# Principles of Digital Communication

March 3, 2021

## Review of Binary Hypothesis testing:

Setup: two hypotheses $H \in \{0, 1\}$,

observation $Y \sim P_{Y|H}(y|i)$ $y \in \mathcal{Y}$, $i \in \{0,1\}$

- $\Pr(H=0) = P_H(0)$, $\Pr(H=1) = P_H(1)$

Need to design a decision rule:

$\hat{H} : \mathcal{Y} \to \{0,1\}$, want to maximize

$\Pr(\hat{H}(Y) = H)$.

We discover (by reading Sect 2.?.?) that

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = \eta .$$

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Recall that

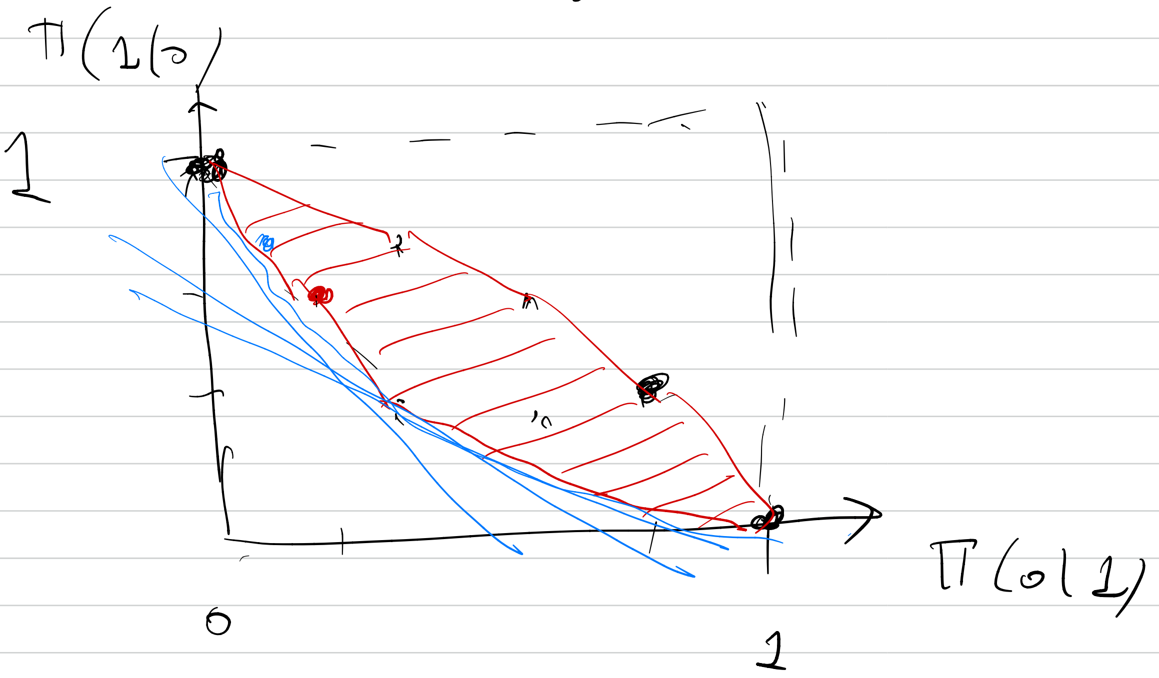
$$\Pr(\hat{H}(Y) \neq H) = \Pr(H=0)\,\Pr(\hat{H}(Y)=1 \mid H=0) + \Pr(H=1)\,\underline{\Pr(\hat{H}(Y)=0 \mid H=1)}.$$

$$\left(\begin{aligned} \pi(1|0) &= \Pr(\hat{H}(Y)=1 \mid H=0) \\ \pi(0|1) &= \Pr(\hat{H}(Y)=0 \mid H=1) \end{aligned}\right)$$

Let $\phi(y) = \{0,1\}$ denote the decision when the observation is $y$. Depending on $\phi$ we will obtain the values $\pi(0|1)$ and $\pi(1|0)$,

$$\pi(0|1) = \sum_y P_{Y|H}(y|\underline{1}) \left[1-\phi(y)\right]$$

$$\pi(1|0) = \sum_y P_{Y|H}(y|0)\, \phi(y)$$




if $y \in \{\underline{a}, \underline{b}, \underline{c}\}$

By going to "randomized decision rules" we are able to "connect the dots", to obtain a <u>convex</u> region for $(\pi(0|1), \pi(1|0))$ pairs.

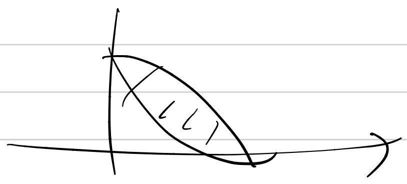
It is easy to see that this randomization corresponds to the following:

- Instead of $\phi : \mathcal{Y} \to \{0,1\}$, we allow

$$\phi : \mathcal{Y} \to [0,1] \quad \equiv$$

if $\phi(y) = 0.7$, we decide 1 with prob. $(0.7)$
" " 0 " " $[1-(0.7)]$

To characterize the lower boundary of the



we need to consider the

minimizer of $\lambda \pi(0|1) + (1-\lambda)\pi(1|0)$

for each $0 \le \lambda \le 1$. now:

$$\underbrace{\lambda \pi(0|1) + (1-\lambda)\pi(1|0)}$$

$$= \sum_y{}' \left( \lambda p_{Y|H}(y|1)\,[1-\phi(y)] + (1-\lambda)\, p_{Y|H}(y|0)\,\phi(y) \right)$$

$$\geq \sum_y{}' \min\{\lambda p_{Y|H}(y|1),\ (1-\lambda) p_{Y|H}(y|0)\}$$

$$= \text{if } \phi(y) = \begin{cases} 0 & \lambda p_{Y|H}(y|1) < (1-\lambda) p(y|0) \\ 1 & \lambda p(y|1) > (1-\lambda) p(y|1) \\ * & = \end{cases}$$

$$\equiv \underbrace{\frac{p_{Y|H}(y|1)}{p_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1-\lambda}{\lambda}}$$

a <u>likelihood ratio test.</u>

(Neyman–Pierson theorem): likelihood ratio tests achive the lower boundary of

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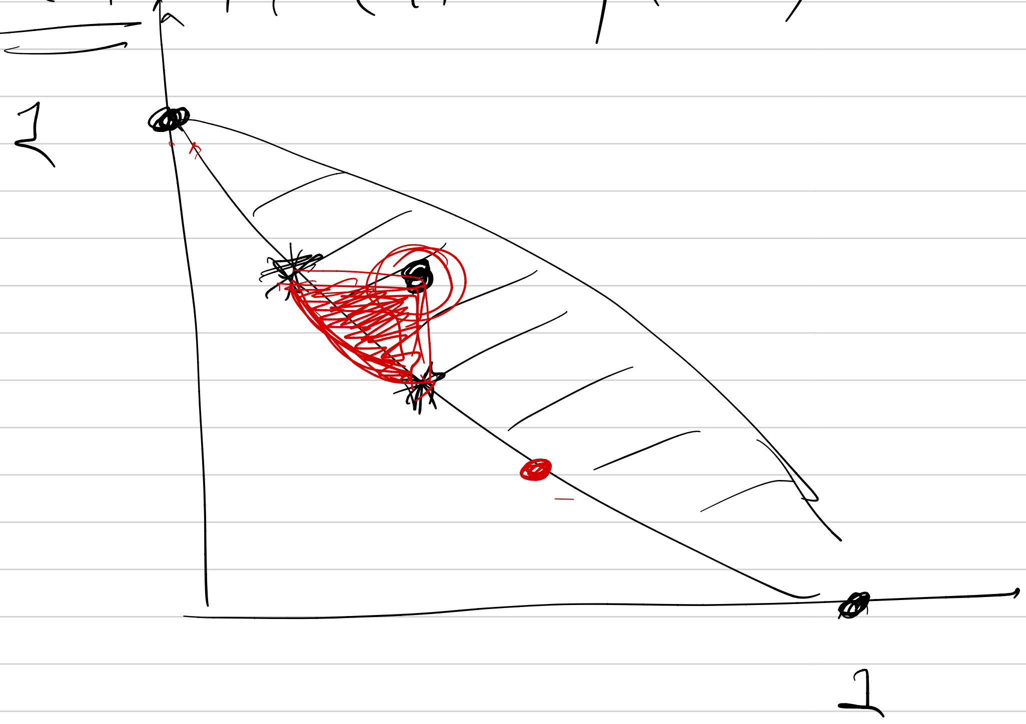
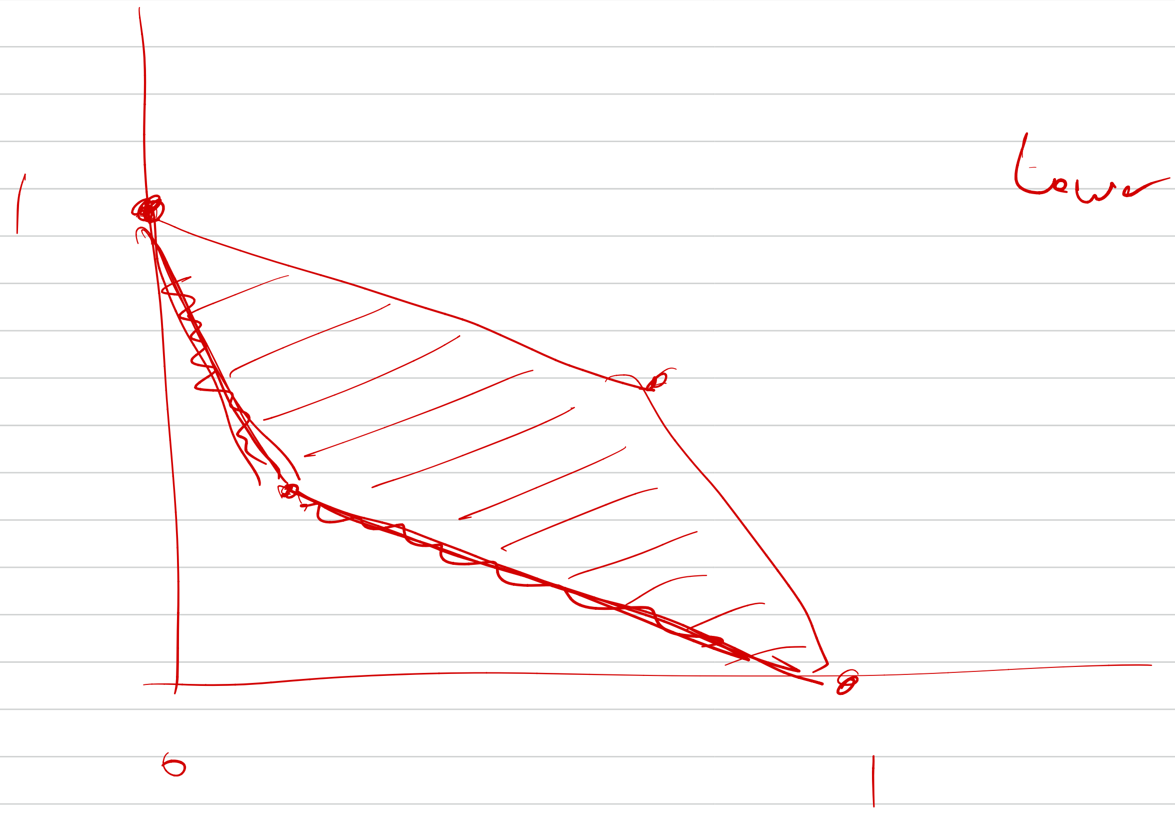
Why lower boundary?

$\Pi(1|0) = \Pr(\hat{H}=1 | H=0)$




$\Pi(0|1) = \Pr(\hat{H}=0 | H=1)$

Lower boundary smooth?

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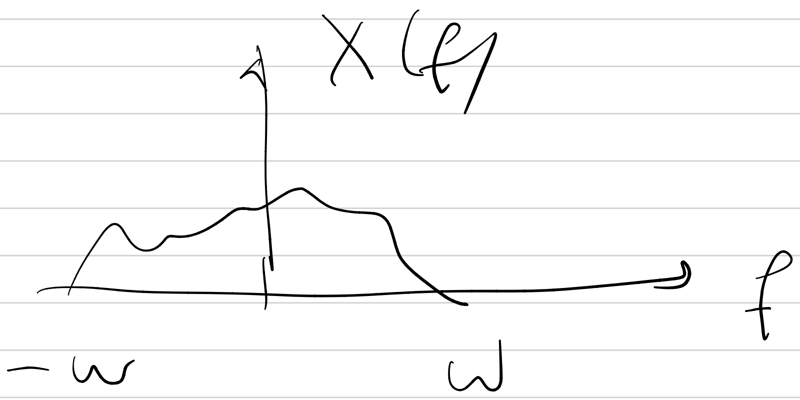
Representation of signals, bandwidth, etc.

$(x_1 \ldots x_n) \longrightarrow x(t)$




Such $x(t)$ can be constructed from its "samples"

$$x_i = x\left(\frac{i}{2W}\right) \qquad i = 0, 1, 2, \pm 1, \pm 2, \ldots$$

So: $x(t)$ bandlimited to $[-W, W]$

$\{ x(t) \quad 0 \le t \le T \}$

$\approx \underbrace{2WT}_{=n}$ samples

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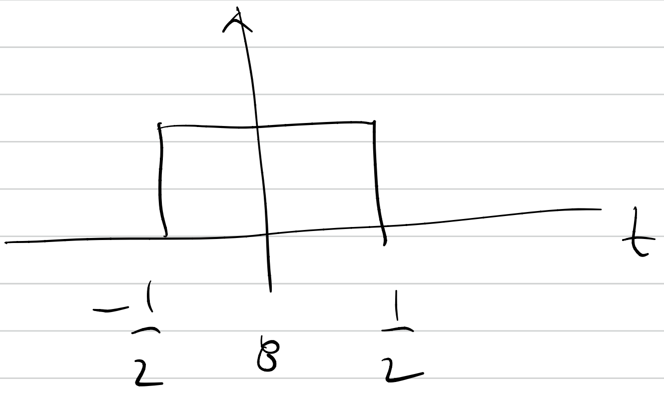
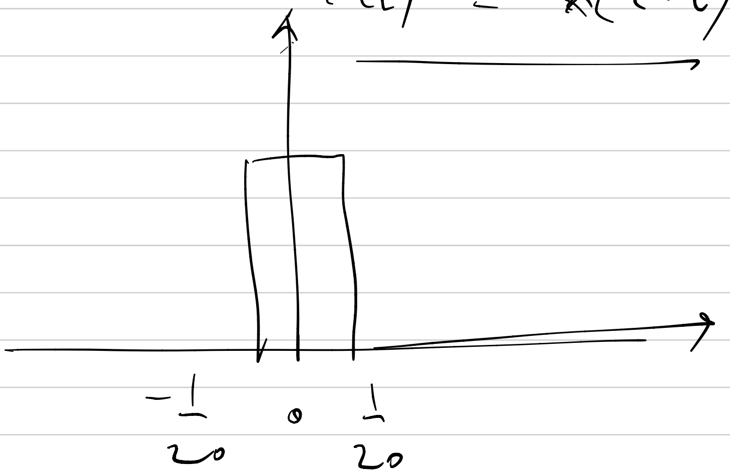
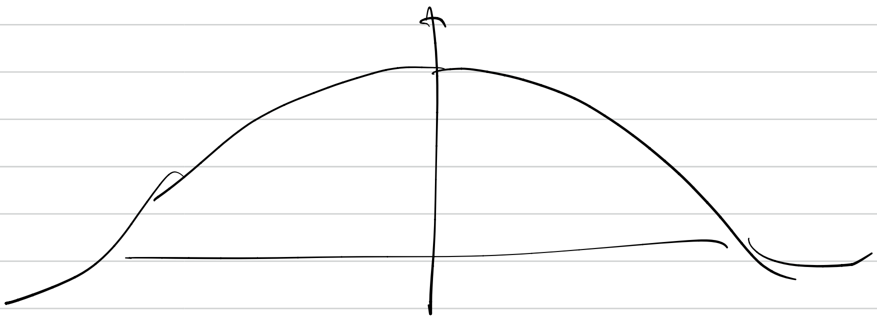
$x(t)$      $X(f)$ : Fourier transform of $x(t)$




$\tilde{x}(t) = x(10t)$      $\tilde{X}(f) = \frac{1}{10} \hat{X}(f/10)$



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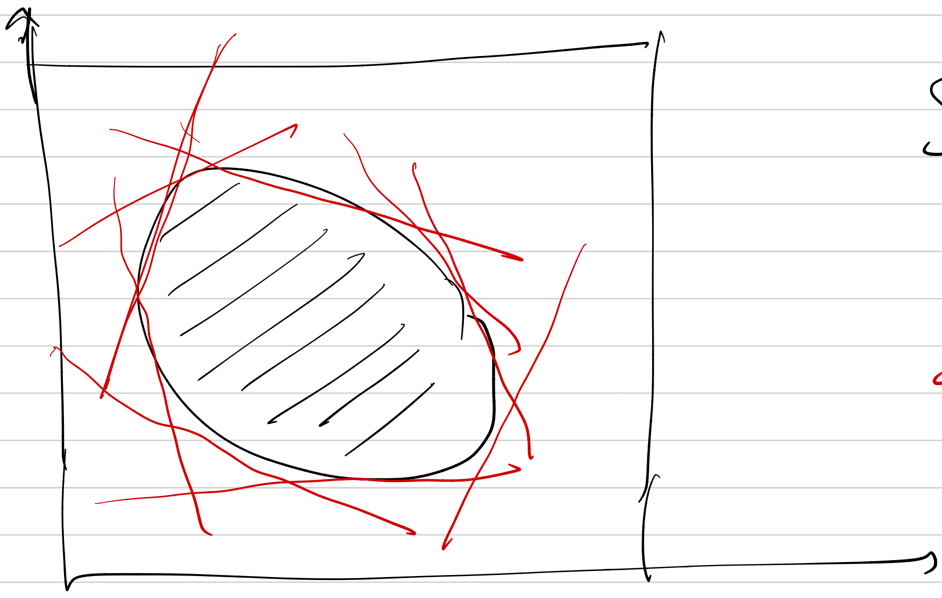
Characterizing 2-D convex sets.

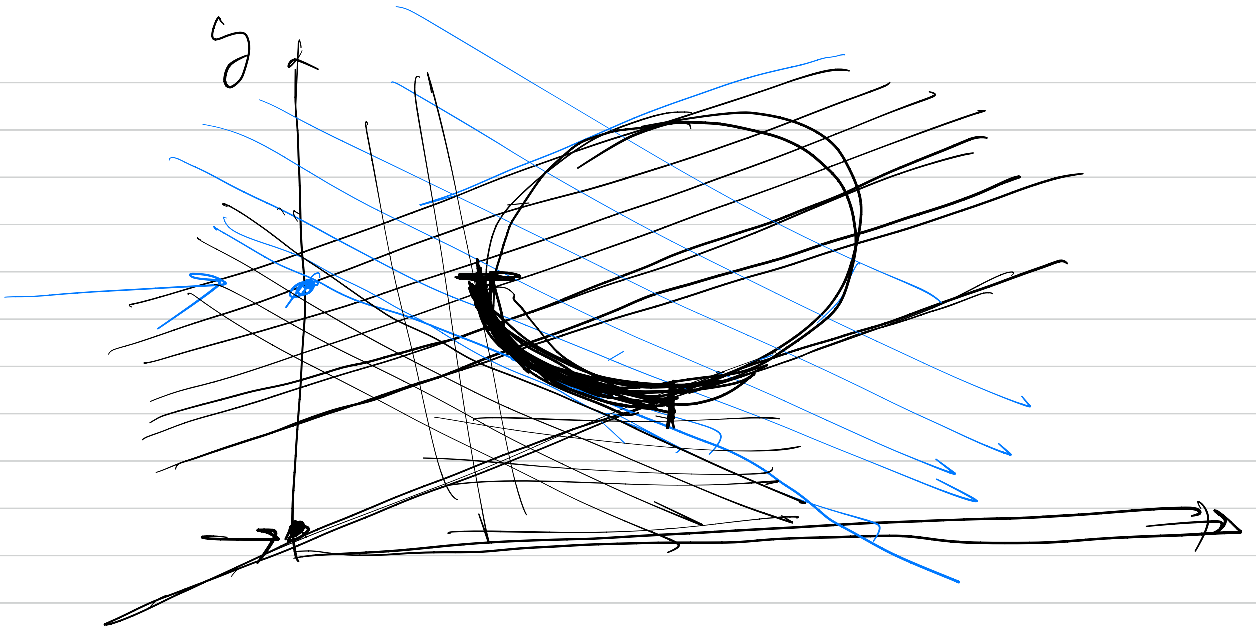


$S$ : convex set in 2D.

$S$ can be described by giving all its tangents.

To do this: fix a slope $\eta$ and consider

$$f(\eta) = \min_{(x,y) \in S} (y - \eta x)$$




$y - \eta x$

$$\min_{(x,y)\in S} (y - \eta x)$$

Thus, if we consider

$$F(\eta) = \min_{(x,y)\in S} y - \eta x, \qquad G(\eta) = \max_{(x,y)\in S} (y - \eta x)$$

$$\equiv \quad J(\lambda,\mu) = \min_{(x,y)\in S} (\lambda y + \mu x).$$

if we are interested in the segment it is sufficient to take $\lambda, \mu \geq 0$.

$\equiv \quad J(\lambda, 1-\lambda)$ is sufficient,

$0 \leq \lambda \leq 1$