Principles of Digital Communication

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"Appendices" of chapter 7, complex valued RVs.
(& random vectors).

Z is a complex RV $\equiv \begin{bmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{bmatrix}$ is a $\mathbb{R}$eal random vector.

$Z \in \mathbb{C}^n$ is a vector $\equiv \begin{pmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{pmatrix}$ is a $2n$ dim $\mathbb{R}$eal R. vector.

Z is a Gaussian $\mathbb{C}^n$

$$\equiv \begin{pmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{pmatrix} \text{ is } \underline{\mathbb{R}^{2n} - \text{Gaussian}}.$$

Then, to specify the statistical properties of Z we should specify

(1). $E\begin{pmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{pmatrix}$ and covariance matrix,

(2). $$E\left[\begin{pmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{pmatrix} \begin{bmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{bmatrix}^T \right] - \left(E\begin{bmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{bmatrix} \right) \left(\;''\; \right)^T$$

Assume, to simplify notation that

$E\begin{bmatrix} \mathrm{Re}(Z) \\ \mathrm{Im}(Z) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so the covariance matrix is

$$E\begin{bmatrix} \boxed{Re(z)\, Re(z)^T} & \boxed{Re(z)\, Im(z)^T} \\ Im(z)\, Re(z)^T & \boxed{Im(z)\, Im(z)^T} \end{bmatrix}$$

we need four $n \times n$ real matrices to specify.

$$E\left[\hat{z}\hat{z}^T\right] \quad \text{with} \quad \hat{z} = \begin{bmatrix} Re(z) \\ Im(z) \end{bmatrix}$$

compare this with

$$E[zz^*] = E\left(\left(Re(z) + j\, Im(z)\right)\left(Re(z)^T - j\, Im(z)^T\right)\right)$$

$$= \boxed{E\left[Re(z)\, Re(z)^T\right] + E\left(Im(z)\, Im(z)^T\right)}$$

$$+ j\left[-E\left(Re(z)\, Im(z)^T\right) + E\left(Im(z)\, Re(z)^T\right)\right]$$

Knowing $E[zz^*]$ is $\equiv$ to knowing:

$$\left\{ \begin{array}{l} E\left(Re(z)\, Re(z)^T\right] + E\left(Im(z)\, Im(z)^T\right] \\ \& \ E\left[Re(z)\, Im(z)^T\right) - E\left(Im(z)\, Re(z)^T\right]. \end{array} \right\}$$

Suppose we also knew

$$E[zz^T] = E\left(\left(Re(z) + j\, Im(z)\right)\left(Re(z)^T + j\, Im(z)^T\right)\right)$$

$$= \left(E\left(Re(z)\, Re(z)^T\right) - E\left(Im(z)\, Im(z)^T\right]\right)$$

$$+ j\left(E\left[Re(z)\, Im(z)^T\right] + E\left[Im(z)\, Re(z)^T\right]\right)$$

So, we also know

$$\left\{ \begin{array}{l} E\left(\mathrm{Re}(z)\mathrm{Re}(z)^T\right) - E\left(\mathrm{Im}(z)\,\mathrm{Im}(z)^T\right) \\ E\left(\mathrm{Re}(z)\,\mathrm{Im}(z)^T\right) + E\left(\mathrm{Im}(z)\,\mathrm{Re}(z)^T\right). \end{array} \right\}$$

So, to specify the statistics of z, all we need is $\underline{E[zz^*]}$ & $\underline{E[zz^T]}$.

We define a special class of RVs:
We say that a RV (or Rvec) is "proper" if

zero mean

$\underline{E[zz^T] = 0}$.

Thus, for a proper Rvec, $E(zz^*)$ is sufficient to specify its statistics.

gaussian, zero-mean

More generally, we say that a Rvector z is "circularly symmetric" if for every $\theta \in [0, 2\pi)$

$$z \sim \underline{z e^{j\theta}}$$

if z is circ. sym, then:

$$E[z] = 0. \quad \left(\underline{E(z)} = E[z e^{j\theta}] = \underline{E(z) e^{j\theta}} \right)$$

$$E(zz^T) = 0 \quad \left(E(zz^T) = E(z e^{j\theta} (z e^{j\theta})^T \right)$$

$$\underbrace{E(ZZ^T)} = \underbrace{E(ZZ^T)} \cdot e^{j2\theta} \Rightarrow E(ZZ^T) = 0.$$

So: a circ-sym RVect is zero-mean & proper.

~~For a Gaussian RVecto~~

Moreover if $Z \in \mathbb{C}^n$ is Gaussian, zero-mean & proper then Z is circ symmetric.

why: $Ze^{j\theta}$ is also Gaussian, has zero mean, $E((Ze^{j\theta})(Ze^{j\theta})^T) =$

$$E(ZZ^T)e^{j\theta 2} = 0 = E(ZZ^T).$$

$$E(\underbrace{(Ze^{j\theta})}(Ze^{j\theta})^*) = E(\underbrace{ZZ^*})$$

So (Z) & $(Ze^{j\theta})$ have the same 1st & 2nd order statistics $\Rightarrow$ they have the same distrib.

↑ Gaussian

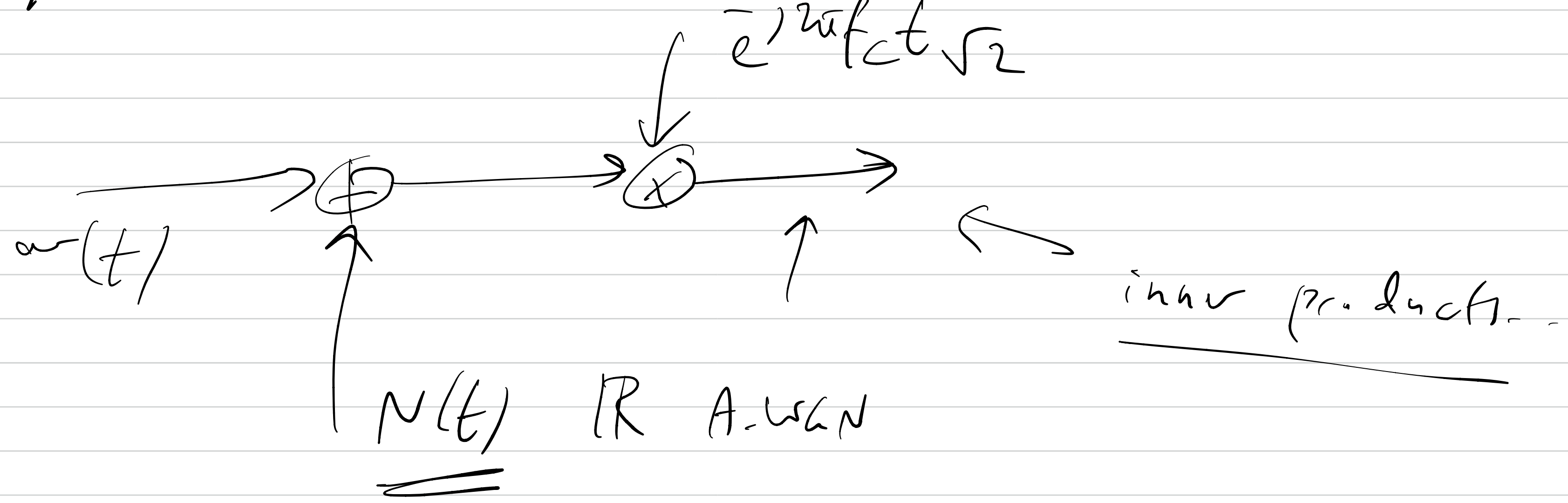
Suppose Z is a circ. sym Gaussian with $K = E[ZZ^*]$, then the pdf of Z is given by:

$$f(z) = (\det(\pi K))^{-1} \exp(-z^* K^{-1} z).$$

[Compare to the real case $Z \sim N(0, K)$ (Real

$$f_Z(z) = \det(2\pi K)^{-1/2} \exp(-\tfrac{1}{2} z^T K^{-1} z).$$
)

Why worry about the circ. sym Gaussian?



the noise after the inner product:

$$Z_j = \left(\underline{N(t)} e^{-j2\pi f_c t}\sqrt{2}, \underline{g_j(t)} \right) \qquad j = 1 \dots n$$

if $N(t)$ is $\mathbb{R}$-WGN

$g_j(t)$ Bandlimited to $[-B, B]$ with $B \leq f_c$

then $\begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$ is circ. sym. Gaussian $\mathbb{C}$ rand. vector

with $\underline{E(ZZ^*)} = \begin{bmatrix} \langle g_1, g_1 \rangle & \cdots & \langle g_1, g_n \rangle \\ \\ \langle g_n, g_1 \rangle, & \cdots & \langle g_n, g_n \rangle \end{bmatrix}$.