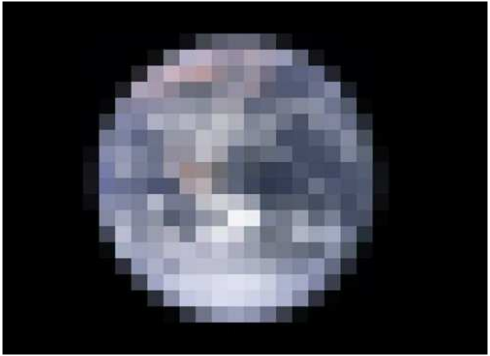
COM-202: Signal Processing

Chapter 7: Continuous-time Signal Processing

Overview:

- The analogue world
- Continuos-time Fourier transform
- Continuous-time LTI theory
- Band-limitied signals

Two models of the world



Analog/continuous versus discrete/digital

Two models of the world

digital worldview:

- arithmetic
- combinatorics
- computer science
- DSP

analog worldview:

- calculus
- distributions
- system theory
- electronics

Two models of the world, two languages

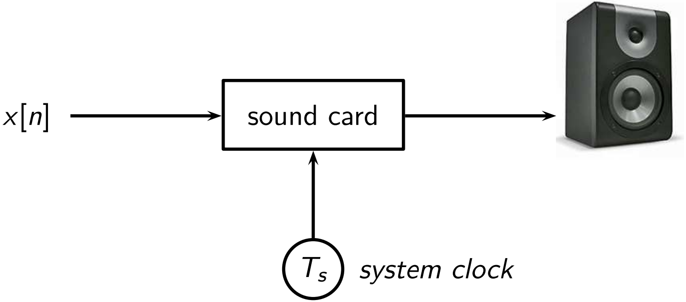
digital worldview:

- countable integer index n
- finite-energy sequences $\mathbf{x} \in \ell_2(\mathbb{Z})$
- frequency $\omega \in [-\pi, \pi]$
- DTFT: $\ell_2(\mathbb{Z}) \mapsto L_2([-\pi, \pi])$

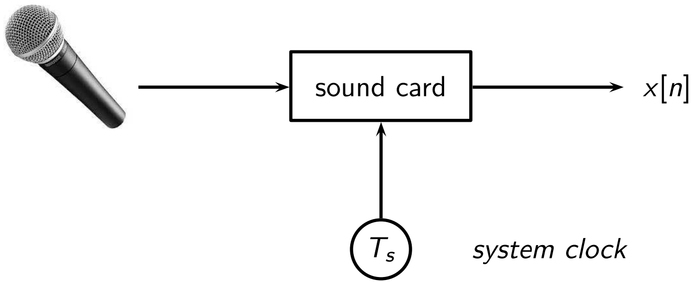
analog worldview:

- real-valued time t (sec)
- finite-energy functions $\mathbf{x} \in L_2(\mathbb{R})$
- frequency $f \in \mathbb{R}$ (Hz)
- CTFT: $L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$

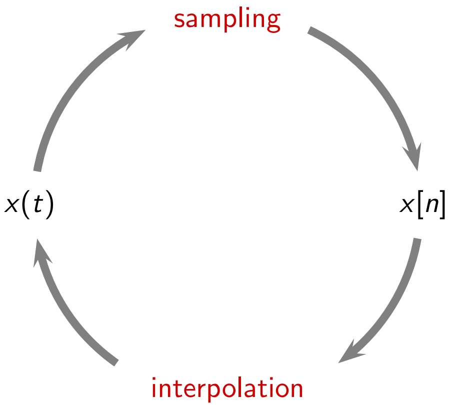
Translating between languages: interpolation



Translating between languages: sampling



Bridging the gap

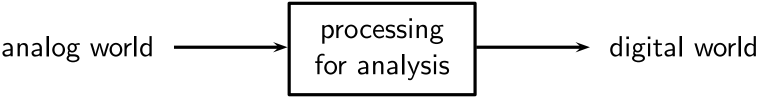


Today, processing is as digital as possible

- analog to digital
- digital to analog
- analog to digital to analog

Digital processing of signals from the analog world

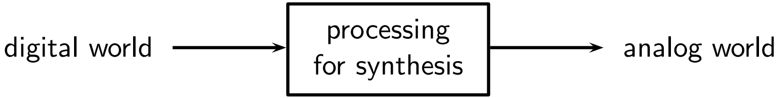
- input is continuous-time: $x(t)$
- output is discrete-time: $y[n]$
- processing is on sequences: $x[n], y[n]$



examples: storage and compression (MP3, JPG), control systems, monitoring

Digital processing of signals to the analog world

- input is discrete-time: $x[n]$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$



examples: telecommunication front-ends, music synthesizers, biomedical

Digital processing of signals from/to the analog world

- input is continuous-time: $x(t)$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$

examples: end-to-end telecommunication, sound effects, digital photography

The continuous-time Fourier transform

About continuous time

- time: real variable t
- signal $x : \mathbb{R} \mapsto \mathbb{C}$: complex functions of a real variable
- finite energy: $\mathbf{x} \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

- energy: $||\mathbf{x}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle$

About continuous time: convolution

- convolution is an operation between two signals that produces a third signal:

$$\mathbf{y} = \mathbf{x} * \mathbf{h}$$

- convolution for the individual output sample

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Real-world frequency

frequency: number of repetitions per *second*

- f expressed in Hz (1/sec)
- alternatively, angular frequency in rad/s: $\Omega = 2\pi f$
- period for periodic signals is $T = \dfrac{1}{f} = \dfrac{2\pi}{\Omega}$

Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continuous time no max frequency: $f \in \mathbb{R}$
- concept is the same: similarity to sinusoidal components

$$X(f) = \langle e^{j2\pi ft}, x(t) \rangle$$

Continuous-time Fourier transform

- Analysis formula

$$X(f) = \langle e^{j2\pi ft}, x(t) \rangle$$
$$= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \qquad \leftarrow \textit{not periodic!}$$

- Synthesis formula

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Fourier analysis (in rad/s)

$$X(j\Omega) = \langle e^{j\Omega t}, x(t)\rangle$$
$$= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \qquad \leftarrow \textit{not periodic!}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$$

- Laplace transform computed on the imaginary axis

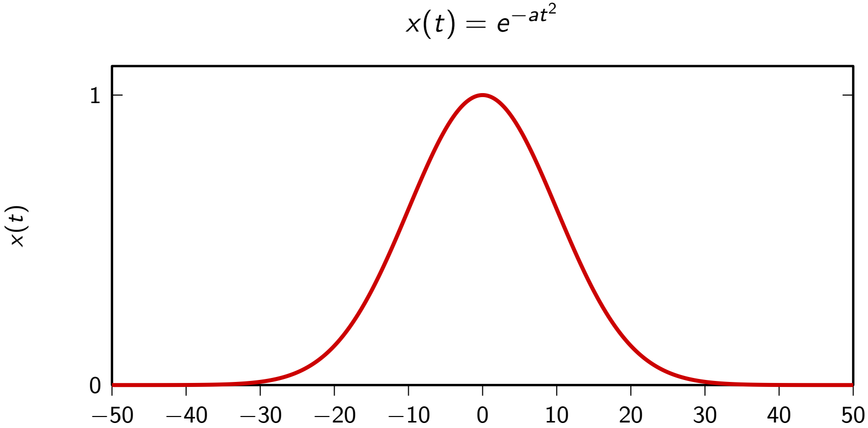
When does CTFT converge?

- For square integrable sequences: $\mathbf{x} \in L_2(\mathbb{R})$
- For absolutely integrable sequences: $\mathbf{x} \in L_1(\mathbb{R}) \leftarrow$ easy to show, lets do it now

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \leq \int_{-\infty}^{\infty} |x(t) e^{-j2\pi ft}| dt$$

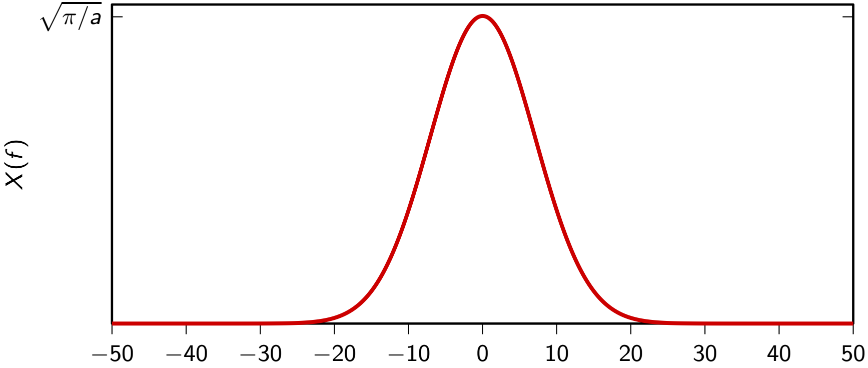
$$\leq \int_{-\infty}^{\infty} |x(t)||e^{-j2\pi ft}| dt \leq \int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Example



Example

$$X(f) = \sqrt{\pi/a}\, e^{-\frac{\pi^2}{a} f^2}$$

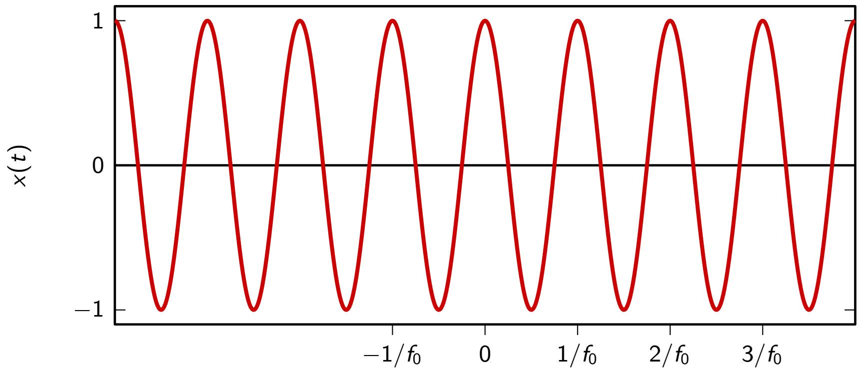


What about other interesting signals?

- Dirac delta: $x(t) = \delta(t)$?
- Constants: $x(t) = 1$?
- Sinusoids: $x(t) = e^{j2\pi f_0 t}$, $x(t) = \cos(2\pi f_0 t)$, $x(t) = \sin(2\pi f_0 t)$?

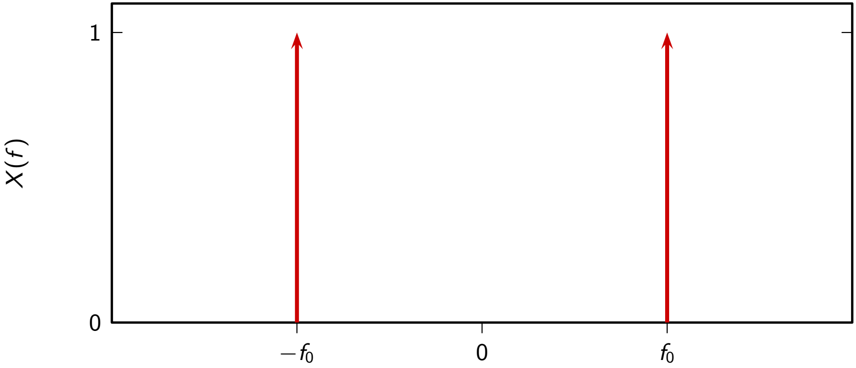
Example

$$x(t) = \cos(2\pi f_0 t)$$



Example

$$X(f) = (1/2)\delta(f \pm f_0)$$



Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

- "Energy is conserved"

Convolution property

$$(x * y)(t) \overset{\text{CTFT}}{\longleftrightarrow} X(f)Y(f)$$

- Convolution in time is multiplication in frequency

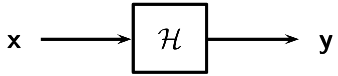
Modulation prorperty

$$x(t)y(t) \overset{\text{CTFT}}{\longleftrightarrow} (X * Y)(f)$$

- Multiplication in time is convolution frequency

Continuous-time LTI theory

Linear, time-invariant systems (in continuous time)



$$\mathcal{H}(\alpha\, \mathbf{x}_1 + \beta\, \mathbf{x}_2) = \alpha\, \mathcal{H}\mathbf{x}_1 + \beta\, \mathcal{H}\mathbf{x}_2$$

$$\mathcal{H}\mathcal{S}^{t_0}\mathbf{x} = \mathcal{S}^{t_0}\mathcal{H}\mathbf{x}$$

Impulse response

$$\mathbf{h} = \mathcal{H}\delta$$

Same fundamental result: Impulse response also fully characterizes a continuous-time LTI system!

Recall: in discrete time...

Every signal is a linear combination of atomic time elements

$$\mathbf{x} = \sum_{k=-\infty}^{\infty} x[k]\, \boldsymbol{\delta}_k,$$

$$\boldsymbol{\delta}_k = \mathcal{S}^{-k} \boldsymbol{\delta}$$

$$\delta_k[n] = \delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k. \end{cases}$$

Recall: in discrete time...

Filter's output from impulse response

$$\mathcal{H}\mathbf{x} = \mathcal{H}\left(\sum_{k=-\infty}^{\infty} x[k]\,\mathcal{S}^{-k}\boldsymbol{\delta}\right)$$ *using linearity...*

$$= \sum_{k=-\infty}^{\infty} \mathcal{H}(x[k]\,\mathcal{S}^{-k}\boldsymbol{\delta})$$ *using linearity...*

$$= \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{S}^{-k}\mathcal{H}\boldsymbol{\delta}$$ *using time invariance...*

$$= \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{S}^{-k}\mathbf{h}$$

Every CT signal is a linear combination of Dirac deltas

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau,$$

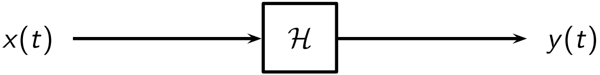
and

$$\mathbf{x} = \int_{-\infty}^{\infty} x(\tau)\mathcal{S}^{-\tau}\boldsymbol{\delta}d\tau$$

CT filter's output from impulse response

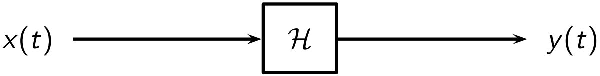
$$\begin{aligned}\mathcal{H}\mathbf{x} &= \mathcal{H}\left(\int_{-\infty}^{\infty} x(\tau)\mathcal{S}^{-\tau}\boldsymbol{\delta} d\tau\right) \\ &= \int_{-\infty}^{\infty} \mathcal{H}\left(x(\tau)\mathcal{S}^{-\tau}\boldsymbol{\delta}\right) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{S}^{-\tau}\mathcal{H}\left(\boldsymbol{\delta}\right) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{S}^{-\tau}\mathbf{h} d\tau\end{aligned}$$

Analog LTI filters



$$
\begin{aligned}
y(t) &= (\mathbf{h} * \mathbf{x})(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
\end{aligned}
$$

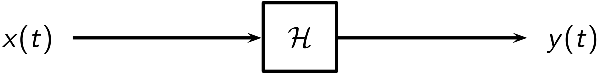
Example: integrator



$\mathcal{H}\mathbf{x}$ is defined by

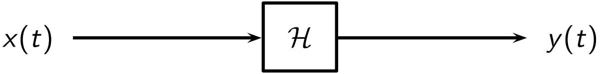
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Example: time shift



$$\mathcal{H}\mathbf{x} = \mathcal{S}^{t_0}\mathbf{x}$$

Example: "echo"



$$\mathcal{H}\mathbf{x} = \mathbf{x} + a_0 \mathcal{S}^{t_0}\mathbf{x} + a_1 \mathcal{S}^{t_1}\mathbf{x}$$

Continuous vs discrete time LTI systems

Just like in discrete time:

- Impulse response completely characterizes a continuous-time LTI system
- Continuous-time convolution relates the input to the output via the impulse response

Continuous vs discrete time LTI systems

Many other parallels:

- Frequency response is the CTFT of the impulse response for stable systems
- Transfer function is the Laplace transform of the impulse response
- A continuous-time system is BIBO stable if an only if its impulse response is *absolutely integrable*