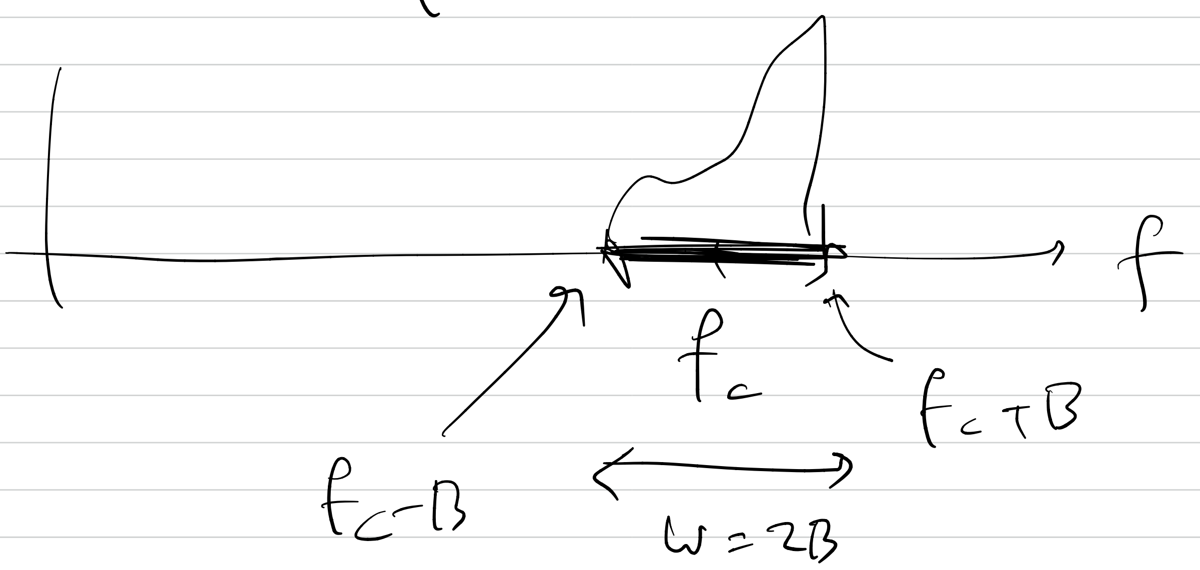
Principles of Digital Communication

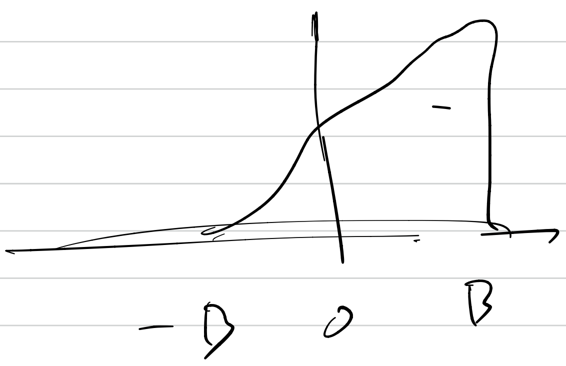
May 19, 2021

Reading war: "passband communication" 7.(1-4).

We wish to communicate using signals that live on a frequency band:



Q: Can we design "everything" in base-band



and "simply" upconvert just before transmission

Ans: yes.

We notice: if $w(t)$ is a real waveform such that $w_F(f) = 0$ for $\big||f|-f_c\big| > B$,

then $w(t)$ can be written as

$$w(t) = \sqrt{2}\,\mathrm{Re}\{w_E(t)\, e^{j2\pi f_c t}\}$$

where: $w_E(t)$ is low-pass ($w_{E,F}(f) = 0$ when $|f| > B$)

$w_E(t)$ is the "baseband equivalent" of $w(t)$ and is (in general) a $\mathbb{C}$-valued signal

$$x(t) \sim x_E(t)$$

$$y(t) \sim y_E(t)$$

$$\langle x, y \rangle = \mathrm{Re}\{\langle x_E, y_E \rangle\}$$

$$w(t) = \sqrt{2}\,\mathrm{Re}\{w_E(t)\}\cos(2\pi f_c t) - \sqrt{2}\,\mathrm{Im}\{w_E(t)\}\sin(2\pi f_c t)$$

in-phase part of $w(t)$

quadrature-phase part of $w(t)$

So: TX architecture:

$$i \in \{0, \dots, m-1\}$$

$$\downarrow$$

$$c_i \in \mathbb{C}^n$$

$$\downarrow$$

$$w_{E,i}(t) = \sum_{j=1}^{n} c_{ij}\,\underline{\psi_j(t)}$$

complex, $\perp$-normal system of waveforms, baseband

$$\downarrow$$

$$w_i(t) = \mathrm{Re}\{\sqrt{2}\,w_{E,i}(t)\,e^{j2\pi f_c t}\}.$$

Note:

$$w_i(t) = \sqrt{2}\,\mathrm{Re}\left\{ \sum_{j=1}^{n} c_{ij}\, \psi_j(t)\, e^{j2\pi f_c t} \right\}$$

$$= \sum_{j=1}^{n} \mathrm{Re}\{c_{ij}\}\, \mathrm{Re}\{\psi_j(t) e^{j2\pi f_c t}\sqrt{2}\}$$

$$- \mathrm{Im}\{c_{ij}\}\, \mathrm{Im}\{ \quad '' \quad \}$$

So, each $w_i(t)$ is a linear combination of $2n$ real waveforms:

$$\mathrm{Re}\{\psi_j(t) e^{j2\pi f_c t}\sqrt{2}\} \quad j=1 \dots n$$

$$\mathrm{Im}\{ \quad '' \quad \} \quad j=1 \dots n.$$

Surprisingly these $2n$ waveforms are an $\perp$-normal basis (because ψ_j's are $\perp$-normal).

Because of this we know what to do at the RCVR: we should form

$$\langle R(t), \mathrm{Re}\{\sqrt{2}\,\psi_j(t)\, e^{j2\pi f_c t}\}\rangle \quad j=1 \dots n$$

and $\langle \quad \mathrm{Im}\{ \quad \} \rangle \quad ''$.

This is equivalent to:

$$\mathrm{Re}\left\{\left\langle R(t),\ \sqrt{2}\,\psi_j(t)\,e^{+j2\pi f_c t}\right\rangle\right\},$$
$$\mathrm{Im}\left\{\left\langle R(t),\ \sqrt{2}\,\psi_j(t)\,e^{j2\pi f_c t}\right\rangle\right\}$$

$$\equiv \quad \mathrm{Re}\left\{\left\langle R(t)\,e^{-j2\pi f_c t}\sqrt{2},\ \psi_j(t)\right\rangle\right\},$$
$$\mathrm{Im}\left\{\left\langle \quad '' \quad ,\ '' \right\rangle\right\}$$

$$\equiv \quad \langle \underbrace{R(t)\,e^{-j2\pi f_c t}\sqrt{2}}_{},\ \underbrace{\psi_j(t)}_{} \rangle \qquad j = 1 \dots n.$$

$$\equiv \quad \begin{tikzcd} R(t) \arrow[r] & \otimes \arrow[r] & \langle \cdot, \psi_j \rangle \\ & \sqrt{2}\,e^{-j2\pi f_c t} \arrow[u] & = Y_j \end{tikzcd} \qquad j = 1 \dots n.$$

TX		RX
i		$\hat{i}$
$\downarrow$		$\uparrow$
$c_i \in \mathbb{C}^n$		$Y = c_i + Z \in \mathbb{C}^n$
$\downarrow$		$\uparrow$
$w_{E,i}(t)$ complex, baseband		$r_E(t) = w_{E,i}(t) + \tilde{N}(t)$
$\downarrow$		$\uparrow$
$w_i(t)$ real, passband	$\longleftrightarrow$	$R(t) = w_i(t) + N(t)$

iid Real & im parts $N(0, \frac{N_0}{2})$

The baseband equivalent noise is

$$\tilde{N}(t) = N(t)\, e^{-j 2\pi f_c t} \sqrt{2}$$

$$= \underline{N(t)\sqrt{2}\cos(2\pi f_c t)} - j\, \underline{N(t)\sqrt{2}\sin(2\pi f_c t)}$$

What we will do with $\tilde{N}(t)$ is to take its inner products with base-band waveforms $g(t)$.

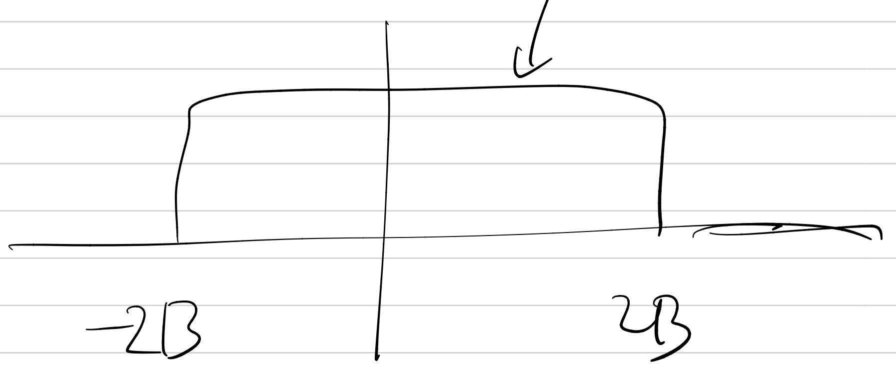
$$\langle \tilde{N}, g\rangle = \langle \underline{N(t)\sqrt{2}\cos(2\pi f_c t)}, g\rangle$$

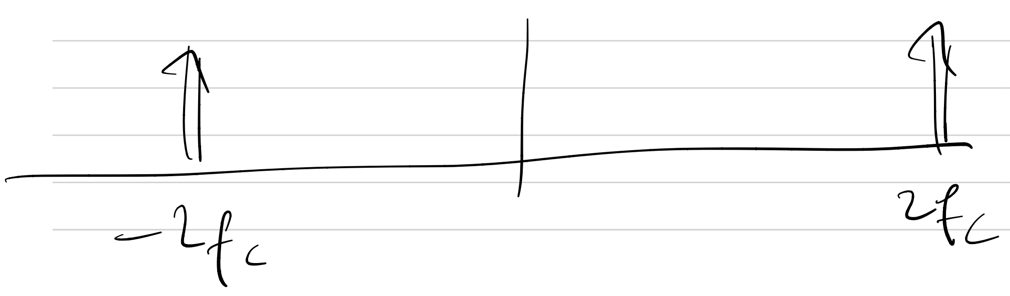
$$- j\, \langle \underline{N(t)\sqrt{2}\sin(2\pi f_c t)}, g\rangle$$

$$\int N(t)\underbrace{\left[g^*(t)\sqrt{2}\cos(2\pi f_c t)\right]}_{a(t)} dt = \langle N, a\rangle$$

$$\int N(t)\underbrace{\left[g^*(t)\sqrt{2}\sin(2\pi f_c t)\right]}_{b(t)} dt = \langle N, b\rangle$$

$$\langle a, b\rangle = \int |g(t)|^2 \sin(4\pi f_c t)\, dt = 0$$





$\Rightarrow$ if we replace

$$\tilde{N}(t) = N(t)\sqrt{2}\cos(2\pi f_c t) - jN(t)\sqrt{2}\sin(2\pi f_c t)$$

by

$$\tilde{\tilde{N}}(t) = N_1(t) - jN_2(t)$$

independent AWGN's

the receiver architecture will not see any difference

$\Rightarrow$ (7.4): baseband equivalent noise:

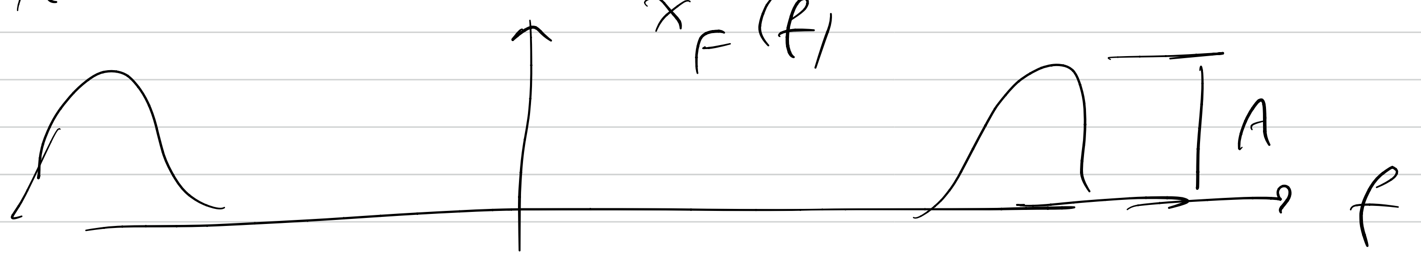
$$R_E(t) = w_E(t) + \tilde{\tilde{N}}(t)$$

complex AWGN, with independent $\mathrm{Re}\{\}$ and $\mathrm{Im}\{\}$ parts each of intensity $N_0/2$.

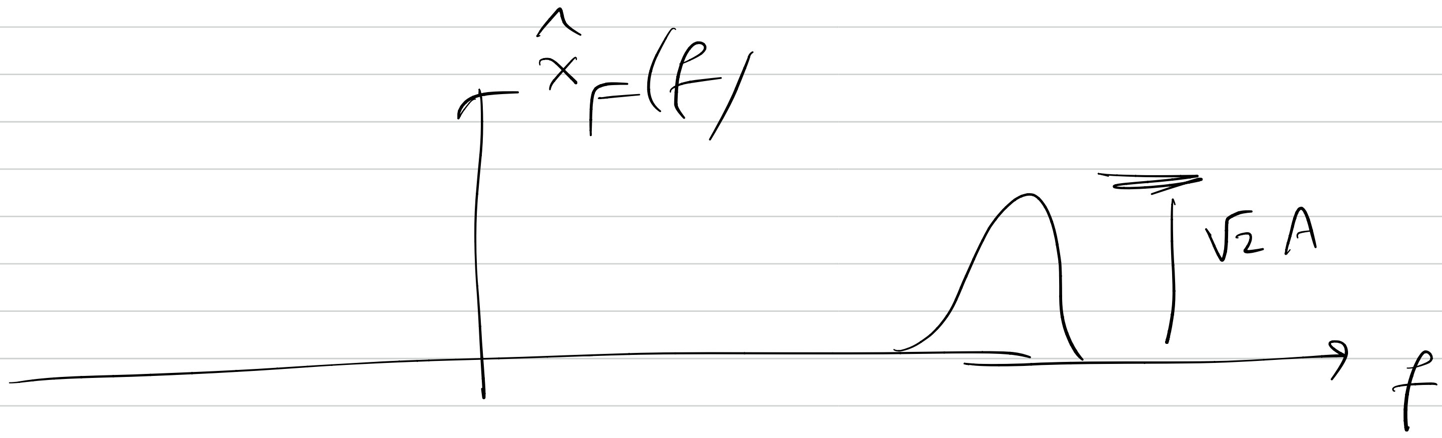
i	$\hat{i}$
c_i	$Y = c_i + Z$
$w_{E,i}(t)$	$r_E(t) = w_{E,i}(t) + \tilde{N}(t)$
$w_i(t)$	$R(t) = w_i(t) + N(t)$.

Q: where are the $\sqrt{2}$'s coming from?

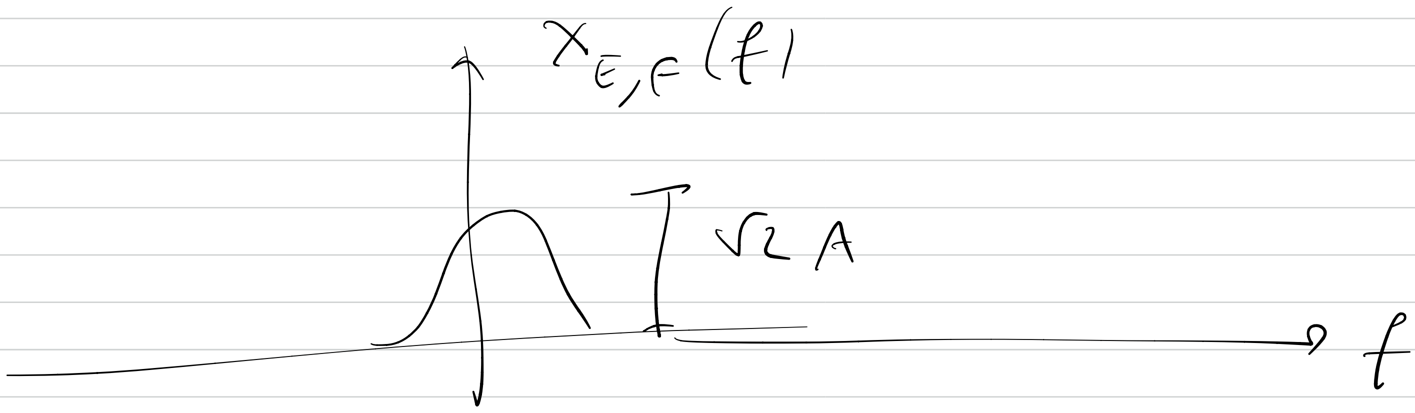
A: we have



: band pass waveform



analytic signal



baseband equiv.

$$\|x\|^2 = \int |x(t)|^2 \, dt$$

$$= \|x_F\|^2 = \int |x_F(f)|^2 \, df$$

$$= 2\int_0^{\infty} |x_F(f)|^2 \, df$$

$$= \cancel{1/4}\int_{-\infty}^{\infty} \|\hat{x}_F(f)\|^2 \, df$$

$$= \int_{-\infty}^{\infty} |\underbrace{\cancel{\sqrt{2}\,\hat{x}_F(f)}}_{|\hat{x}_F(f)|^2}|^2 \, df$$