Principles of Digital Communication

March 10, 2021

Reading way.

- m-ary hypothesis testing
- Q-function
- Hypothesis testing with Additive Gaussian Noise

= m-ary Hyp. Test. :

$H \in \{0, \ldots, m-1\}$

Prob. $p_H(0) \ldots p_H(m-1)$.

observation Y specified by $p_{Y|H}(y|i)$

- we need to design $\hat{H}(Y)$ so as to maximize

$$\Pr(H = \hat{H}(Y))$$

- the "best" $\hat{H}$ is the "MAP" rule

$$\hat{H}(y) = \underset{0 \le i < m}{\arg\max} \Pr(H = i \mid Y = y).$$

- Properties of the Gaussian distribution.

· "Standard" Gaussian ($\equiv$ Standard Normal)

Z : gaussian $\underline{0}$ mean variance $= \underline{1}$

$$\Pr(Z > x) \stackrel{\Delta}{=} Q(x).$$

· General scalar Gaussian

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

mean variance

$$\Rightarrow \frac{X-\mu}{\sigma} \text{ is a standard Gaussian}$$

$$\Pr(X > a) = \Pr\left(\frac{X-\mu}{\sigma} > \frac{a-\mu}{\sigma}\right)$$

$$= Q\left(\frac{a-\mu}{\sigma}\right).$$

Hypothesis testing with Gaussian noise.

· Binary, scalar.

$$H=0 \Rightarrow Y = c_0 + Z$$

$$H=1 \Rightarrow Y = c_1 + Z$$

By computing the MAP rule

$$y(c_1 - c_0) \underset{0}{\overset{1}{\gtrless}} \text{threshold}$$

& prob. error expressed in terms of the Q funct.

Binary Hypothesis test $Y \in \mathbb{R}^n$.

$H=0 \Rightarrow Y = c_0 + Z$

$H=1 \Rightarrow Y = c_1 + Z$

$c_0 \in \mathbb{R}^n, \quad c_1 \in \mathbb{R}^n$

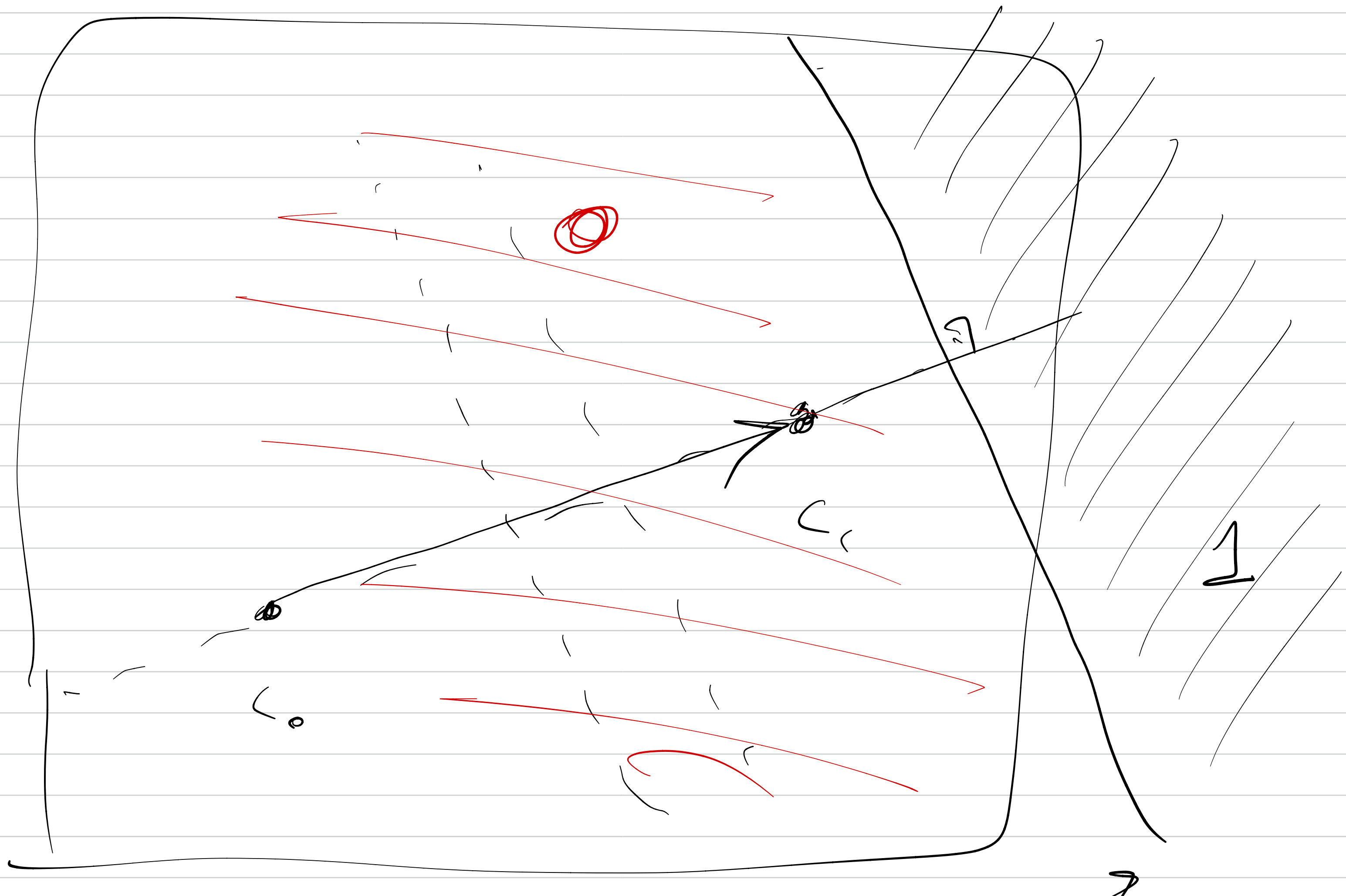
$$\underbrace{Z \sim N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \sigma^2 I\right)}$$

$$\equiv \left(\begin{array}{ll} E[Z_i] = 0 & \forall i \\ E(Z_i Z_j) = 0 & i \neq j \\ E(Z_i^2) = \sigma^2 & \end{array} \right)$$

$\equiv Z_1, \ldots, Z_n$ iid $N(0, \sigma^2)$

$\Rightarrow$ MAP rule:

$$\langle y, c_1 - c_0 \rangle \underset{0}{\overset{1}{\gtrless}} \text{threshold}$$



$\langle y, c_1 - c_0 \rangle$ = threshold

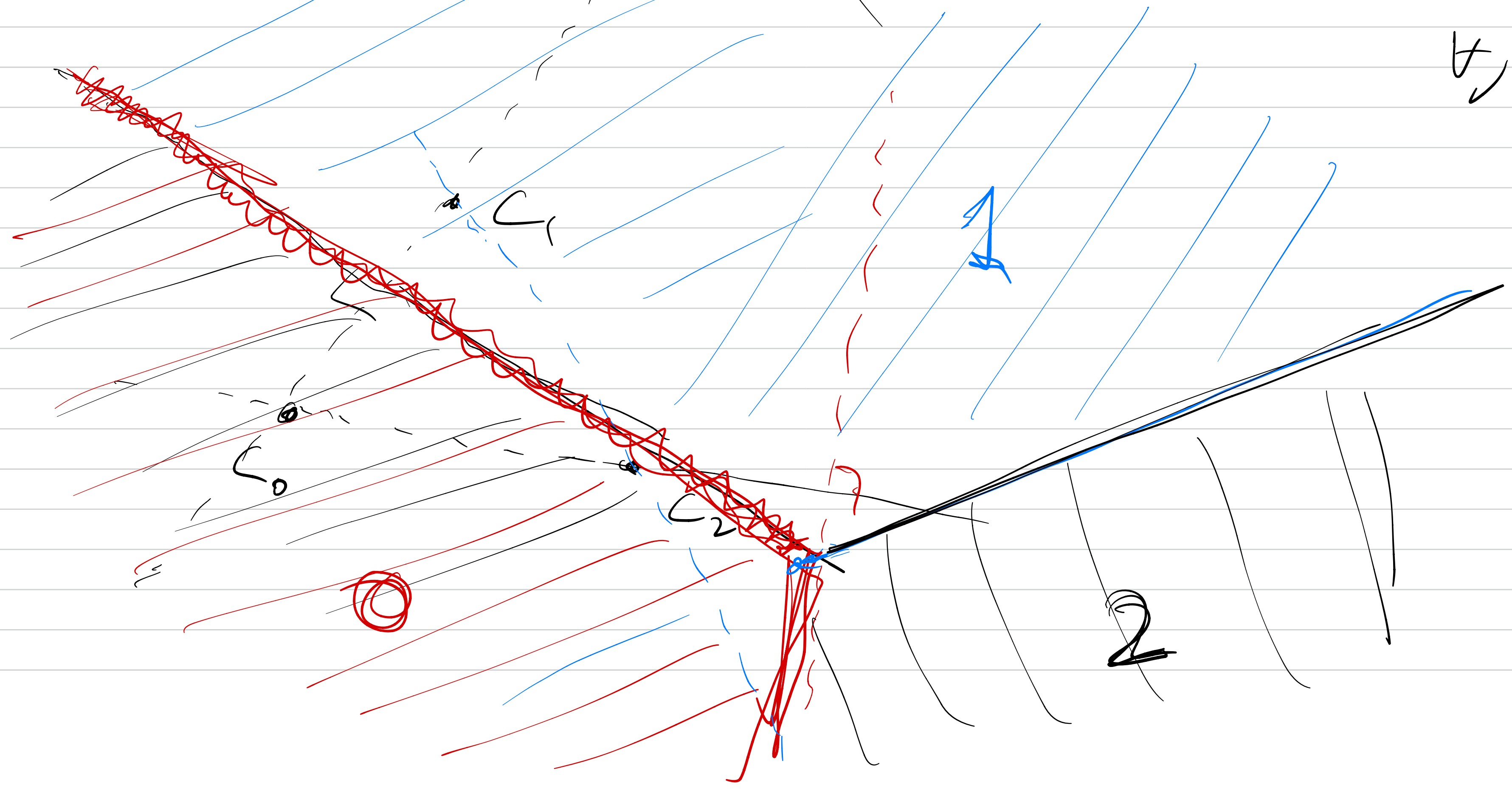
For m-ary Hyp. test, n-dim Gaussian observation

$H = i \Longrightarrow Y = c_i + Z$ $\quad c_i \in \mathbb{R}^n$

$Z \sim N(0, \sigma^2 I)$

MAP rule:

decide i iff $\Pr(H=i \mid Y=y) > \Pr(H=j \mid Y=y) \quad \forall j$



How do we find the threshold..

when is $\Pr(H=i \mid Y=y) > \Pr(H=j \mid Y=y)$

$$\equiv \underline{f_{Y|H}(y|i)}\ \underline{p_H(i)} > \underline{f_{Y|H}(y|j)}\ \underline{p_H(j)}$$

$f_{Y|H}(y|i) = ?$

↳ given $H=i$, $Y = \begin{bmatrix} c_{i1} + z_1 \\ c_{i2} + z_2 \\ \vdots \\ c_{in} + z_n \end{bmatrix}$

So given $H=i$, $Y_1 \ldots Y_n$ are <u>independent</u>,

$Y_k \sim N(c_{ik}, \sigma^2)$

$$f_{Y_k|H}(y_k|i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_k - c_{ik})^2\right).$$

$$f_{Y|H}(y|i) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-1}{2\sigma^2} \underbrace{\sum_{k=1}^{n} (y_k - c_{ik})^2}_{\text{“} \|y - c_i\|^2 \text{”}}\right)$$

So "i is better than j"

$$\equiv \quad p_H(i) \cancel{\frac{1}{(2\pi\sigma^2)^{n/2}}} \exp\left(-\frac{1}{2\sigma^2}\|y-c_i\|^2\right)$$

$$> \quad p_H(j) \cancel{\frac{1}{(2\pi\sigma^2)^{n/2}}} \exp\left(-\frac{1}{2\sigma^2}\|y-c_j\|^2\right)$$

$$\equiv \quad \exp\left(+\frac{1}{2\sigma^2}\|y-c_j\|^2 - \frac{1}{2\sigma^2}\|y-c_i\|^2\right) > \frac{p_H(j)}{p_H(i)}$$

$$\equiv \quad \frac{1}{2\sigma^2}\left[\|y-c_j\|^2 - \|y-c_i\|^2\right] > \ln \frac{p_H(j)}{p_H(i)}$$

Recall: $\|a-b\|^2 = \|a\|^2 - 2\langle a,b\rangle + \|b\|^2$

$$\equiv \quad \frac{1}{2\sigma^2}\left[2\langle y, c_i - c_j\rangle + \|c_j\|^2 - \|c_i\|^2\right] > \ln \frac{p_H(j)}{p_H(i)}$$

$$\equiv \quad \langle y, c_i - c_j\rangle > \sigma^2 \ln \frac{p_H(j)}{p_H(i)} + \frac{1}{2}\left[\|c_i\|^2 - \|c_j\|^2\right]$$

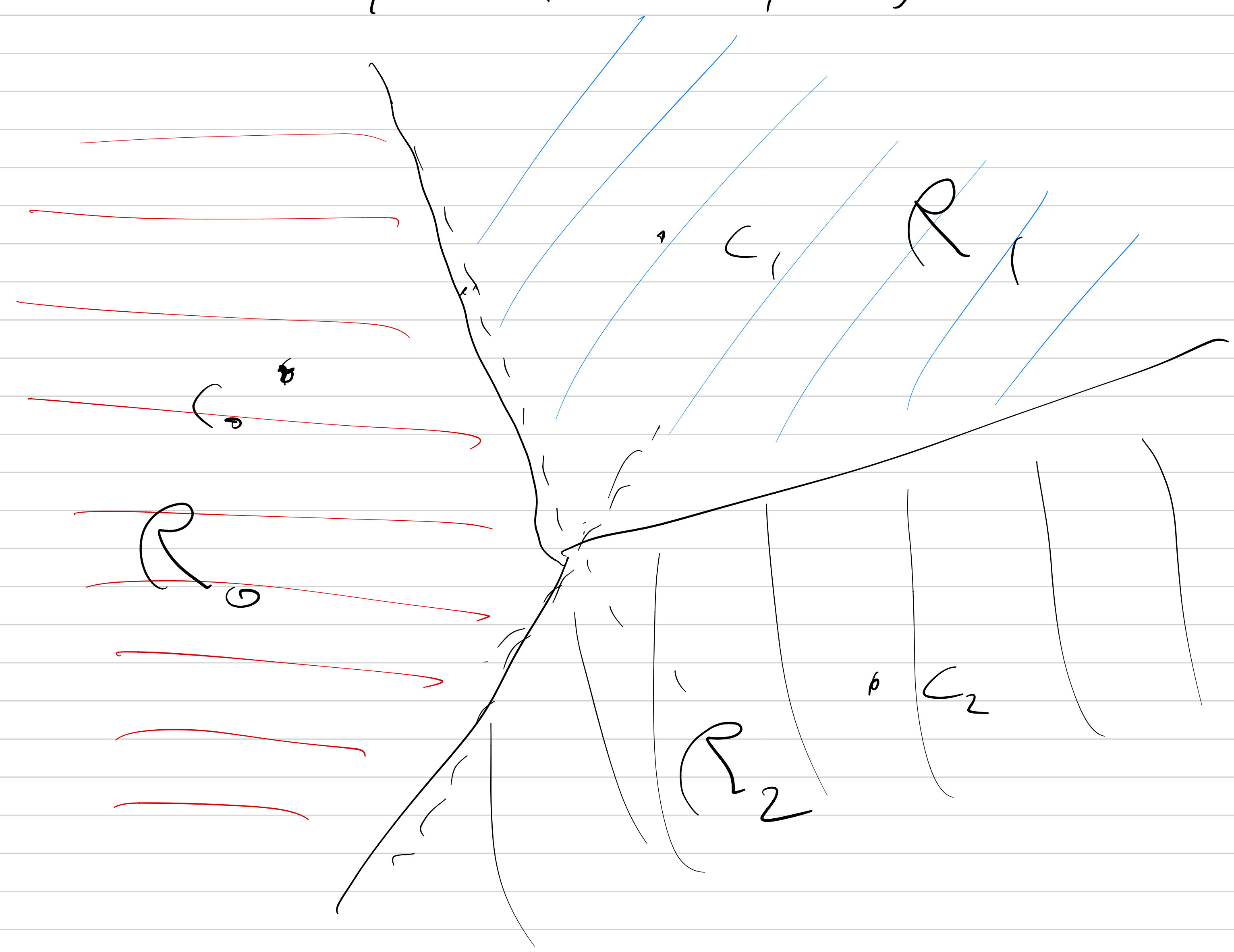
Question: can the point c_i fall into the decision region of j?

Answer: yes it is possible, but (not when the hypotheses are a-priori equally likely.)

when $P_H(i) = \frac{1}{m}$ for $i \in \{0, \dots, m-1\}$,

then "c_i is better than j"

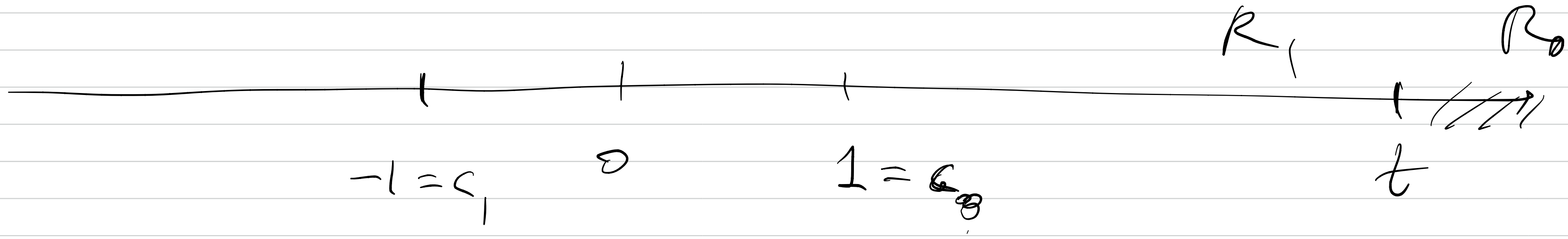
$$\equiv \|y - c_i\| < \|y - c_j\|.$$



To give an example when $c_i \in R_j$ $j \neq i$

let $n = 1$ (all scalars)

$c_0 = 1$, $c_1 = -1$ $\sigma^2 = 1$.



the boundary that separates R_1 & R_0 is
$= t$

$$p_H(0)\, e^{-\frac{1}{2}(t-1)^2} = p_H(1)\, e^{-\frac{1}{2}(t+1)^2}$$

$$= \frac{p_H(0)}{p_H(1)} = \exp\left(\frac{1}{2}(t-1)^2 - \frac{1}{2}(t+1)^2\right)$$

$$= \exp(-2t)$$

$$t = -\frac{1}{2} \ln \frac{p_H(0)}{p_H(1)}$$

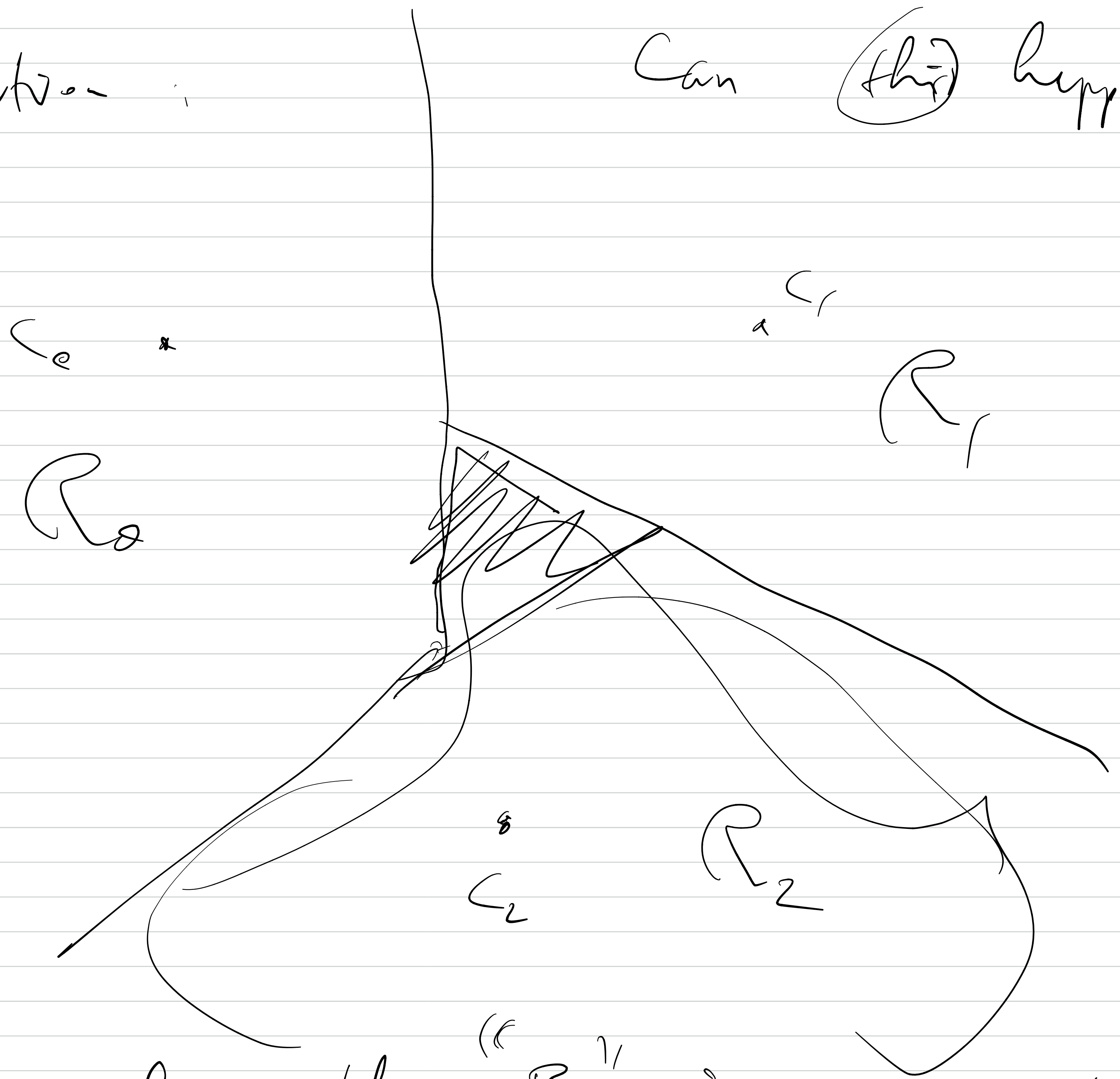
t can take on any value in $\mathbb{R}$ if we choose $p_H(0)$, $p_H(1)$ appropriately.

e.g. to make $t = 10$ we need to choose

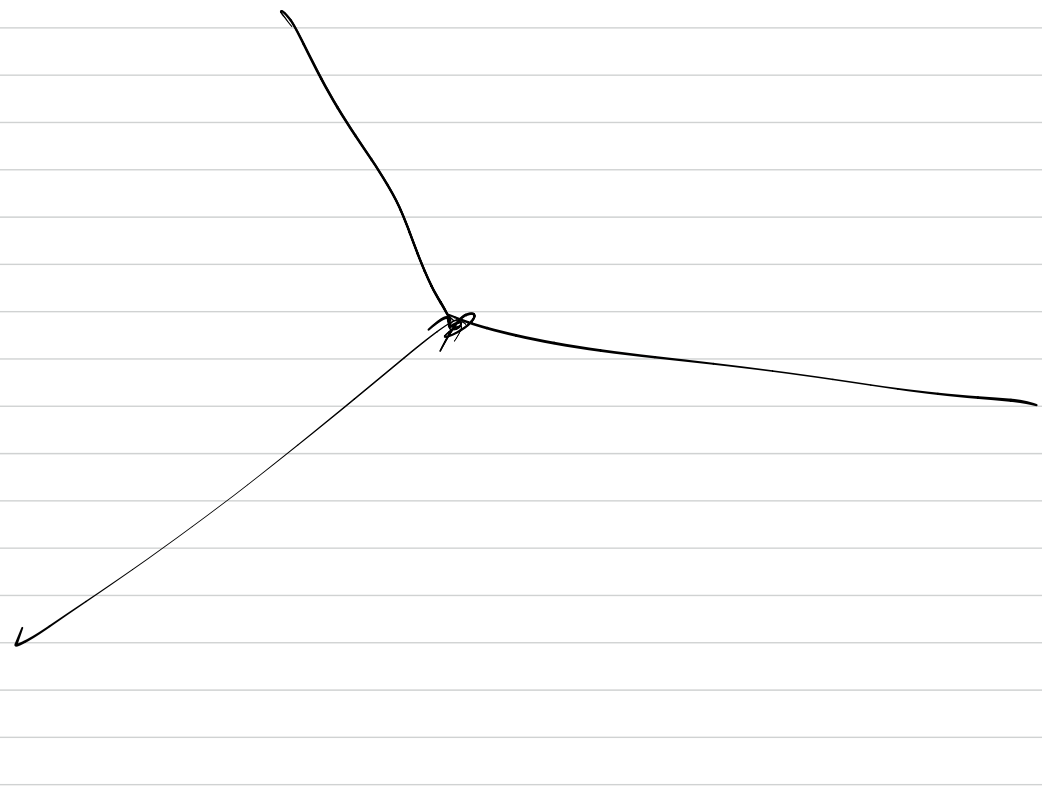
$$\frac{p_H(1)}{p_H(0)} = e^{20} \implies p_H(1) = \frac{e^{20}}{e^{20}+1}$$

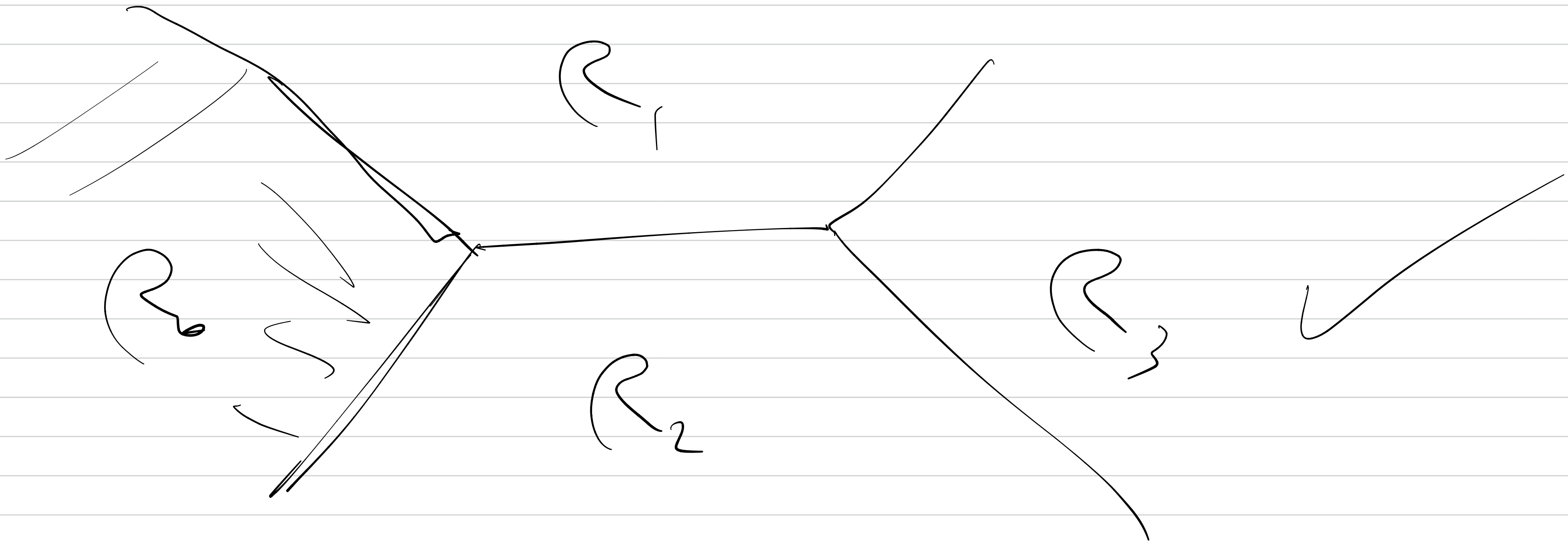
$$p_H(0) = \frac{1}{e^{20}+1}$$

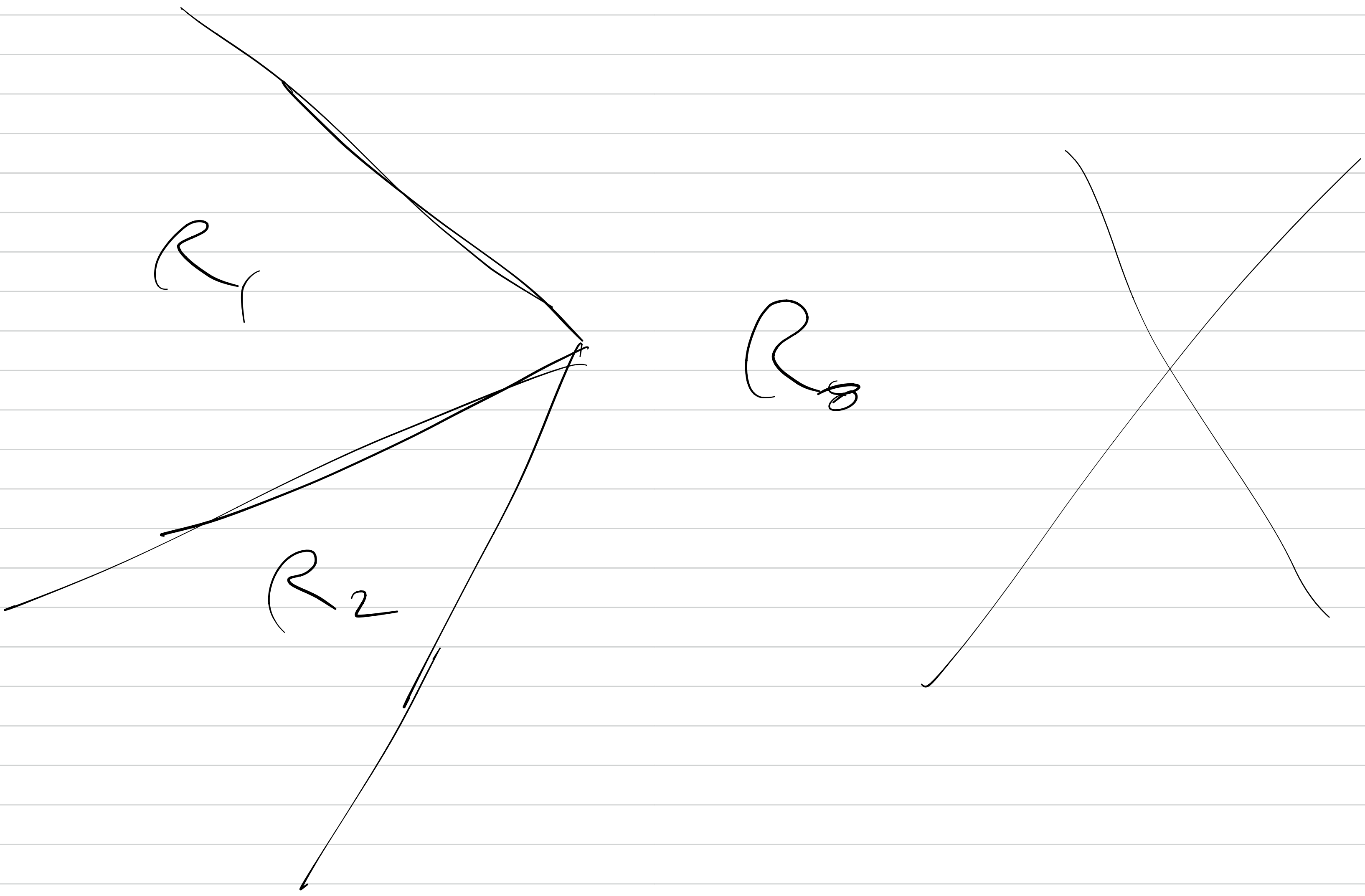
Question: Can (this) happen?

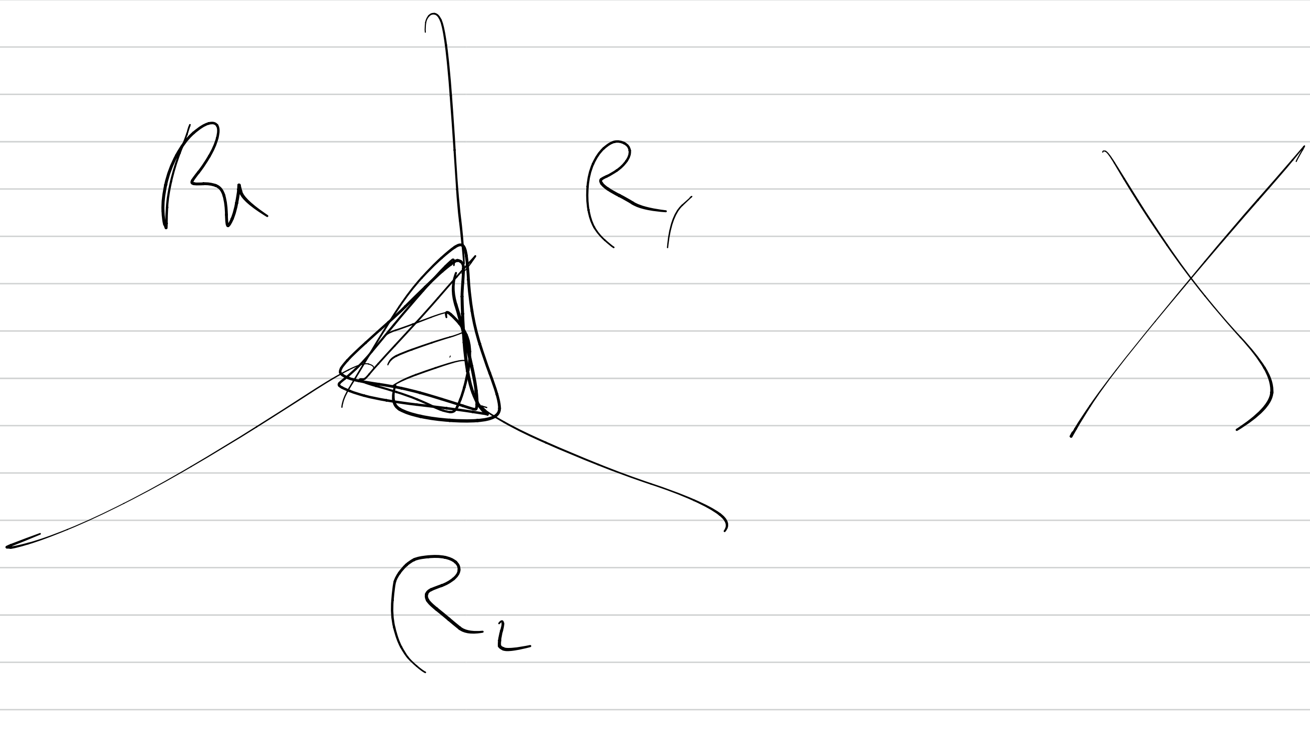


We saw that the "R_i" is an intersection of Half-spaces.




R1
R6
R2
R3


R1
R8
R2


R4
R1
R2

Modify our Hypothesis test problem:

$H \in \{0, 1\}$, but we allow

$\hat{H} \in \{0, 1, ?\}$

$Pr(\hat{H} = 1 \mid H = 0)$, $Pr(\hat{H} = ? \mid H = 0)$

$Pr(\hat{H} = 0 \mid H = 1)$, $Pr(\hat{H} = ? \mid H = 1)$.