Principles of Digital Communication

March 16, 2022

Reading now: 2.5, 6, 7.

2.5 → "Sufficient statistics" *

2.6 → "Bounds on Error Probability" *

2.7 Summary of Ch. 2.

Sufficient statistics:

$H \in \{0, \dots, m-1\}$ $\quad$ $Y \in \mathcal{Y}$

$P_H(i)$ $\quad$ $p_{Y|H}(y|i)$

given y $\quad$ $P_{H|Y}(i|y)$ $\quad$ $i = 0, \dots, m-1$

$T = \underline{\text{funct}(Y)}$ is a suffic statistic

if the value t of T lets us

compute $\{\underline{P_{H|Y}(i|y)} : i = 0 \dots m-1\}$

Thm: Fisher-Neyman factorization:

If $p_{Y|H}(y|i) = \underline{h(y)\, g(i,t)}$

$t = \underline{\text{funct}(y)}$

then $T = \text{funct}(Y)$ is a suff. statistic. (And conversely).

Pf:

$$p_{H|Y}(i|y) = \frac{p_{HY}(i,y)}{p_Y(y)}$$

$$= \frac{p_{HY}(i,y)}{\sum_j p_{HY}(j,y)} = \frac{p_H(i)\, p_{Y|H}(y|i)}{\sum_j p_H(j)\, p_{Y|H}(y|j)}$$

$$= \frac{p_H(i)\, \cancel{h(y)}\, g(i,t)}{\sum_j p_H(j)\, \cancel{h(y)}\, g(j,t)} = f(t)$$

Examples:

Binary Hyp. Test.

$H \in \{0,1\} \rightsquigarrow Y$

$p_H(0),\ p_H(1) \qquad p_{Y|H}(y|i)$

$$\boxed{p_{H|Y}(0|y)} = \frac{p_H(0)\,\boxed{p_{Y|H}(y|0)}}{p_H(0)\,p_{Y|H}(y|0) + p_H(1)\,p_{Y|H}(y|1)}$$

$$= \frac{p_H(0)}{p_H(0) + p_H(1)\,\Lambda(y)}$$

$$\boxed{\Lambda(y) \stackrel{\Delta}{=} \frac{p_{Y|H}(y|1)}{p_{Y|H}(y|0)}}$$

$p_{H|Y}(1|y) = 1 - p_{H|Y}(0|y)$

So $T = \Lambda(Y)$ is a suff. statistic.

Moreover suppose T is some suff. statistic.

$\Rightarrow$ it lets us compute

$$\frac{P_{H|Y}(0|y)}{P_{H|Y}(1|y)} = \frac{P_H(0)}{P_H(1)} \frac{P_{Y|H}(0)}{P_{Y|H}(1)}$$

so T lets us compute $\Lambda(y)$.

On Markov Chains:

"$A \leftrightarrow B \leftrightarrow C$" means

$$P_{ABC}(a,b,c) = P_A(a)\, P_{B|A}(b|a)\, \cancel{P_{C|BA}(c|ba)}$$
$$P_{C|B}(c|b)$$

"$A_1 \leftrightarrow A_2 \leftrightarrow A_3 \leftrightarrow \ldots \leftrightarrow A_n$"

is a Markov chain means

when we condition on A_i,

the variables $(A_1, \ldots, A_{i-1})$, $(A_{i+1}, \ldots, A_n)$

are independent.

"In

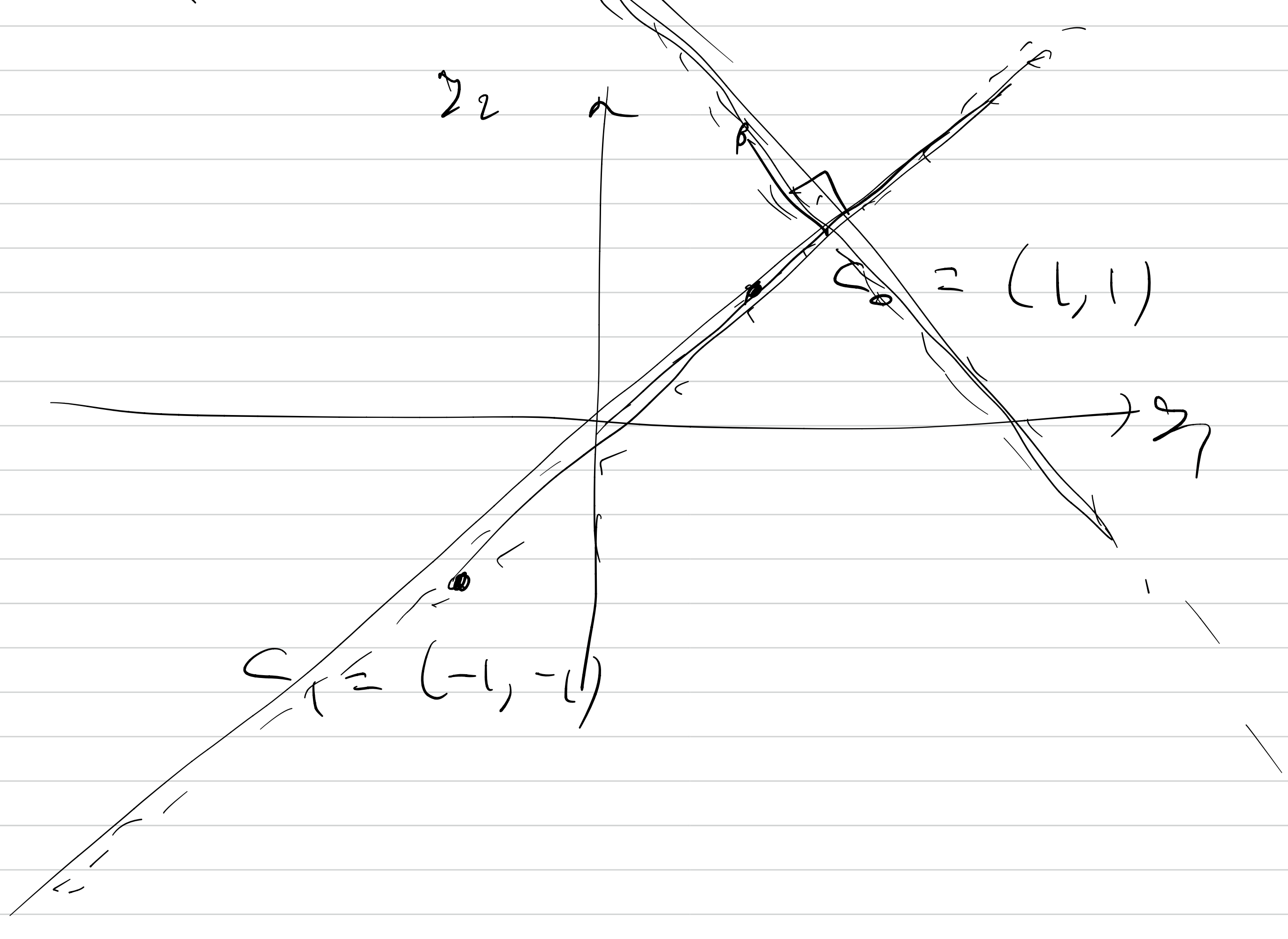
$$H \longrightarrow Y \longrightarrow T = \text{funct}(Y)"$$

we always have $H - Y - T$

for T to be a suff. statistic we need

$$H - T - Y$$

Example: "AWGN" case $n = 2$



$$\Lambda(y) = \frac{f_{Y|H}(y|1)}{f_{Y|H}(y|0)}$$

$$f_{Y|H}(y|0) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\|y - c_0\|^2\right)$$

$$f_{Y|H}(y|1) \quad \text{''} \quad \ldots \quad \|y - c_1\|^2$$

$$\sigma^2 \log \Lambda(y) = \frac{\sigma^2}{2\sigma^2}\left[\|y - c_0\|^2 - \|y - c_1\|^2\right)$$

$$= \langle y, c_1 - c_0 \rangle$$

$$= -2(y_1 + y_2)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longleftrightarrow \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \longleftarrow$$

if y and y' have

$$y_1 - y_2 = y_1' - y_2' \Rightarrow \Lambda(y) = \Lambda(y').$$

Bounds on Error probability:

$$p(\text{Error} \mid 0) = \sum_{y \in \mathcal{D}_1} \underline{p_{Y|H}(y|0)}$$

$$p(\text{Error} \mid 1) = \sum_{y \in \mathcal{D}_0} p_{Y|H}(y|1).$$

$$p(\text{Error}) = \sum_{\underline{y \in \mathcal{D}_1}} p_H(0)\, p_{Y|H}(y|0) + \sum_{\underline{\underline{y \in \mathcal{D}_0}}} p_H(1)\, p_{Y|H}(y|1)$$

$$= \sum_{y \in \mathcal{Y}} \begin{cases} \underline{p_H(0)\, p_{Y|H}(y|0)} & \text{if } y \in \mathcal{D}_1 \\ \underline{p_H(1)\, p_{Y|H}(y|1)} & \text{if } y \in \mathcal{D}_0 \end{cases}$$

$$\geq \sum_{y \in \mathcal{Y}}' \min \{ p_H(0) p_{Y|H}(y|0), p_H(1) p_{Y|H}(y|1) \}$$

= if we use "MAP" to find $\mathcal{D}_0$ & $\mathcal{D}_1$

$$\Rightarrow P_{MAP}(ERR) = \sum_{y}' \min \{ \quad , \quad \}$$

$$\min\{a, b\} \leq \sqrt{ab}$$

$$\Rightarrow P_{MAP}(Error) \leq \sqrt{p_H(0) p_H(1)} \sum_{y}' \sqrt{p_{Y|H}(y|0) p_{Y|H}(y|1)}$$

Bhattacharyya bound.

For the m-ary case:

Suppose $H=i$ is the true hypothesis, y is observed

an error only if for some j

$$P_H(j)\,P_{Y|H}(y|j) \geq P_H(i)\,P_{Y|H}(y|i)$$

$$\mathbb{1}\{\text{error}\} \leq \sum_{j\neq i} \mathbb{1}\{\quad\quad\}$$

$$\mathbb{1}\{P_H(j)\,P_{Y|H}(y|j) \geq P_H(i)\,P_{Y|H}(y|i)\}$$

$$\leq \sqrt{\frac{P_H(j)}{P_H(i)} \cdot \frac{P_{Y|H}(y|j)}{P_{Y|H}(y|i)}}$$

$$P(\text{error} \mid H=i) = \sum_{y} P_{Y|H}(y|i)\,\mathbb{1}\{\text{error}\}$$

$$\leq \sum_{j\neq i} \sum_{y} \sqrt{\frac{P_H(j)}{P_H(i)}} \sqrt{\frac{P_{Y|H}(y|j)}{P_{Y|H}(y|i)}} \cdot P_{Y|H}(y|i)$$

$$= \sum_{j \neq i}{}' \sqrt{\frac{p_H(j)}{p_H(i)}} \sum_{y}{}' \underbrace{\sqrt{p_{Y|H}(y|i)\, p_{Y|H}(y|j)}}$$

$$\underbrace{P(E(x)) \leq \sum_{\substack{i,j \\ i \neq j}}{}' \sqrt{p_H(i)\, p_H(j)} \cdot \sum_{y} \underline{\qquad\qquad}}$$

"Union – Bhattacharyya Bound."