Principles of Digital Communication

March 18, 2022

Reading Assignment:

3.1 – 3.4

So far:

$i \in \mathcal{H} \in \{0, \ldots, m-1\}$

$$Y \in \mathbb{R}^n, \quad \underline{Y} = c_i + Z$$

$$Z = \begin{bmatrix} z_1 \\ \\ z_n \end{bmatrix} \qquad \begin{array}{l} c_i \in \mathbb{R}^n \\ z_i \sim N(0, \sigma^2) \\ \text{independent.} \end{array}$$

We found: $\{\langle y, c_i \rangle : i = 0, \ldots, m-1\}$

Lets us make the MAP decision

moreover $t = \underline{(\langle y, c_0 \rangle, \ldots, \langle y, c_{m-1} \rangle)}$

$\underline{P_{H|Y}(i|y)}$

$$\underline{f_Z(z) = e^{-\|z\|^2/2\sigma^2}}$$

Setup for Chap 3

$i \in \{0, \dots, m-1\}$

$w_0(t)$ $\mathbb{R}$-valued function

$w_1(t)$

$\vdots$

$w_{m-1}(t)$

$w_i(t)$, $R(t) = w_i(t) + \underline{Z(t)}$

transmitted

"additive noise White Gaussian"

"White Gaussian Noise in $\mathbb{R}^n$"

$$Z = \begin{bmatrix} Z_1 \\ \\ Z_n \end{bmatrix} \qquad Z_i \text{ iid } N(0, \sigma^2).$$

Consider $a \in \mathbb{R}^n$ and the RV

$$X_a = \langle a, Z \rangle = \sum_{i=1}^{n} a_i Z_i$$

$$E(X_a^2) = E\left(\sum a_i Z_i \sum a_j Z_j\right)$$

$$= \sum_{i,j} a_i a_j \underbrace{E(z_i z_j)}_{= \begin{cases} \sigma^2 & i=j \\ 0 & i \neq j \end{cases}}$$

$$= \sigma^2 \sum a_i^2 = \sigma^2 \|a\|^2 = \langle a, a \rangle \sigma^2$$

$$X_b = \langle b, z \rangle \sim N(0, \sigma^2 \langle b, b \rangle)$$

$$E(X_a X_b) = E \sum a_i z_i \sum b_j z_j$$

$$= \sum_{i,j} a_i b_j E(z_i z_j)$$

$$= \sigma^2 \sum_i a_i b_i = \sigma^2 \langle a, b \rangle$$

$$\begin{pmatrix} X_a \\ X_b \\ X_c \\ X_d \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} \langle a,a \rangle & \langle a,b \rangle & \cdots & \langle a,d \rangle \\ \vdots & & & \vdots \\ \langle d,a \rangle & \cdots & & \langle d,d \rangle \end{pmatrix} \right)$$

So if Z is "white Gaussian in $\mathbb{R}^n$"

and $a_1, a_2, \ldots$ are vectors in $\mathbb{R}^n$

then $\begin{pmatrix} X_1 \\ \\ X_k \end{pmatrix} = \begin{pmatrix} \langle a_1, Z\rangle \\ \\ \langle a_k, Z\rangle \end{pmatrix}$ $(**)$

will be $N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} \langle a_1, a_1\rangle & \cdots & \langle a_1, a_k\rangle \\ \vdots & & \\ \langle a_k, a_1\rangle & \cdots & \langle a_k, a_k\rangle \end{bmatrix}\right)$.

Conversely if some $W \in \mathbb{R}^n$ has the property $(**)$ then

$$W \sim N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \sigma^2 I\right).$$

$$\begin{pmatrix} w_1 \\ \\ w_n \end{pmatrix} = \begin{pmatrix} \langle e_1, w\rangle \\ \langle e_2, w\rangle \\ \vdots \\ \langle e_n, w\rangle \end{pmatrix} \Rightarrow W \text{ is white Gaussian in } \mathbb{R}^n$$

A random waveform $Z(t)$ is said to be white Gaussian with intensity σ^2 if for any functions $a_1(t), a_2(t), \dots, a_k(t)$,

$$X_1 = \int a_1(t) Z(t)\,dt = \langle a_1, Z \rangle$$

$$\vdots$$

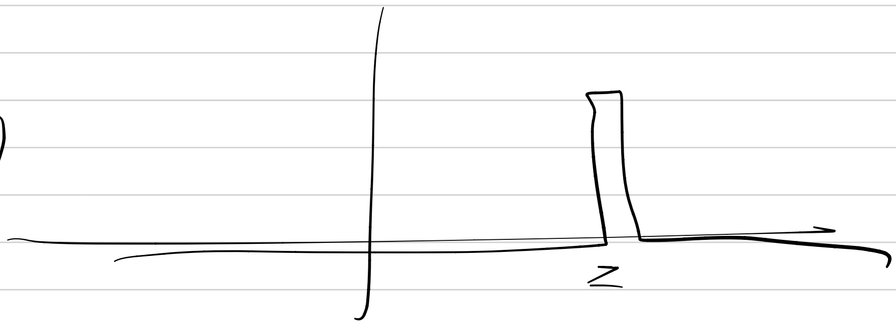
$$X_k = \int a_k(t) Z(t)\,dt = \langle a_k, Z \rangle$$

is

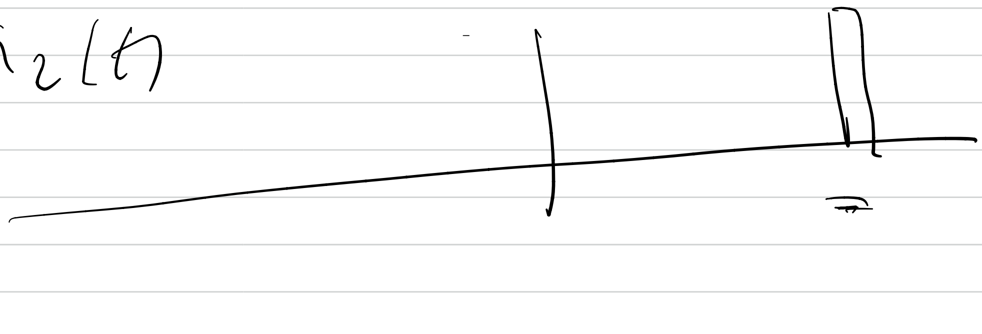
$$\mathcal{N}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} \langle a_1, a_1 \rangle & \dots & \langle a_1, a_k \rangle \\ \vdots & & \\ \langle a_k, a_1 \rangle & \dots & \langle a_k, a_k \rangle \end{bmatrix} \right)$$

$$\langle a, b \rangle = \int a(t) b(t)\,dt$$

$\Rightarrow$ take $a_1(t)$



$a_2(t)$



$$\frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} z(t)\,dt \sim N\left(0, \frac{\sigma^2}{\varepsilon}\right)$$

$$\overset{\shortparallel}{\langle z, a\rangle} \qquad a(t) = \begin{cases} 1/\varepsilon & |t| < \varepsilon/2 \\ 0 & \text{else} \end{cases}$$

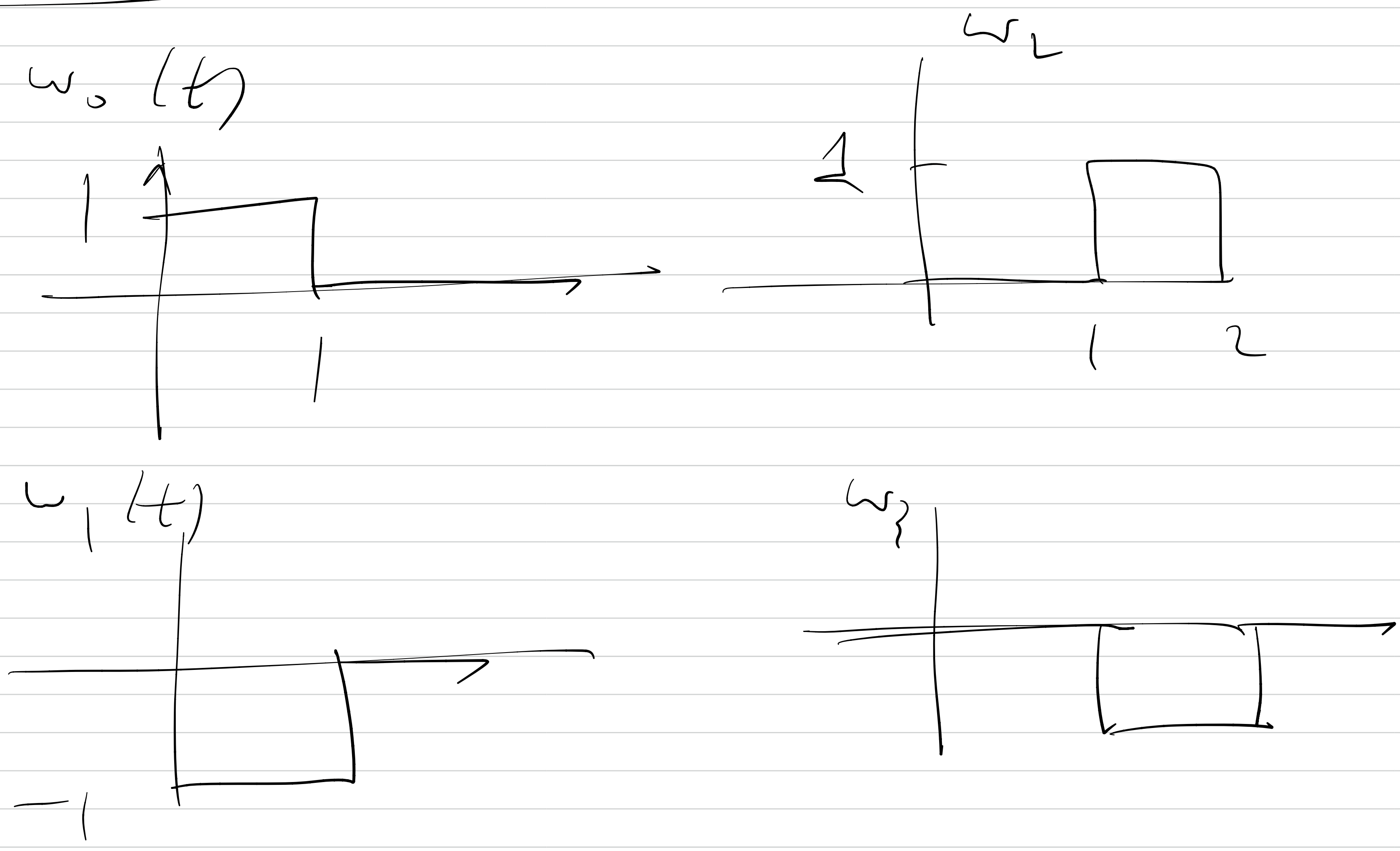
$$\|a\|^2 = 1/\varepsilon$$

This property of AWGN lets us conclude that

$$t = \left[\langle R, w_0\rangle, \ldots, \langle R, w_{m-1}\rangle\right]$$

is a sufficient statistic.

Example:



$(\langle R, w_0 \rangle, \langle R, w_2 \rangle)$ is a suff. statistic

we have $\left\{ \begin{array}{l} \psi_1(t) = w_0(t) \\ \psi_2(t) = w_2(t) \end{array} \right\}$

span all the $w_i(t)$'s.

$\Rightarrow$ given $w_0(t), \ldots, w_{m-1}(t)$

find $\psi_1(t), \ldots, \psi_n(t)$ such that

1. each w_i is a linear comb. of $\psi_1, \ldots, \psi_n$ $(n \leq m)$

2. $\langle \psi_i, \psi_j \rangle = \delta_{ij}$

then:

$$w_i(t) = \sum_{j=1}^{n} c_{ij} \, \psi_j(t)$$

$$\langle R(t), \psi_j \rangle =: Y_j$$

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$ is a suff statistic

At the transmitter side

$$i \longrightarrow c_i = (c_{i1} \ldots c_{in}) \in \mathbb{R}^n$$

$$\rightarrow w_i(t) = \sum_j c_{ij} \varphi_j(t)$$

transmit the signal $w_i(t)$.

Receiver side:

$$R(t) \longrightarrow \text{compute } Y \in \mathbb{R}^n$$

$$\begin{aligned} Y_j &= \langle R, \varphi_j \rangle \\ &= \langle w_i + Z, \varphi_j \rangle \\ &= \underbrace{\langle w_i, \varphi_j \rangle}_{c_{ij}} + \langle Z, \varphi_j \rangle \end{aligned}$$

$$Y = c_i + W \qquad W_j = \langle Z, \varphi_j \rangle$$

$$N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \sigma^2 I_n\right)$$

$\Rightarrow$ Model in Ch 3 reduces to that in Ch2.