# Principles of Digital Communication

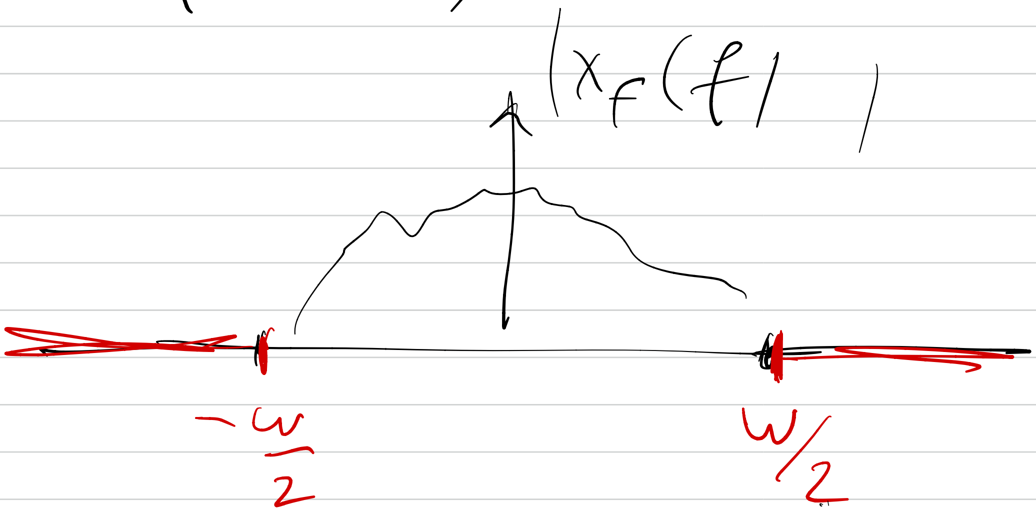
April 21, 2021

Reading was 4.4.? – Summary of Ch 4.

- physical meaning of the signal-space dimension "n".

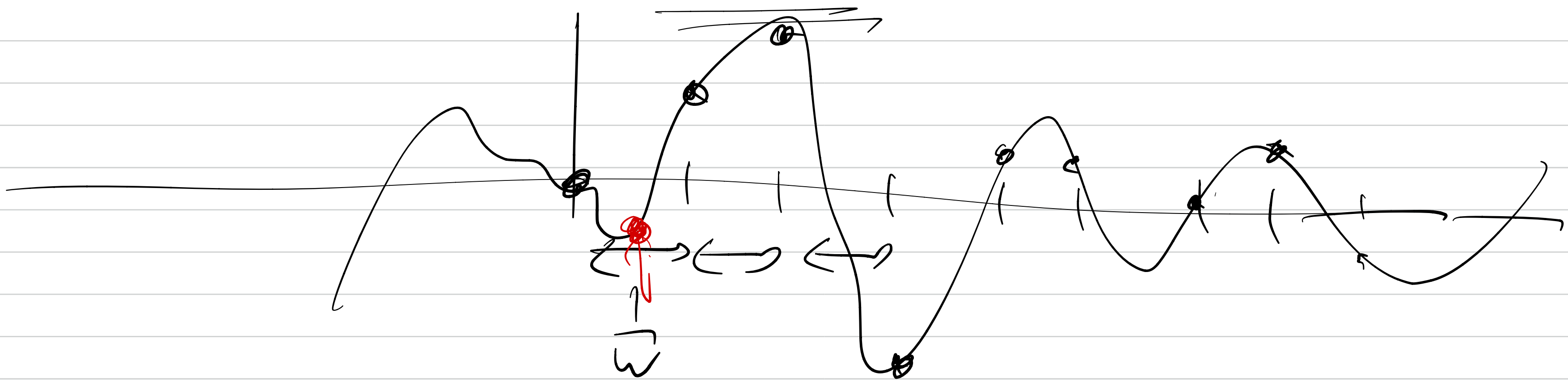
Recall the Nyquist Sampling Theorem:

if $x(t)$ is a waveform, bandlimited



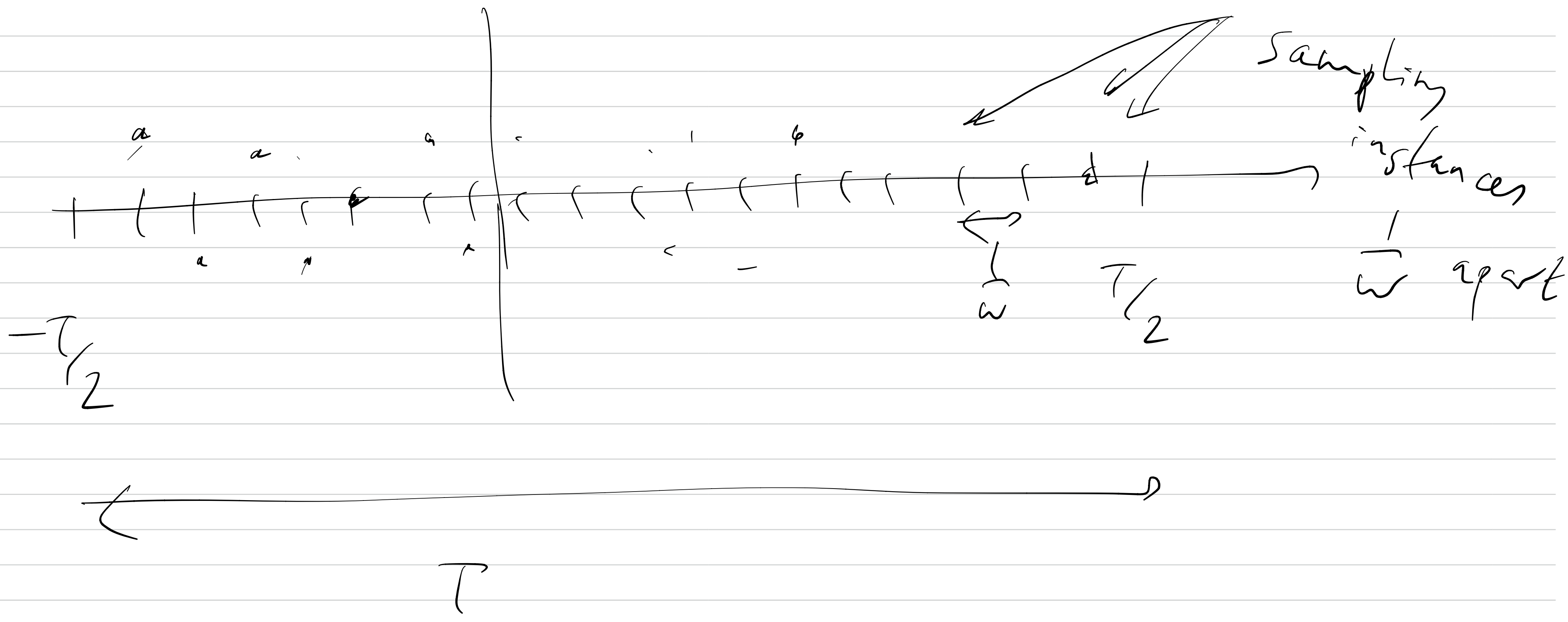

then $x(t)$ can be reconstructed from its "Nyquist samples":

$$x_n = x\left(\frac{n}{W}\right) \qquad n = \dots, -2, -1, 0, 1, 2, \dots$$




$$x(t) = \sum_{n=-\infty}^{+\infty} x_n \operatorname{sinc}(Wt - n)$$

$$= \begin{cases} 1 & Wt = n \\ 0 & Wt = m \neq n \end{cases}$$




we have $WT$ samples in this interval

we would "expect" that

$\boxed{\left(x(t) : -\frac{T}{2} < t < \frac{T}{2}\right)}$ is determined by the

samples $x_n$ taken inside the interval

(not really true) [$WT$ samples]

$$\equiv \left(x(t) : |t| < T/2\right) \longleftrightarrow \mathbb{R}^{(WT)}$$

So intuitively speaking, from the Nyquist

theorem we expect that

{functions of duration $T$ & fourier

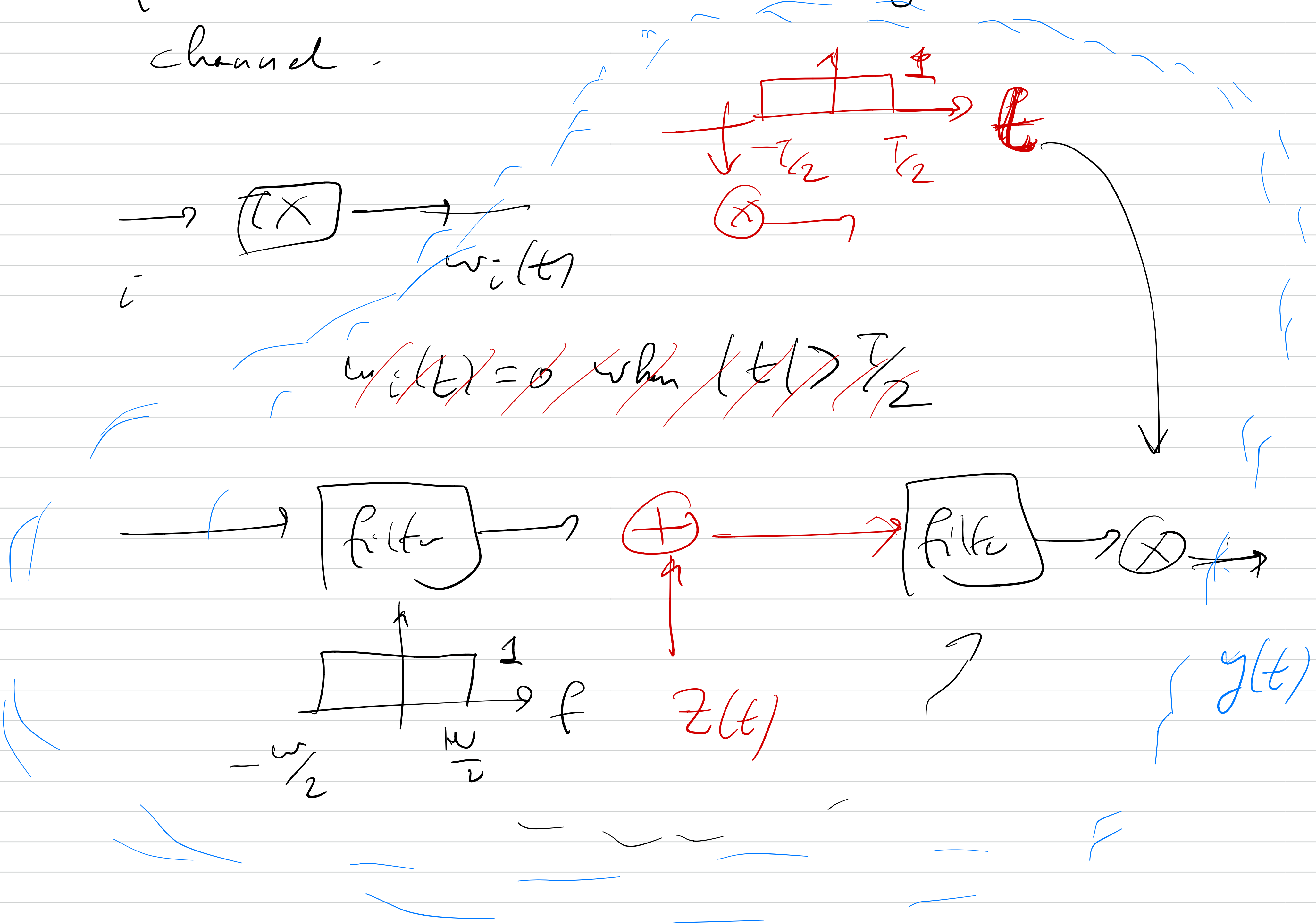
footprint $W$} form a $WT$

dimensional space.

This engineering intuition is made precise in section 4.5. (due to a Thm by D. Slepian, 1970s)

---

we try to design a communication system for a bandlimited additive gaussian noise channel.




$$y(t) = \int K(t,\tau)\, w(\tau)\, d\tau + \tilde{z}(t)$$

$$y_i = \sum_j K_{ij} x_j$$

General theorem

$$K(t,\tau) = \sum_\ell \sigma_\ell \, u_\ell(t) \, v_\ell(\tau)$$

Matrices: Singular Value Decomposition

$$K = U D V^T$$

$(u_1, \dots, u_n)$ ⊥-nc

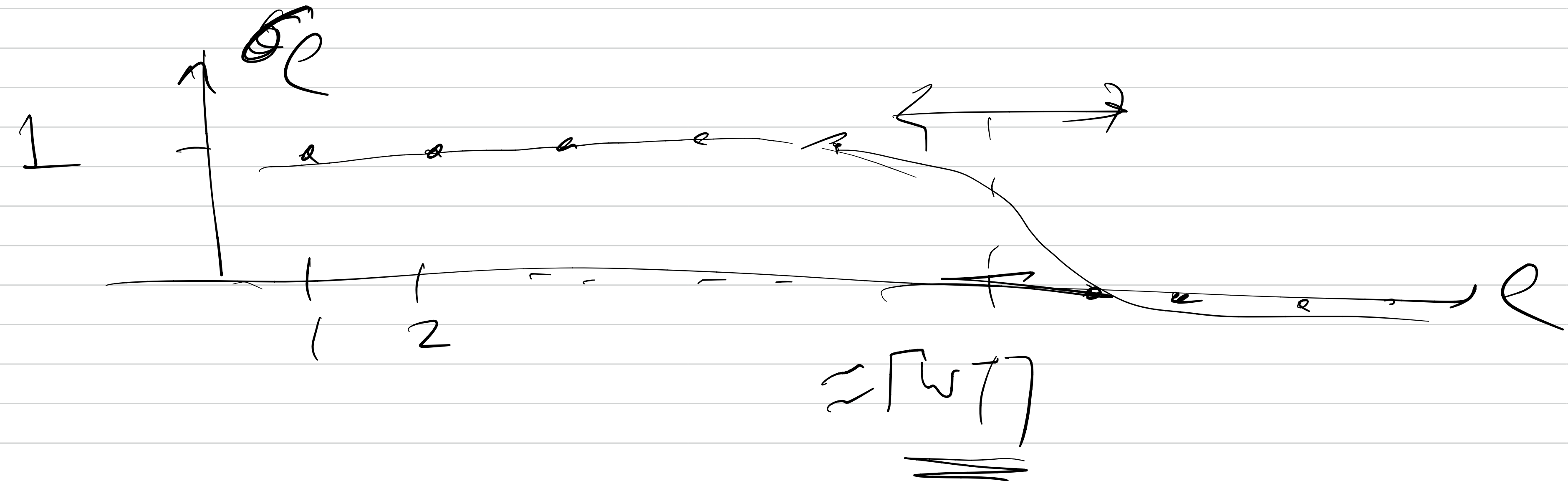
$$K = \sum_\ell \sigma_\ell \, u_\ell \, v_\ell^T$$

$(v_1 \dots v_n)$ ⊥-n

If we write $w(\tau) = \sum_\ell c_\ell \, v_\ell(\tau)$

$\Rightarrow$ output $y(t) = \sum_\ell \sigma_\ell c_\ell \, u_\ell(t) + \tilde{Z}(t)$

"Slepian": for the channel model above



Moral of the story:

signal space is spanned by

$\{v_1(t), v_2(t) \ldots \}$

but it makes sense to use only

$v_1(t), v_2(t), \ldots, v_n(t)$

$n \approx WT$

4.4 (& 4.6) trade-off between

$k$: # of data bits

$n$: # of dimen

$\mathcal{E}_b$: energy per bit, $\sigma^2$: noise power.

Suppose we insist that our signals have

energy $\mathcal{E} = \underline{k\,\mathcal{E}_b}$ so $\|c_i\|^2 \leq k\,\mathcal{E}_b$

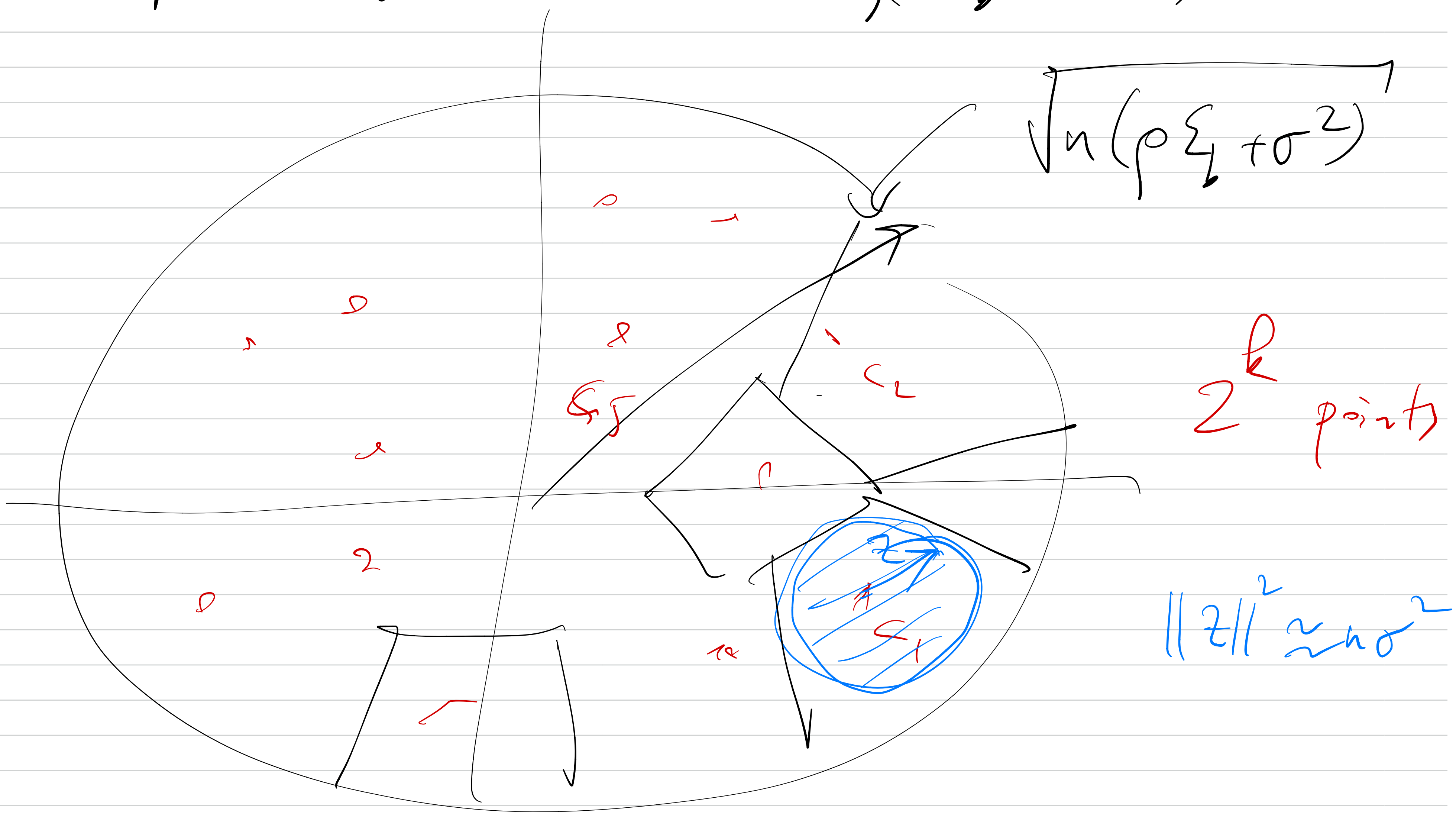
if $c_i$ is transmitted we receive

$$Y = c_i + Z \quad , \quad \|Y\|^2 = \underbrace{\|c_i\|^2}_{\leq k\,\mathcal{E}_b} + \underbrace{\|Z\|^2}_{\approx n\sigma^2} + \underbrace{2\langle c_i, Z\rangle}_{N(0,\,4\sigma^2 k\,\mathcal{E}_b)}$$

Suppose $k$ & $n$ are proportional to each other: $\frac{k}{n} = \rho = $ # of bits/dimension

$$\|Y\|^2 \le n\rho\mathcal{E}_b + n\sigma^2 + \underbrace{\mathcal{N}(0, \sigma^2\rho n\mathcal{E}_b)}_{O(\sqrt{n})}$$

$Y$ will, with probability $\approx 1$ fall inside a sphere of radius$^2 = n(\rho\mathcal{E}_b + \sigma^2)$.




to have a reliable system ($P(\text{Error}) \approx 0$) we better have the decision region $R_i$ includes the "blue ball" of radius$^2 = n\sigma^2$ around $c_i$

So we need each $R_i$ has to have

volume $\geq$ volume of the blue ball

$$\alpha (n\sigma^2)^{n/2}$$

total volume of $R_0, \dots, R_{m-1}$

$$\geq \alpha\, 2^k (n\sigma^2)^{n/2}$$

but since these are all inside the large sphere

of radius$^2 = n(\mathcal{E}_b \rho + \sigma^2)$ we find that

$$(\cancel{n}\sigma^2)^{n/2} \cdot 2^k \leq \left(\cancel{n}(\mathcal{E}_b \rho + \sigma^2)\right)^{n/2}$$

$$\Rightarrow \quad 2^{k/n} \leq \left(1 + \rho \frac{\mathcal{E}_b}{\sigma^2}\right)^{1/2}$$

$$\equiv \quad 2^{\rho} \leq \left(1 + \rho \frac{\mathcal{E}_b}{\sigma^2}\right)^{1/2}$$

$$\Rightarrow \quad \frac{\mathcal{E}_b}{\sigma^2} \geq \frac{2^{2\rho} - 1}{\rho}$$

Moral: if we aim for dimensional efficiency $\rho$

we need to use energy / bit at least

$$\sigma^2 \cdot \left[ \frac{2^{2\rho}-1}{\rho} \right].$$

Also notice that $\frac{2^{2\rho}-1}{\rho} \geq (2 \ln 2)$.

So, for a reliable system we need $\frac{\mathcal{E}_b}{\sigma^2} \geq 2 \ln 2$

If we read ~~4.6~~ 4.4.3 in this light we see that:

$\rho \simeq 0$, but $\mathcal{E}_b \simeq \sigma^2 \, 2 \ln 2$

Regards $\frac{2^{2\rho}-1}{\rho} = \frac{e^{2\rho \ln 2} - 1}{\rho}$

$e^x = 1 + x + \frac{1}{2} x^2 \ldots$ so

$e^x - 1 \geq x \Rightarrow \frac{2^{2\rho}-1}{\rho} \geq \frac{2\rho \ln 2}{\rho} = 2 \ln 2$

$\simeq$ if $\rho$ is small.

Question about 4.4.3 — How to compute the error probability of the "orthogonal transmission"?

$c_0 = [1, 0, \ldots, 0]\sqrt{\mathcal{E}}$ $\qquad \mathcal{E} = k\,\mathcal{E}_b$

$c_1 = (0, 1, \ldots, 0)\sqrt{\mathcal{E}}$ $\qquad k = \log_2 m$

$c_{m-1} = (0, \ldots, 0, 1)\sqrt{\mathcal{E}}$

$Y = c_i + Z$ if $i$ is the message

$$\begin{bmatrix} Y_1 \\ \\ \\ Y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\sqrt{\mathcal{E}} + \begin{bmatrix} Z_1 \\ \\ \\ Z_n \end{bmatrix}$$

Suppose $\underline{\underline{i = 0}}$ is sent message

$Y_1 = \sqrt{\mathcal{E}} + Z_1$

$Y_2 = Z_2$

$\vdots$

$Y_n = Z_n$

The RCVR will decide correctly if and only if $Y_1 > Y_2,\ Y_1 > Y_3, \ldots Y_1 > Y_n$.

$$P(\text{correct} \mid i=0) = \Pr(Y_1 > Y_2, Y_1 > Y_3 \ldots, Y_1 > Y_n \mid i=0)$$

$$= \Pr\left(\sqrt{\mathcal{E}} + z_1 > z_2,\ \sqrt{\mathcal{E}} + z_1 > z_3, \ldots, \sqrt{\mathcal{E}} + z_1 > z_n\right)$$

$$n = m = 2^k$$

$$\Pr(\text{correct} \mid z_1 = z)$$

$$= \Pr\left(z_2 < \sqrt{\mathcal{E}} + z,\ z_3 < \sqrt{\mathcal{E}} + z, \ldots, z_n < \sqrt{\mathcal{E}} + z\right)$$

$$= \left(1 - \varphi\left(\frac{\sqrt{\mathcal{E}} + z}{\sigma}\right)\right)^{n-1}$$

$$\Rightarrow P(\text{correct}) = \int \left( \quad \right)^{n-1} \frac{e^{-z^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dz$$