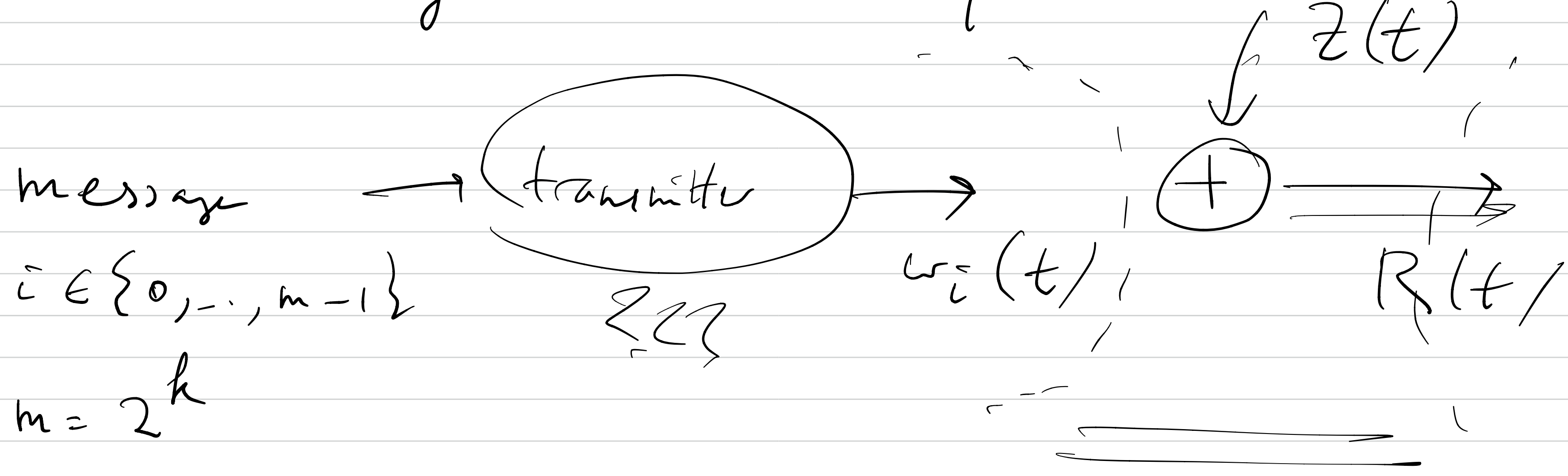
Principles of Digital Communication

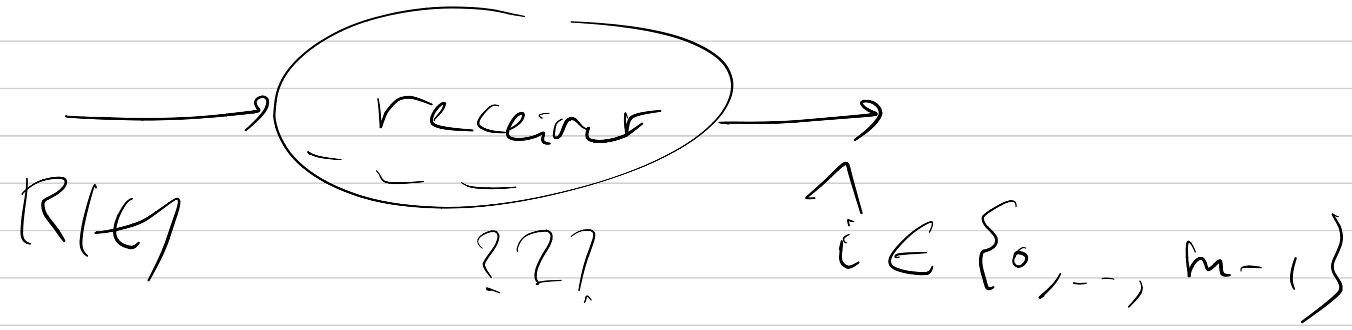
April 14, 2021

Reading assignment was:

- Chapter 4.(1, 2, 3, ..., 4.4.2).

Isometries of transmitted waveforms.





We observed that, ① we can represent $w_0, \dots, w_{m-1}$ in an $\perp$-normal basis $\{\psi_1, \dots, \psi_n\}$ $\psi_j(t)$

$$w_i(t) = \sum_{j=1}^{n} c_{ij} \psi_j(t). \qquad c_0, \dots, c_{m-1} \in \mathbb{C}^n$$

② the optimal receiver (MAP rule) can be implemented as Ⓐ. form $Y_j = \langle R, \psi_j \rangle = \int R(t) \psi_j^*(t) dt$

$j = 1, \dots, n$

and our system is equivalent to the following

$i \longrightarrow c_i \in \mathbb{C}^n$

if i is sent

$Y \longrightarrow \hat{i} \qquad Y \in \mathbb{C}^n, \quad Y = c_i + Z$

$Z \in N(0, I_n \sigma^2)$

In case of equally likely messages

$$\hat{i} = \arg\min_i \|Y - c_i\|^2.$$

or

(B) form $\underline{\langle R, w_i \rangle}$ $\quad i = 0, \ldots, m-1.$

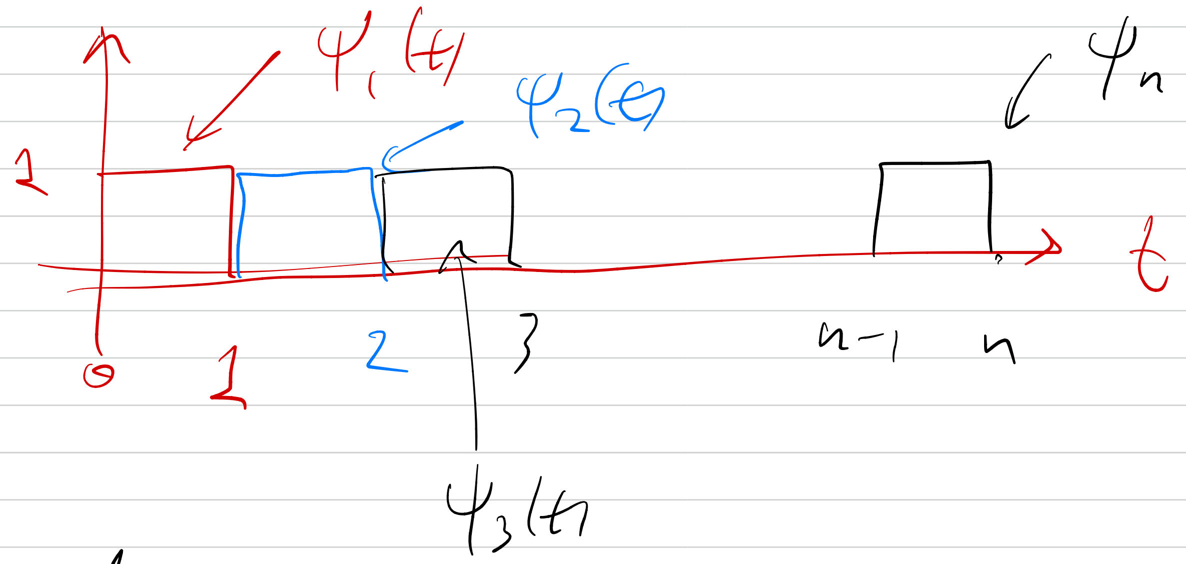
do ~~an~~ $\arg\max_i$

Global parameters of our design:

k $(m = 2^k)$ bits are sent

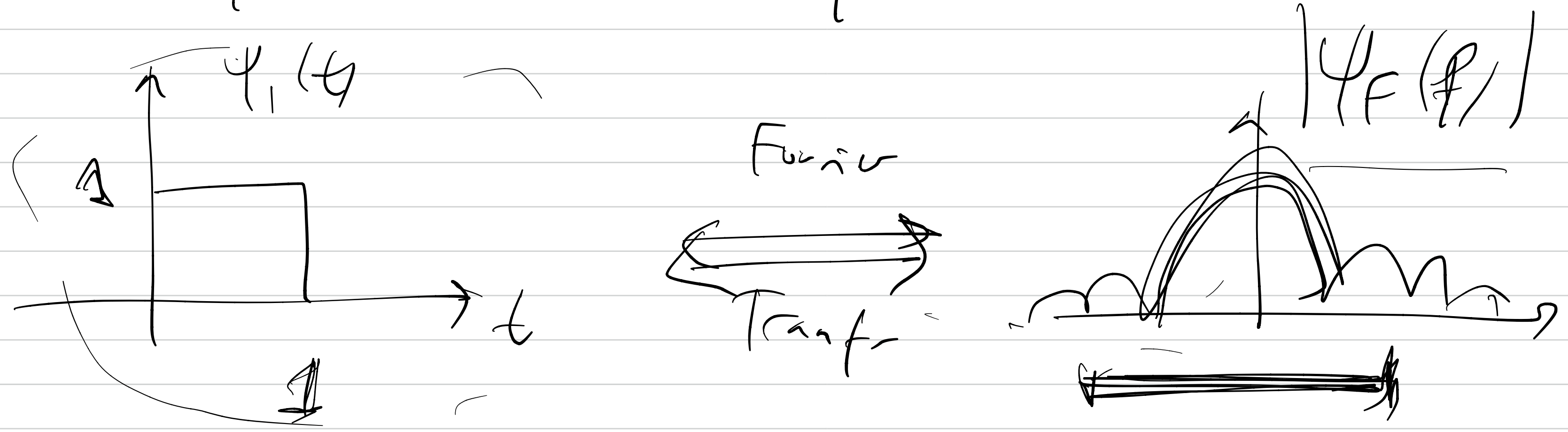
n (dimensionality of the set of waveforms)

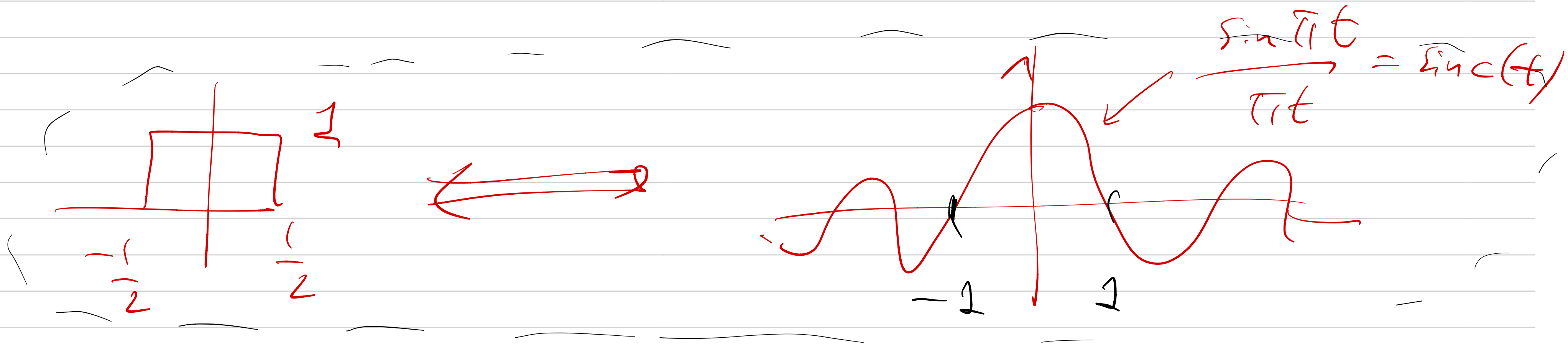
????

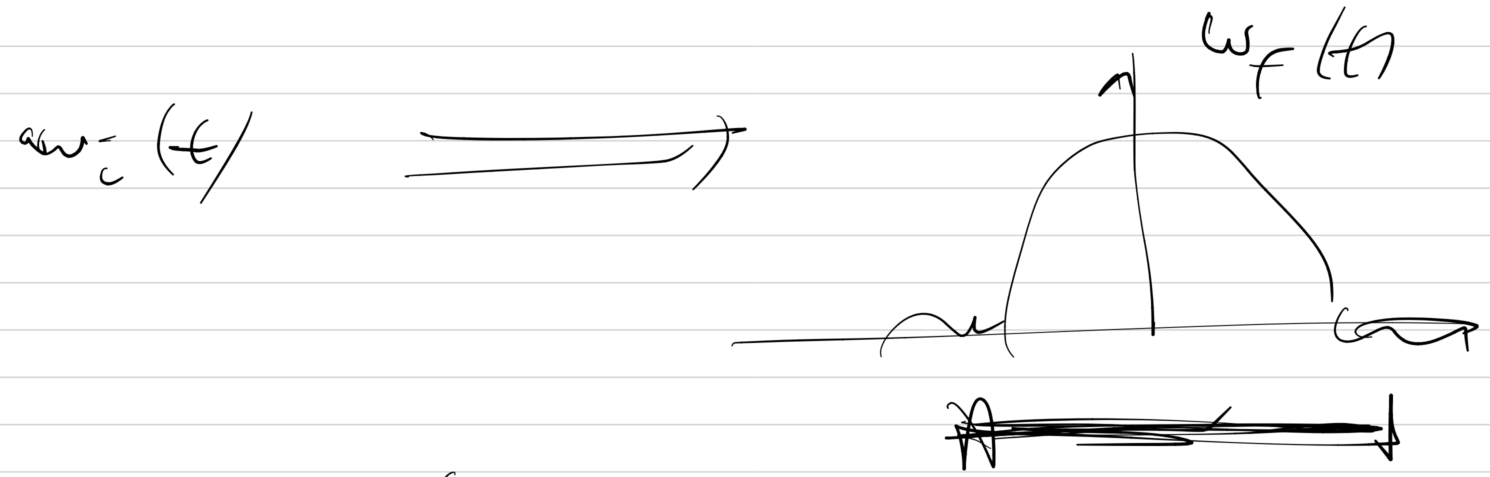


each $w_i = \sum_{1}^{n} c_{ij} \varphi_j$ lives in the time interval $[0, n]$

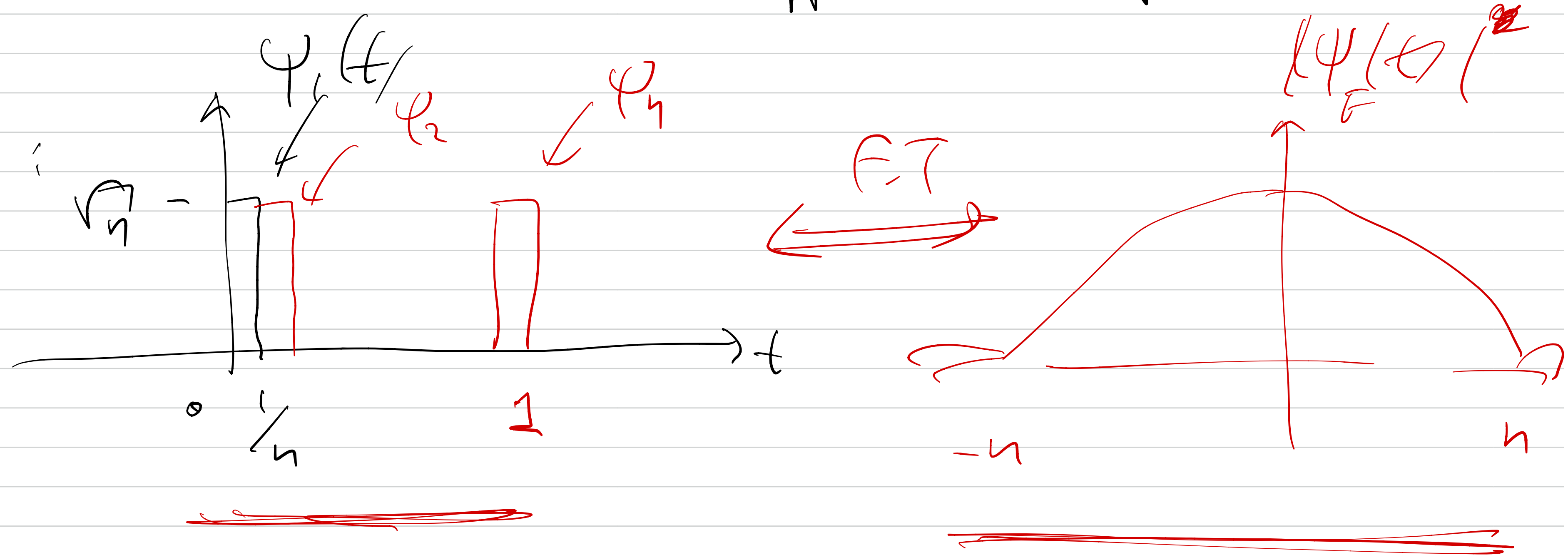
The picture in the Fourier domain







Ex:



These examples suggest n is related to

(footprint in time) $\times$ (footprint in frequency).

$k/n \approx$ bit/sec/Hz (a measure of efficiency of resource utilization)

suppose $\mathcal{E} = \frac{1}{m} \sum_{i=0}^{m-1} \|w_i\|^2$, we can also

define $\mathcal{E}_b \equiv \mathcal{E}/k = \mathcal{E}/\log_2 m$ energy per transmitted bit.

we see

$(n, k, \mathcal{E}_b)$ is efficiency parameters of our system

P(Error) is a reliability measure

what we want is

- P(Error) : small
- k/n high
- $\mathcal{E}_b$ low.

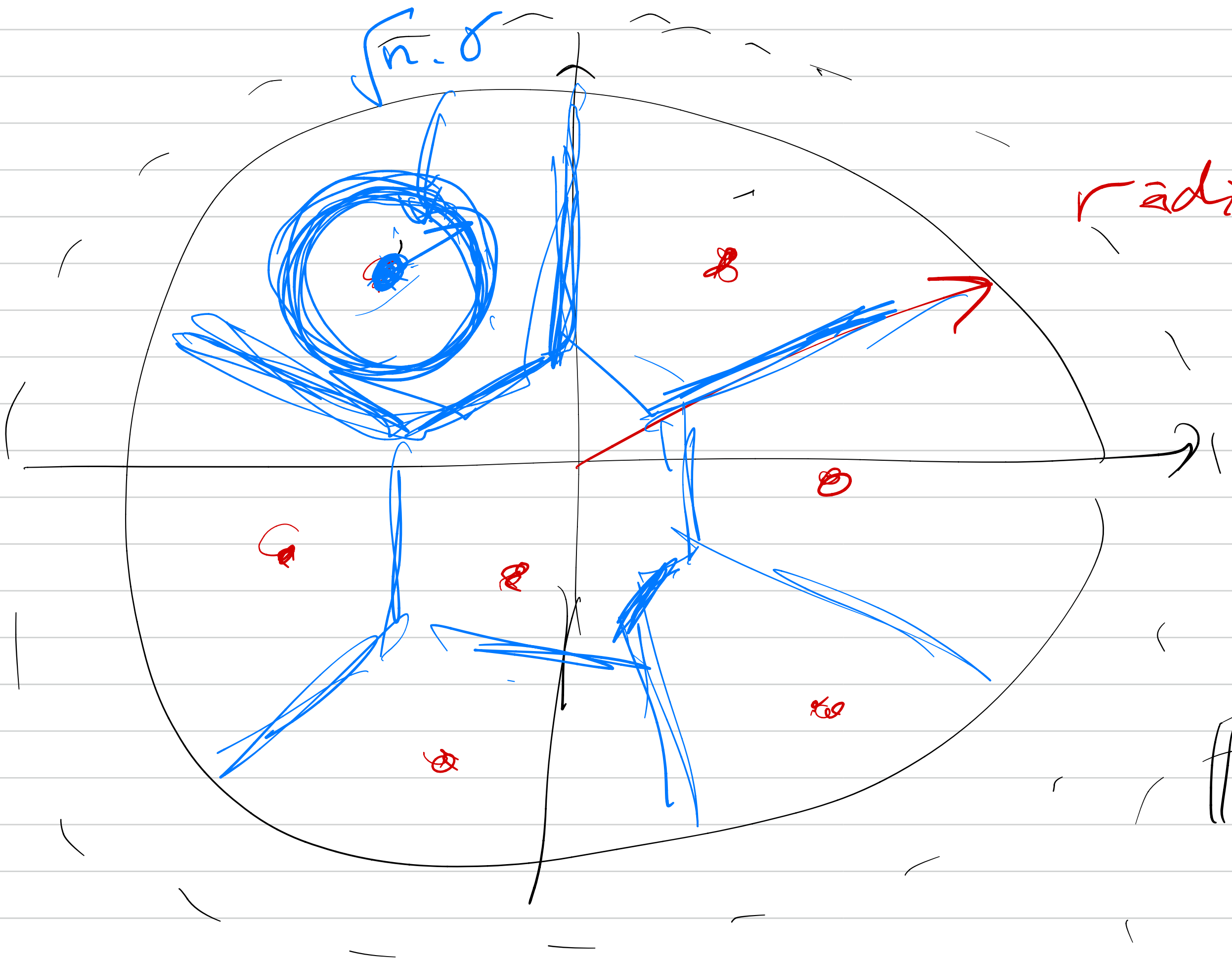
Q: what are the trade-offs ??

Reading (4.4.1, 4.4.2) was about trying to develop some intuition towards this.

Pictorial Representation:

We have $m = 2^k$ signals $\equiv c_0, \ldots, c_{m-1}$

points in $\mathbb{R}^n$, suppose each $\|c_i\|^2 \le \mathcal{E}$
$= k\,\mathcal{E}_b$



$Y = c_i + Z$

$\|Y\|^2 = \|c_i\|^2 + \|Z\|^2 + \underbrace{\langle c_i, Z\rangle 2}_{N(0,\ \sigma^2\|c_i\|^2)}$

$\sigma^2 k\,\mathcal{E}_b$.

$$\|Z\|^2 = z_1^2 + \cdots + z_n^2 = n\sigma^2 + O(\sqrt{n})$$

$$\frac{\|Z\|^2 - n\sigma^2}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i^2 - \sigma^2) \longrightarrow \text{gaussian}(0, V).$$

0-mean

$3\sigma^4$ = variance

$$\|Y\|^2 \simeq k\,\mathcal{E}_b + \sigma^2 n + \cancel{2}\sqrt{\sigma^2 k\,\mathcal{E}_b} + O(\sqrt{n}).$$

We can now perform a volume calculation.

- each decoding region needs to contain a ball of radius $(\sigma\sqrt{n})$ around c_i (for reliability)

$\Rightarrow$ Volume (decoding region) $\geq$ Vol$(\underbrace{\text{Ball}(\sigma\sqrt{n})})$

$\Rightarrow$ Total volume of the decoding region

$$\geq 2^k \, \text{Vol}(\text{Ball}(\sigma\sqrt{n}))$$

- But the received vector Y has

$$\|Y\|^2 \leq \underbrace{k\mathcal{E}_b + \sigma^2 n + 2\sqrt{k\mathcal{E}_b\sigma^2}}$$ with high prob

The volume of the decoding regions

$$\leq \text{Vol}(\text{Sphere}(\sqrt{\underbrace{\qquad\qquad}}))$$

$\Rightarrow$ restrict $(k, n, \mathcal{E}_b)$ to certain values.

$$\left[\; 2^k (\sigma\sqrt{n})^n \leq \left(k\mathcal{E}_b + n\sigma^2 + 2\sqrt{k\mathcal{E}_b\sigma^2}\right)^{n/2}\right.$$

given $\mathcal{E}_b$, there is an upper bound on k/n in terms of $\mathcal{E}_b/\sigma^2$.