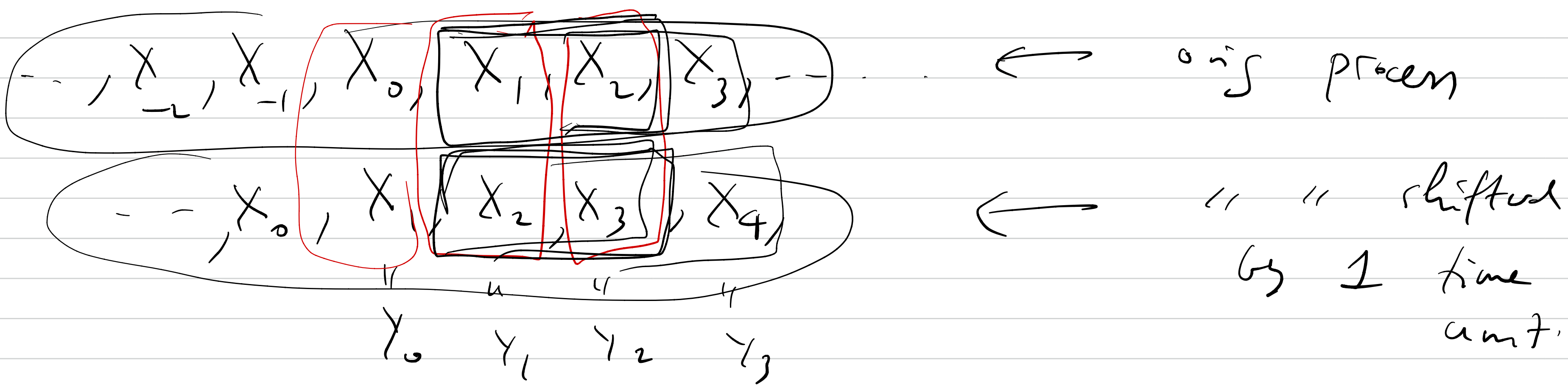
Let us restrict to the case $E(X_i) = 0 \quad \forall i$

$$\left(X_i = \underline{(X_i - E(X_i))} + \underline{E(X_i)} \right)$$

1's order statistics are $E(X)$'s

2nd " " are $E(X_i X_j^*) = \underline{K_{ij}}$

"stationarity":



if statistics of Y is the same as stats of X we call the process X to be stationary.

Stationarity requires: $\sim X_1 \sim X_2 \sim X_3 \sim X_4 \sim \ldots$

& $(X_1 X_2) \sim (X_2 X_3) \sim (X_3 X_4) \ldots$

& $(X_1 X_2 X_3) \sim (X_2 X_3 X_4) \sim \ldots$

in general the statistics of

$$(X_{i+1}, X_{i+2}, X_{i+3} \ldots X_{i+n})$$ should not change with i

$$\sim (X_1, \ldots X_n).$$

In particular: if X is stationary then

$$E(X_i X_j^*) = E(X_0 X_{j-i}^*) = \underline{R_{j-i}}.$$

A process that is not necessarily stationary but has the property that $E(X_i) = \text{const}$ & $E(X_i X_j^*) = R_{j-i}$ is called <u>Wide-Sense-Stationary.</u>

if X is a WSS process (or stationary)

$$\ldots, X_1, X_2, \ldots \longrightarrow \boxed{h} \longrightarrow \ldots, Y_0, Y_1, Y_2, \ldots$$

the Y is also WSS (stationary).

$$Y_i = \sum_n h_n X_{i-n} \qquad E(Y_i) = 0. \quad (\text{if } E(X_i) = 0).$$

$$\underbrace{Y_i Y^*_{i+k}} = \sum_n h_n X_{i-n} \sum_m h^*_m X^*_{i+k-m}$$

$$= \sum_{n,m} h_n h^*_m \underbrace{X_{i-n}} \underbrace{X^*_{i+k-m}}$$

$$E[\quad] = \sum_{n,m} h_n h^*_m R_X[k+n-m]$$

$$\underline{R_Y[k]}$$

let us define $S_X(\theta) = \sum_k R_X(k) e^{-j2\pi\theta k}$

$$S_Y(\theta) = \sum_k R_Y(k) e^{-j2\pi\theta k}$$

$$S_Y(\theta) = \sum_{k,n,m} h_n h^*_m R_X(\underline{k+n-m}) e^{-j2\pi\theta(\underline{k+n-m} - n + m)}$$

$$= \sum_{l,n,m} \underline{h_n e^{+j2\pi\theta n}} \; \underline{h^*_m e^{-j2\pi\theta m}} \; \underline{R_X(l) e^{-j2\pi\theta l}}$$

$$= S_X(\theta) H(-\theta) H^*(-\theta). = S_X(\theta) |H(\theta)|^2.$$

Had we defined $R_X(k) = E(X_i^* X_{i+k})$,

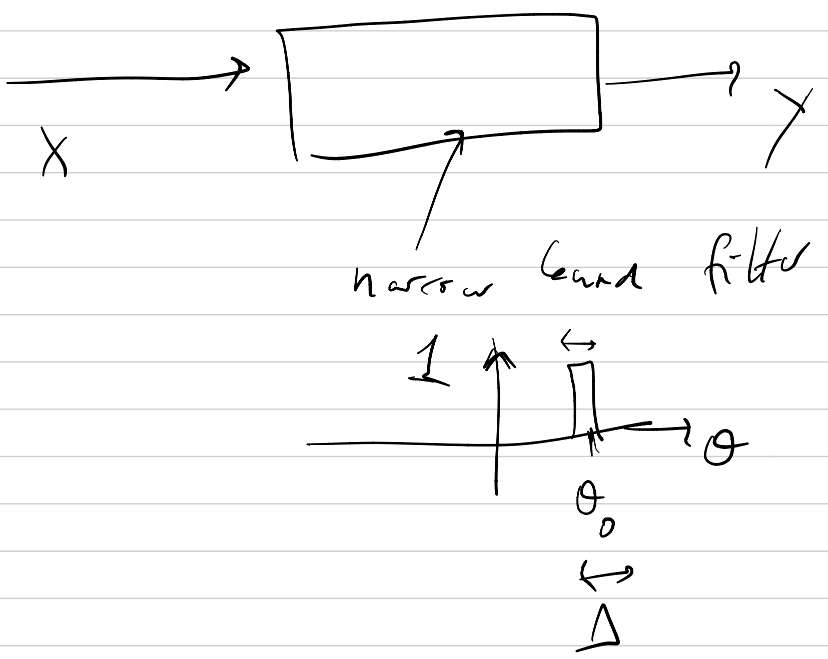
~~we would~~ $R_Y(k) = E(Y_i^* Y_{i+k})$

we would have found

$$S_Y(\theta) = S_X(\theta)\, |H(\theta)|^2.$$

Note: $R_X[0] = E(|X_i|^2)$ is the "power" in X process

$$= \int_{-1/2}^{1/2} S_X(\theta)\, d\theta$$



$$S_Y(\theta) = \begin{cases} S_X(\theta). & \theta \in [\theta_0 - \frac{\Delta}{2}, \theta_0 + \frac{\Delta}{2}) \\ 0 & \text{else} \end{cases}$$

power of Y = $R_Y(0) = \int S_Y(\theta)\, d\theta \approx \Delta S_X(\theta_0)$

$\Rightarrow$ the amount of power in X in the frequency band $[\theta_0 - \frac{\Delta}{2}, \theta_0 + \frac{\Delta}{2})$ is $\approx \Delta S_X(\theta_0)$.

So we call (on the basis of this computation)

$S_X(\theta)$ the power spectral density of the random process X.

There is a complete analogy in continuous time

$$R_X(\tau) = E\left(X^*(t)\, X(t+\tau)\right) \quad (\text{for a WSS } X \text{ of zero mean})$$

$$X(t) \longrightarrow \boxed{h} \longrightarrow Y(t)$$

Y is also WSS $\quad S_Y(f) = S_X(f)\,|H(f)|^2$.

$$S_X \xleftrightarrow{F.T} R_X$$

When we design a communication system:

$$\longrightarrow \boxed{} \longrightarrow w_i(t)$$

i - random in $\{0, \dots, m-1\}$

stoch process.

"stay below the blue line"

When

$$w_i(t) = \sum' c_{ij}\, \psi(t-j)$$

we will find that we can compute $S_w(f)$ in terms of $\Psi_F(f)$ & the statistics of $\{c_{ij}\}$.