Principles of Digital Communication

March 12, 2021

Reading for next week:

- [2.5 – 2.7].
  - ↪ 2.5 Sufficient Statistics. (conceptually diff.)
  - 2.6. Bounds on Error Probabilities (technically diff.)
  - 2.7: Summary of Chapt 2

On sufficient statistics:

Setup: $H \in \{0, \dots, m-1\}$ $\qquad P_H(i)$ //

$Y \in \mathcal{Y} \qquad P_{Y|H}(y|i)$

What we do: observe the value y of Y, we compute $P_{H|Y}(i|y) = P_H(i)\, P_{Y|H}(y|i) / P_Y(y)$.

$$P_Y(y) = \sum_i P_H(i)\, P_{Y|H}(y|i)$$

The "MAP" decision then chooses $\hat{H}(y) = \arg\max_i P_H(i)\, P_{Y|H}(i)$

A sufficient statistic T is a function $t(Y)$ of the observation such that knowing the value of T, we can compute $\left(P_H(i)\, P_{Y|H}(i) / P_Y(y) \; : i = 0 \dots m-1 \right)$

without the knowledge of Y.

Example: $H \in \{0,1\}$

$$Y = c_i + Z \quad \text{if} \quad H = i \qquad Z \sim N(0, \sigma^2 I_n)$$

then
$$P_{Y|H}(y|1) = \dots \exp\left(-\frac{1}{2\sigma^2}\|y - c_1\|^2\right).$$
$$P_{Y|H}(y|0) = \dots \exp\left(-\frac{1}{2\sigma^2}\|y - c_0\|^2\right).$$

So
$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|1)} = \exp\left(\frac{1}{\sigma^2}\langle c_1 - c_0, y\rangle - \frac{1}{2\sigma^2}\|c_1\|^2 - \frac{1}{2\sigma^2}\|c_0\|^2\right)$$

Thus to compute all I need to know about y is $\langle c_1 - c_0, y\rangle =: t(y)$.

once $t(y)$ is given, we can evaluate

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \Longrightarrow \text{evaluate} \quad \frac{P_{H|Y}(1|y)}{P_{H|Y}(0|y)}$$
$$= \frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \cdot \frac{P_H(1)}{P_H(0)}$$
$$=: \alpha$$

So we know that

$$\left[P_{H|Y}(1|y) = \frac{\alpha}{1+\alpha}, \quad P_{H|Y}(0|y) = \frac{1}{1+\alpha} \right]$$

We conclude that $t(y) = \langle c_1 - c_0, y\rangle$ is a suff. stat.

You will read about how to check if a given $t(y)$ is a sufficient statistic

Attn: just because a $t(y)$ lets us find $\hat{H}_{MAP}(y)$ does not mean that it is a suff. statistic.

Ex: $t(y) = \text{sign}\left(p_{H|Y}(0|y) - p_{H|Y}(1|y)\right)$,

lets us find $\hat{H}_{MAP}(y) = \begin{cases} 1 & \text{if } t<0 \\ 0 & \text{if } t>0 \\ * & \text{if } t=0. \end{cases}$

but not $\left(p_{H|Y}(0|y),\ p_{H|Y}(1|y)\right)$.

$\Rightarrow$ $t(y)$ is <u>not</u> a suff. stat. //

<u>Error Probability Bounds</u>:

Let us re-examine the binary case:

$H \in \{0,1\}$ $\quad p_H(0), p_H(1)$ $\quad Y \sim p_{Y|H}(y|i)$ if $H=i$.

decision mechanism $\mathcal{Y} = D_0 \cup D_1$, $\quad D_0 \cap D_1 = \phi$

$$P(\text{Err} \mid H = \underline{\underline{0}}) = \sum_{y \in D_1} p_{Y|H}(y|\underline{\underline{0}}).$$

$$P(\text{Error} \mid H=1) = \sum_{y \in D_0} P_{Y|H}(y|1).$$

$$P(\text{Error}) = P_H(0)\, P(\text{Error} \mid 0) + P_H(1)\, P(\text{Error} \mid 1).$$

$$= \sum_{y \in D_1} P_H(0) P_{Y|H}(y|0) + \sum_{y \in D_0} P_H(1) P_{Y|H}(y|1)$$

$$= \sum_{y \in \mathcal{Y}} \begin{cases} P_H(0) P_{Y|H}(y|0) & \text{if } y \in D_1 \\ P_H(1) P_{Y|H}(y|1) & \text{if } y \in D_0. \end{cases}$$

$$\geq \sum_{y \in \mathcal{Y}} \min\{P_H(0) P_{Y|H}(y|0),\ P_H(1) P_{Y|H}(y|1)\}$$

= if D_0 & D_1 are determined by the MAP rule

Conclusion: for the binary case & MAP rule,

$$P(\text{Error}) = \sum_{y \in \mathcal{Y}} \min\{P_H(0) P_{Y|H}(y|0),\ P_H(1) P_{Y|H}(y|1)\}$$

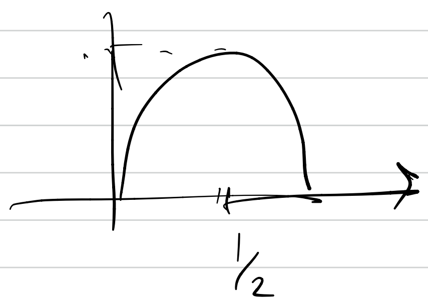
trivial observation: $\min\{a, b\} \leq \sqrt{ab}$, so

$$\boxed{P(\text{Error}) \leq \sqrt{P_H(0) P_H(1)} \cdot \sum_{y \in \mathcal{Y}} \sqrt{P_{Y|H}(y|0)\, P_{Y|H}(y|1)}.}$$

Bhattacharyya bound.

For the binary case $p_H(0) + p_H(1) = 1$

$$\Rightarrow \sqrt{p_H(0)\, p_H(1)} \le \tfrac{1}{2}.$$



So $$P(\text{Err}) \le \frac{1}{2} \sum_y \sqrt{p(y|0)\, p(y|1)} \quad (\text{discrete } Y)$$

$$P(\text{Err}) \le \frac{1}{2} \int \sqrt{f(y|0)\, f(y|1)}\, dy \quad (\text{cont. } Y).$$

Let us examine the m-ary case; MAP rule:

$$\text{Decide } \hat{H}(y) = \arg\max_i \underbrace{p_H(i)\, p_{Y|H}(y|i)}.$$

$$\text{Let } B_{j \ge i} = \{ y : p_H(j)\, p_{Y|H}(y|j) \ge p_H(i)\, p_{Y|H}(y|i) \}$$

$$\{ \hat{H}(Y) \ne i \} \subseteq \bigcup_{j \ne i} \{ Y \in B_{j \ge i} \}.$$

$$\Rightarrow \Pr(\hat{H}(Y) \ne i \mid H = i) \le \sum_{j : j \ne i} \Pr(Y \in B_{j \ge i} \mid H = i).$$

by the same techniques as above

$$\le \left(\sum_y \sqrt{p(y|i)\, p(y|j)} \right) \cdot \sqrt{p_H(j) / p_H(i)}$$

$$\Rightarrow \boxed{P(\text{Error}) \le \sum_{\substack{i,j \\ i \ne j}} \sqrt{p_H(i)\, p_H(j)} \cdot \sum_y \sqrt{p(y|i)\, p(y|j)}}$$

Union-B. bound.