ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 31 Principles of Digital Communications
Final exam June 19, 2025

4 problems, 44 points, 180 minutes.
2 sheets (4 pages) of notes allowed.

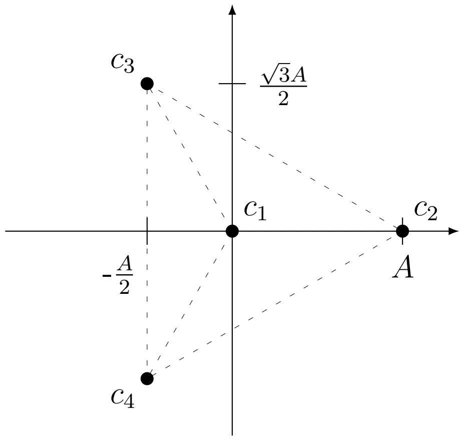
Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. (8 points)
Answer the following questions with proper justification.

(a) (2 pts) Assuming equally likely messages, is it true that the energy of the constellation $\{c_1, c_2, c_3, c_4\}$ given in the figure below cannot be reduced by isometric transformations? If not, give a minimum energy version.



(b) (2 pts) Re-do part (a) with $P_H(1) = 1/2$, $P_H(2) = P_H(3) = P_H(4) = 1/6$ instead.

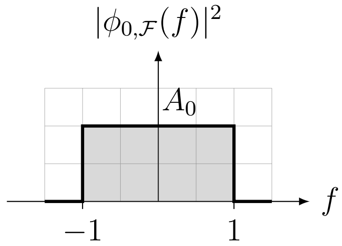
(c) (2 pts) In a binary hypothesis problem with observation Y, suppose $f_{Y|H}(y|0) = \exp(-y)\,\mathbb{1}\{y > 0\}$, and $f_{Y|H}(y|1) = (1/\alpha)\exp(-y/\alpha)\mathbb{1}\{y > 0\}$, with $\alpha > 1$. Is it true that if T is a sufficient statistic, then T determines Y?

(d) (2 pts) Continuing with (c), is it true that no matter what the a priori probabilities of the two hypotheses are, the error probability of the MAP decision rule is upper bounded by $2\sqrt{\alpha}/(1+\alpha)$?
Hint: Bhattacharyya bound.

Problem 2. (9 points)

(a) (2 pts) Suppose $\phi_0(t)$ is a waveform with Fourier transform $\phi_{0,\mathcal{F}}(f)$ satisfying

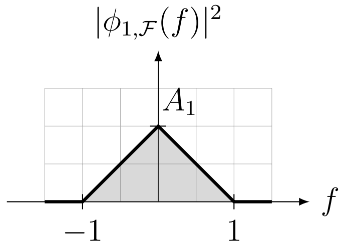
$$|\phi_{0,\mathcal{F}}(f)|^2 = A_0\, \mathbb{1}\{|f| < 1\} \qquad \equiv$$



What is A_0 if ϕ_0 is a unit energy waveform (i.e., $\int_{-\infty}^{\infty} |\phi_0(t)|^2\, \mathrm{d}t = 1$)? Find the smallest T such that $\{\phi_0(t - jT) : j \in \mathbb{Z}\}$ is an orthogonal set of waveforms, and call it T_0.

(b) (2 pts) Repeat (a) for $\phi_1(t)$ whose Fourier transform's square magnitude is given by

$$|\phi_{1,\mathcal{F}}(f)|^2 = A_1(1 - |f|)\, \mathbb{1}\{|f| < 1\} \qquad \equiv$$



That is, find A_1 such that ϕ_1 has unit energy and T_1 such that it is the smallest T that makes $\{\phi_1(t - jT) : j \in \mathbb{Z}\}$ an orthogonal set.

(c) (3 pts) Suppose we have two systems using the same codebook set $\mathcal{C} = \{c_1, \dots, c_m\}$. One system uses as orthonormal basis the shifts of ϕ_0 by T_0 obtained in (a), and the other uses the shifts of ϕ_1 by T_1 obtained in (b), to generate the waveforms $w_1, \dots, w_m$, which are then transmitted over an AWGN channel with noise intensity $\frac{N_0}{2}$. Compare the two systems in terms of their

(i) energy per bit.

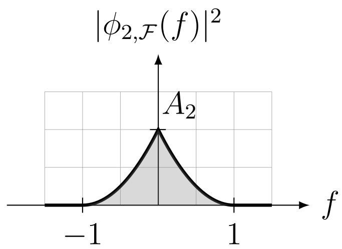
(ii) error probability.

(iii) bandwidth occupied (i.e., the smallest B such that all the w_i's have Fourier transform zero outside of $[-B, B]$).

(iv) rate (i.e., number of bits transmitted per second).

(d) (2 pts) Repeat (a) for $\phi_2(t)$ whose Fourier transform's square magnitude is given by

$$|\phi_{2,\mathcal{F}}(f)|^2 = A_2(1 - |f|)^2\, \mathbb{1}\{|f| < 1\} \qquad \equiv$$

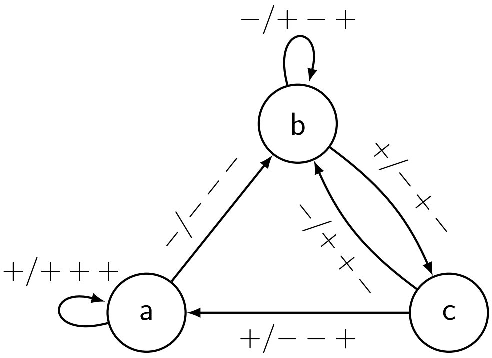


That is, find A_2 such that ϕ_2 has unit energy and T_2 such that it is the smallest T that makes $\{\phi_2(t - jT) : j \in \mathbb{Z}\}$ an orthogonal set (if it exists).

Hint: Look for band-edge symmetry in $|\phi_{2,\mathcal{F}}(f)|^2$ or argue in time-domain.

Problem 3. (11 points)
Consider the following 3-state machine that transforms a binary data sequence $b_1, b_2, \ldots$ (with $b_i \in \{+, -\}$) to a coded sequence $x_1, x_2, x_3, x_4, x_5, x_6, \ldots$ as follows, starting from state a.

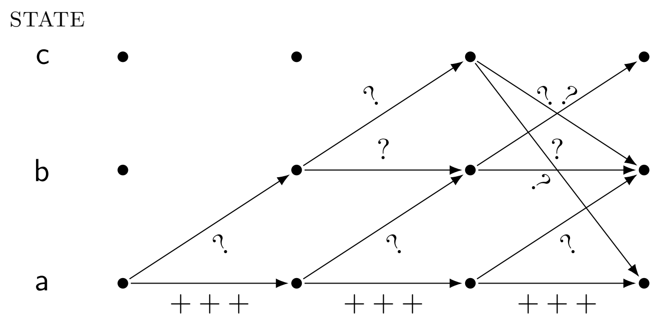
current state	data bit	next state	output
a	$+$	a	$+++$
a	$-$	b	$---$
b	$+$	c	$-+-$
b	$-$	b	$+-+$
c	$+$	a	$--+$
c	$-$	b	$++-$



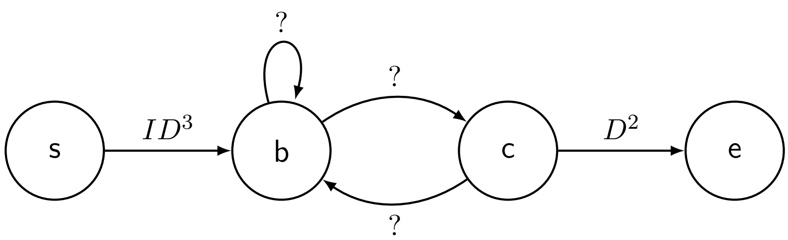
(a) (2 pts) Show that there is a "sentinel sequence of bits" $t_1, t_2, \ldots, t_L$ such that no matter the current state of the machine, after the input $t_1, \ldots, t_L$ the machine will be found in state a. Among all such sequences, find the shortest one.

From now on, assume that the shortest such $t_1, \ldots, t_L$ (as found in (a)) will be appended to the data sequence before transmission, i.e., to send $b_1, \ldots, b_k$, we encode $b_1, \ldots, b_k, t_1, \ldots, t_L$.

(b) (2 pts) Complete the following trellis diagram by filling in the ?'s:



(c) (2 pts) Fill the ?'s in the following detour flow graph formed by splitting state a:



(d) (3 pts) Find the transfer function $T(I, D)$ from s to e.

(e) (2 pts) Consider the communication system described as follows: the bit sequence $b_1, \ldots,$ together with the sentinel bits from (a), are encoded using this 3-state machine to obtain $x_1, x_2, x_3, \ldots$. The receiver observes $Y_1, Y_2, Y_3, \ldots$ where $Y_i = \sqrt{\mathcal{E}_s} x_i + Z_i$, with $Z_i \sim \mathcal{N}(0, \sigma^2)$ i.i.d. A MAP decoder converts the received sequence to decoded bits $\hat{b}_1, \ldots$.

Find an upper bound to the bit error probability of this communication system, when the "all +" data sequence is transmitted.

PROBLEM 4. (16 points)

Hint: For this problem, you may find the following background on Fourier transforms useful:

(i) By Parseval's relation, $\langle a, b \rangle = \langle A_{\mathcal{F}}, B_{\mathcal{F}} \rangle$, where $A_{\mathcal{F}}(f)$ and $B_{\mathcal{F}}(f)$ are the Fourier transforms of $a(t)$ and $b(t)$.

(ii) $a(t)\exp(\mathrm{j}2\pi\Delta t)$ has Fourier transform $A_{\mathcal{F}}(f-\Delta)$.

(iii) $\phi(t) = \operatorname{sinc}(t)$ has Fourier transform $\operatorname{rect}(f) := \mathbb{1}\{|f| < 1/2\}$.

Consider a transmitter that operates with $f_0 > 0$ and $\phi(t) = \operatorname{sinc}(t)$ as follows:

$$\big[i\big] \rightarrow \big[c_i \in \mathbb{C}\big] \rightarrow \big[w_{i,E}(t) = c_i\phi(t)\big] \rightarrow \big[w_i(t) = \sqrt{2}\Re\{w_{i,E}(t)\exp(\mathrm{j}2\pi f_0 t)\}\big].$$

The waveform $w_i(t)$ is then input to an AWGN channel with noise intensity $N_0/2$; the channel's output is $R(t)$.

(a) (3 pts) Consider the following receiver:

$$\big[R(t)\big] \rightarrow \big[R_E(t) = R(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_0 t)\big] \rightarrow \big[Y = \langle R_E, \phi\rangle\big] \rightarrow \big[\hat{\imath} = \arg\min_i |Y - c_i|\big].$$

What are the conditions on f_0 and the probability distribution of the message i so that this receiver is optimal?

For the rest of the problem suppose that there is an error made in the receiver design: while forming $R_E(t)$, an oscillator frequency $f_1 > 0$ instead of f_0 is used, i.e., $R_E(t) = R(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_1 t)$. The rest of the receiver is identical to the design in (a).

(b) (2 pts) Noting that for any complex number z, $2\Re\{z\} = z + z^*$, find f_2 and f_3 such that

$$R_E(t) = c_i\phi(t)\exp(\mathrm{j}2\pi f_2 t) + c_i^*\phi(t)\exp(\mathrm{j}2\pi f_3 t) + N(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_1 t).$$

Suppose we write $Y = d_i + Z$, where $d_i \in \mathbb{C}$ represents the part of Y which depends on the message and Z represents noise.

(c) (3 pts) Find α and β so that

$$d_i = \alpha c_i + \beta c_i^*.$$

(d) (2 pts) Find necessary and sufficient conditions on f_0 and f_1 so that $\beta = 0$. Find necessary and sufficient conditions on f_0 and f_1 so that $\alpha \neq 0$.

(e) (2 pts) Find waveforms $a(t)$ and $b(t)$ so that $\Re\{Z\} = \langle N, a\rangle$ and $\Im\{Z\} = \langle N, b\rangle$. Under what conditions (on f_0 and f_1) will $\Re\{Z\}$ and $\Im\{Z\}$ be independent?

(f) (2 pts) Suppose $f_0 = 3$, $f_1 = 3.2$, and the codewords are $c_0 = \sqrt{\mathcal{E}}$, $c_1 = \mathrm{j}c_0$, $c_2 = -c_0$, $c_3 = -c_1$. What is the error probability?

(g) (2 pts) Suppose $f_0 = 3$, $f_1 = 4$, and the codewords are the same as in part (f). What is the error probability?