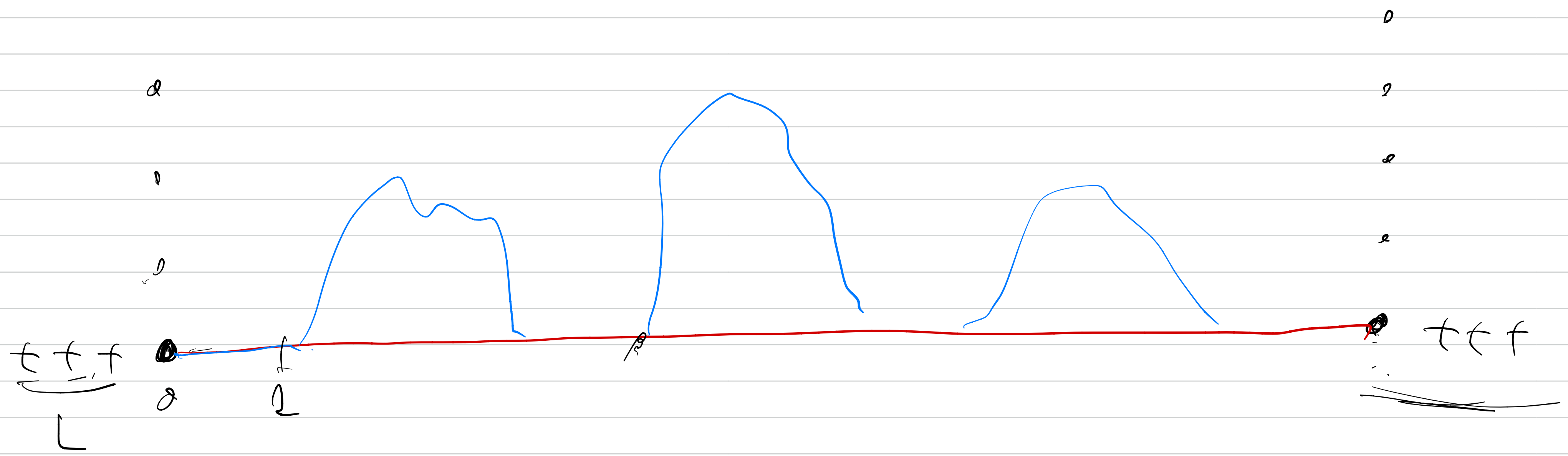
Principles of Digital Communication

May 12, 2021

Reading for today: "Analysis of Convolutional Codes" (6.4).

assume (without loss of generality) that the transmitted

message $= \underbrace{+ + \dots +}_{b_0 - b_{k-1}} \ \underbrace{+ + +}_{L}$



$$\Omega_i = \begin{cases} 0 & \text{if a detour does not start at stage } i \\ \#\text{ of erroneous decoded bits due to the detour that starts at stage } i \end{cases} \qquad i = 0, \dots, k-1$$

Bit error probability $=$

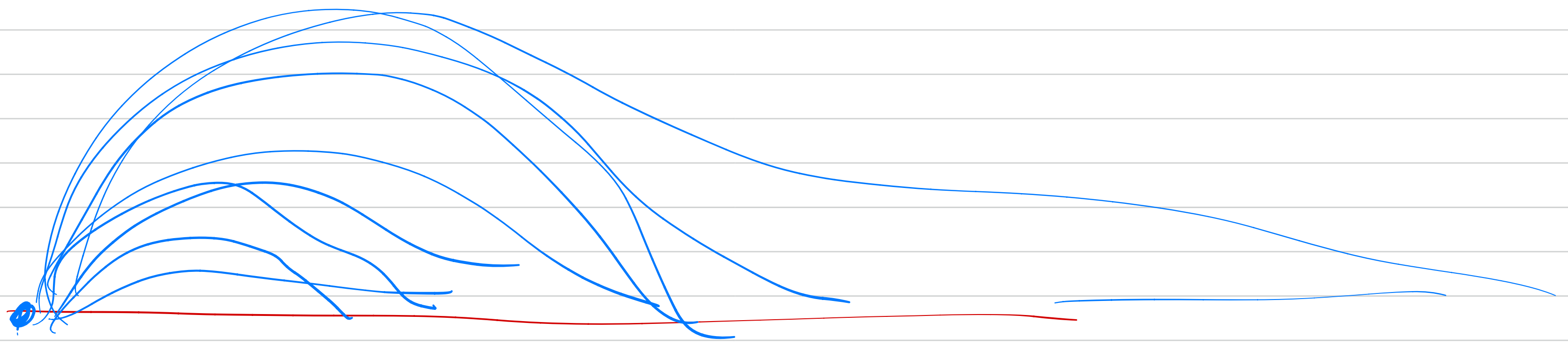
$$\frac{1}{k} \sum_{i=0}^{k-1} E(\Omega_i)$$

We upper bound $E(\Omega_i)$ by removing the time-horizon

$\Rightarrow E(\Omega_0) = E(\Omega_1) = \dots = E(\Omega_{k-1})$

$\Rightarrow$ Bit error rate $\leq \underline{E(\Omega_0)}$

To understand this :



$$E(\Omega_0) = \sum_{\substack{\text{all paths } h \\ \text{that start at} \\ +++\text{, and at} \\ +++}} \Pr(\text{path } h \text{ is taken by the decoder}) \cdot \left(\begin{array}{l}\#\text{ of bit errors} \\ \text{on the path } h.\end{array}\right)$$

$$\text{Probability}(h \text{ is taken}) \leq Q\left(\frac{\text{distance of path } h \text{ to path} \text{ ——— }}{2\sigma}\right)$$

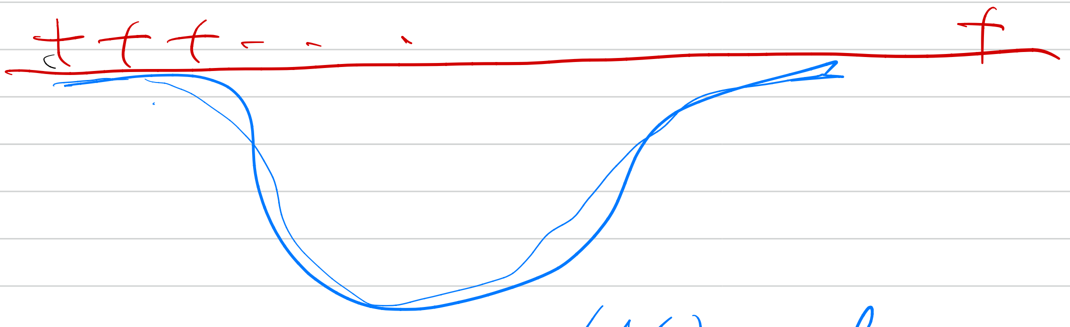
If I knew

$$\left\{ \underline{a(i,d)} = \#\text{ of paths } h \text{ with "x" distance } d \text{ from ——— } \text{ \& "b" distance } i \text{ from } ++++\text{-- bits.} \right\}$$

Then
$$E(\Omega_0) \leq \sum_{i,d} a(i,d)\, Q\left(\frac{d}{2\sigma}\right) \cdot i$$

Question: why is $\Pr(\text{taking path } h) \le Q(\cdot)$?

$x(b) = + + + \cdots +$ $\qquad b = (e e e)$



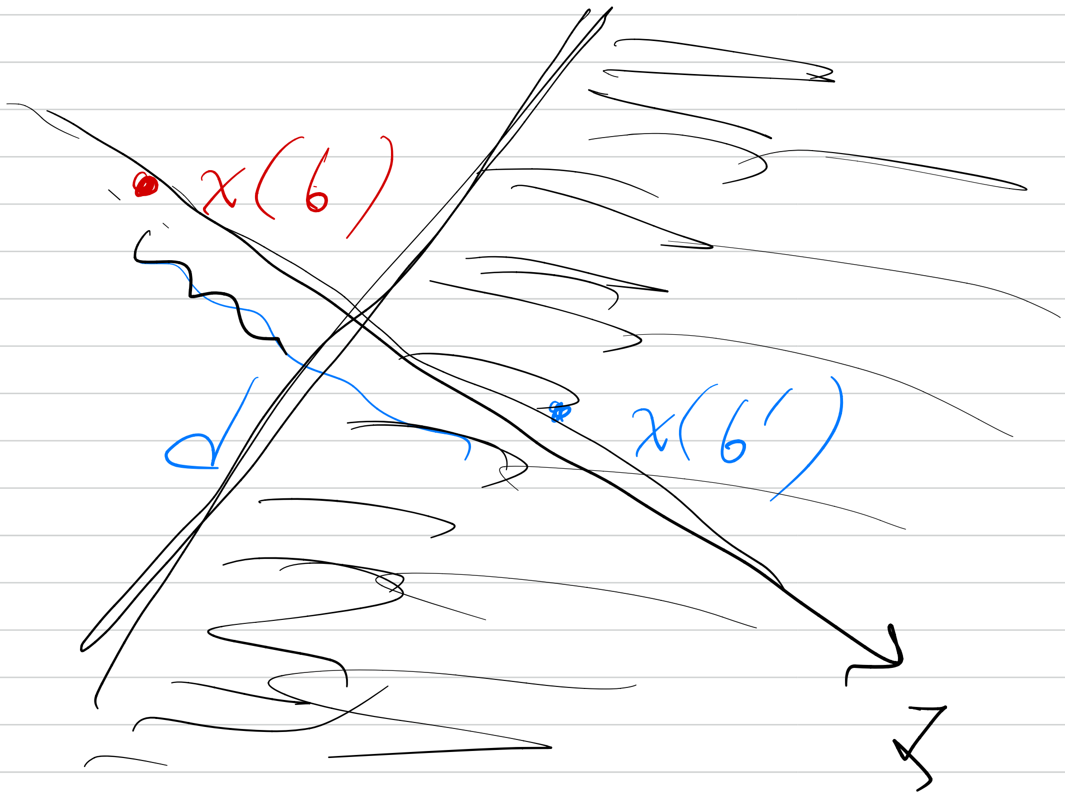
$x(b') = h$

$\Pr(\text{decoder takes } h)$

$= \Pr(h \text{ is close to all other possible paths})$

the received $y_1 \ldots y_n$ than

$\le \Pr(h \text{ is close to } y_1 \ldots y_n \text{ than } \underline{\qquad})$



$= \Pr\left(Z_1 > \frac{d}{2}\right)$

$= Q\left(\frac{d}{2\sigma}\right)$

distance d between $\;+\;+\;\cdots\;+\; = x(b)$

$\qquad\qquad\qquad\quad -\;-\;+\;+\;\cdots\;- = x(b')$

$= \sqrt{\sum_{j=1}^{n} 4\,\mathbb{1}\{x(b')_j \ne \cdot\}}$ $\quad n$

$$d^2 = 4 \boxed{\sum_{j=1}^{n} \mathbb{1}\{ x(b')_j = - \}} \mathcal{E}_s$$

because what is sent is not $+, -$ but is $+\sqrt{\mathcal{E}_s}$ or $-\sqrt{\mathcal{E}_s}$.

# of places the wrong path h is differing from from $++++$ path.

$$P(\text{taking path } h) \le Q(\quad) \le e^{-\frac{1}{2}\frac{d^2}{4\sigma^2}}$$

$$= \exp\left(-\frac{\mathcal{E}_s}{2\sigma^2} \underbrace{(\#\text{ of places } h \text{ has a } -)}\right)$$

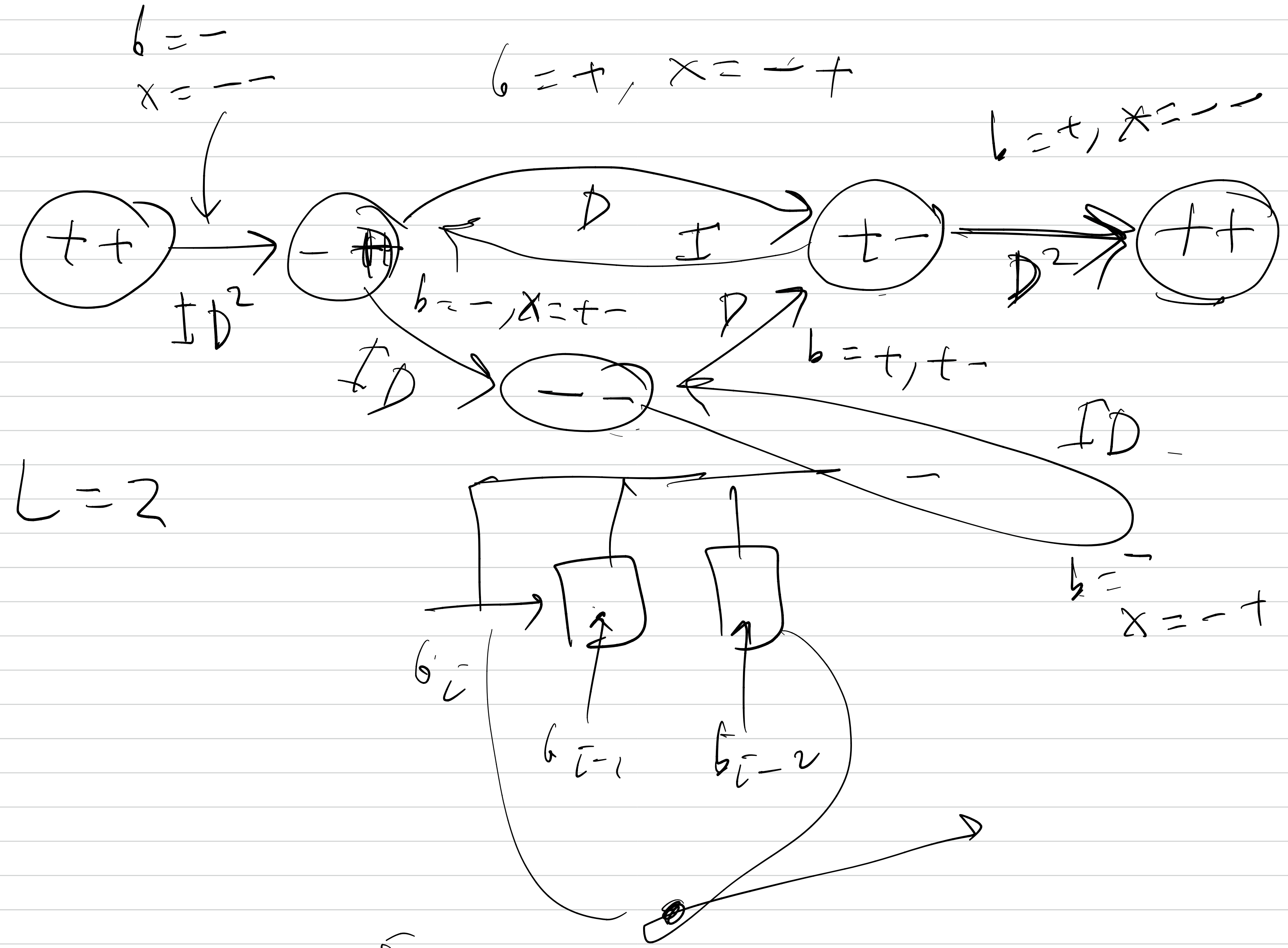
Q: "4 ?" :

$$\overbrace{(\sqrt{\mathcal{E}_s}, \sqrt{\mathcal{E}_s}, \ldots, \sqrt{\mathcal{E}_s})}^{n} = \text{———}$$

$$(\pm\sqrt{\mathcal{E}_s}, \pm\sqrt{\mathcal{E}_s} \ldots \pm\sqrt{\mathcal{E}_s}) = h$$

the squared euclidean distance between ——— and h

$$= \sum_{j=1}^{n} \underbrace{(2\sqrt{\mathcal{E}_s})^2}_{4\mathcal{E}_s} \mathbb{1}\{\text{position } j \text{ in } h \text{ is } -\sqrt{\mathcal{E}_s}\}$$



paths : ID^5, I^2D^5, ...

We are interested in

$$\sum_{h} I^{i(h)} D^{d(h)} = A(I,D).$$

h : from "++" left to "++" right.

$$= \sum_{i,d} a(i,d) I^i D^d$$

How to compute $A(I,D)$:

Let $A_{+-}(I,D) = \sum_{\substack{\text{all } h \text{ that} \\ \text{end in the} \\ \text{state } +-}}' I^{i(h)} D^{d(h)}$

$A_{-+}(I,D) = \sum_{\substack{\text{all } h \text{ that} \\ \text{end in state} \\ -+}}' I^{i(h)} D^{d(h)}$

$A_{--}(I,D) = \sum_{\substack{\text{all } h \text{ that} \\ \text{end in } --}}' I^{i(h)} D^{d(h)}$

$\underline{A(I,D)} = D^2 \underbrace{A_{+-}(I,D)} \leftarrow$

$$\left(\begin{aligned} A_{+-}(I,D) &= \underbrace{A_{-+}(I,D)}\cdot D + \underbrace{A_{--}(I,D)}\cdot D \\ A_{-+}(I,D) &= ID^2 + A_{+-}(I,D)\, I \\ A_{--}(I,D) &= A_{--}(I,D)\, ID + A_{-+}(I,D)\cdot ID \end{aligned}\right)$$

Solve these for $A_{+-}(I,D)$. then find $A(I,D)$

$$\begin{bmatrix} 1 & -D & -D \\ -I & 1 & 0 \\ 0 & -ID & (1-ID) \end{bmatrix} \begin{bmatrix} A_{+-} \\ A_{-+} \\ A_{--} \end{bmatrix} = \begin{bmatrix} 0 \\ ID^2 \\ 0 \end{bmatrix}$$

will find $A(I,D) = \dfrac{\text{polynomial }(I,D)}{\text{polynomial }(I,D)}$

Perhaps (as an illustration)

$$A(I,D) = \frac{I^3 D^7}{1 - 2I^2 D^3}$$

$$= I^3 D^7 \left(1 + 2I^2 D^3 + 4I^4 D^6 + 8I^6 D^9 + \cdots\right)$$

$$= I^3 D^7 + 2I^5 D^{10} + 4I^7 D^{13} + \cdots$$

It appears that we should expand $A(I,D)$, find $(a(i,d) : i, d \in \mathbb{N})$ & then plug this into

$$P(\text{error}) \le \sum_{i,d}' a(i,d) \underbrace{e^{-\frac{d \mathcal{E}_s}{2\sigma^2}}} \cdot i$$

$$A(I,D) = \sum_{i,d}' a(i,d) I^i D^d$$

$$\frac{\partial A(I,D)}{\partial I}\Bigg| = \sum_{i,d}' a(i,d)\, i\, I^{i-1} D^d \Bigg|$$

$I = 1,\ D = \underline{e^{-\mathcal{E}_s/2\sigma^2}}$

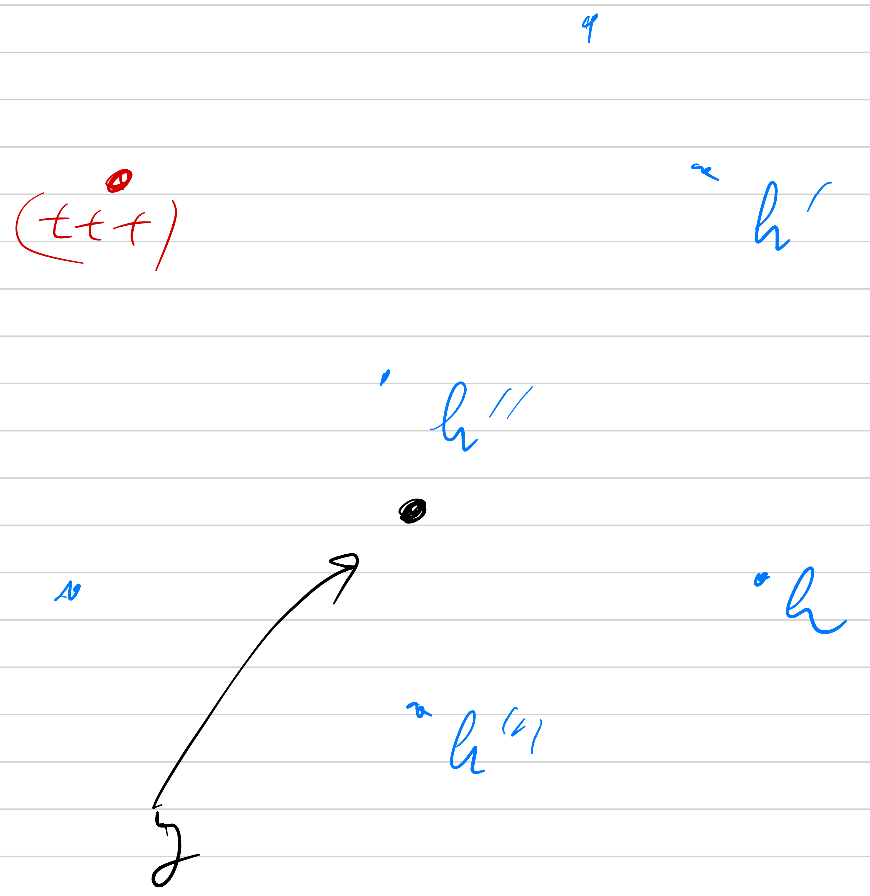
$$= \sum_{i,d}' a(i,d)\, i\, e^{-\frac{\mathcal{E}_s d}{2\sigma^2}} = \text{our upper bound in } P(\text{error}).$$

So we find that we do not need to go and find $a(i,d)$ by power series expansion. All we need is

$$\left(\frac{\partial A(I,D)}{\partial I}\right)\Bigg|_{I=1,\ D=\exp\left(-\frac{\varepsilon_s}{2\sigma^2}\right)}$$

and we are done.

Back to $\Pr(\text{taking the path } h)$: In the $\mathbb{R}^n$ picture



y is closer to h than it is to •

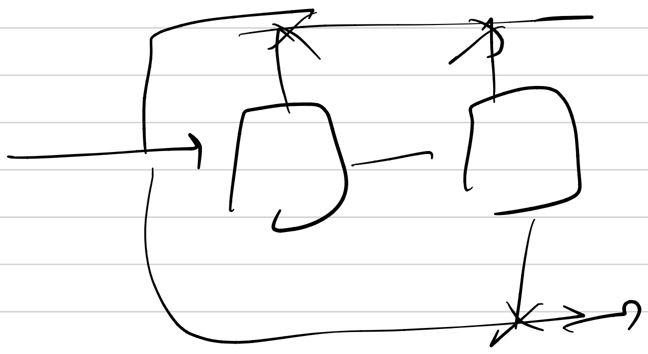
but the decoder does not follow h (it follows h'').

$\Rightarrow Q(\ )$ is only an upper bound).

Question "k_0"?

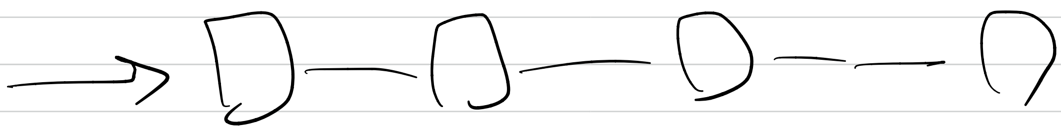
"k_0"

In the examples: at each "stage" 1 bit comes in and some number $n_0 = 2$ bits go out



the "rate" of these encoder $= 1/n_0$ ($1/2$ in the diagram)

In general



at each stage
k_0 bits is accepted
& n_0 bits are produced

Rate $= k_0/n_0$