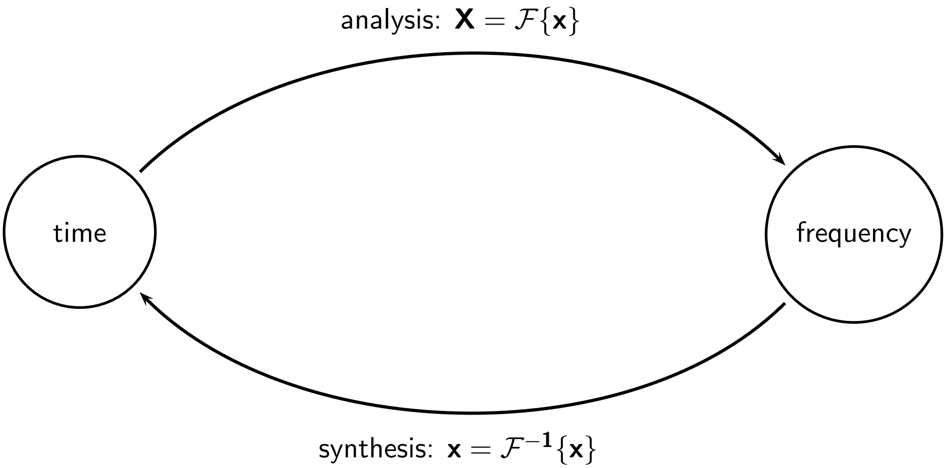
COM-202: Signal Processing

Chapter 5.b: The DTFT and its properties

Overview:

- the DTFT as the limit of a DFT
- DTFT properties
- DTFT of power signals
- relationships between transforms

Fourier Analysis in general



DFT

- finite-length signals
- time domain: $\mathbb{C}^N$, canonical basis
- frequency domain: $\mathbb{C}^N$, Fourier basis
- analysis: compute similarity with the N Fourier basis vectors:

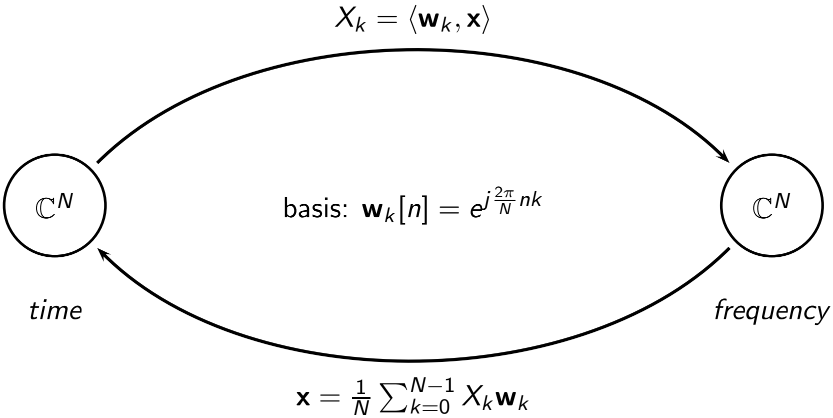
$$X_k = \langle \mathbf{w}_k, \mathbf{x} \rangle$$

- synthesis: build a signal as a linear combination of Fourier basis vectors

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}_k$$

- the DFT is an *algorithm*: we can always compute it numerically since it requires a finite number of arithmetic operations

The DFT as an orthogonal change of basis



All the Fourier Transforms in Signal Processing

	discrete time	continuous time
finite length	DFT/DFS	FS (math)
infinite length	DTFT (this week)	CTFT (later on in the course)

DTFT

- infinite-length signals with finite energy
- time domain: $\ell_2(\mathbb{Z})$, canonical basis
- frequency domain: $L_2([-\pi, \pi])$, Fourier basis
- analysis: compute similarity with an infinite set of oscillations:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- synthesis: build a signal from an infinite set of oscillations:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

DTFT as a formal basis expansion

The DTFT formula looks like an inner product in $\ell_2(\mathbb{Z})$:

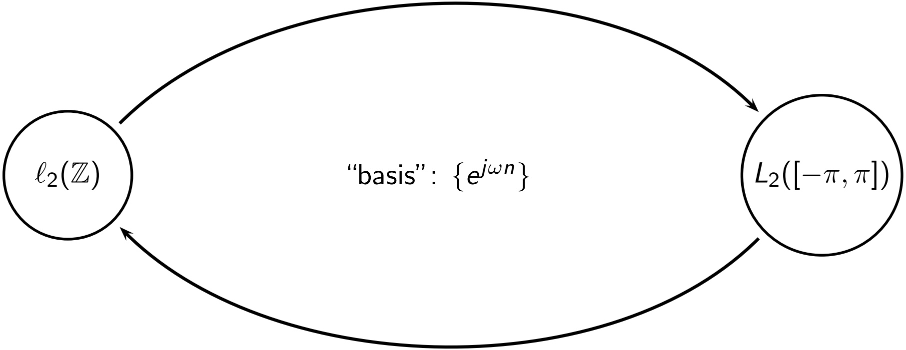
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

However:

- the set $\{e^{j\omega n}\}_{\omega}$ is not countable
- the "basis vectors" $e^{j\omega n}$ don't even belong to $\ell_2(\mathbb{Z})$

DTFT

$$X(\omega) = \sum_n x[n]\, e^{-j\omega n}$$



$$x[n] = \frac{1}{2\pi} \int X(\omega) e^{j\omega n} d\omega$$

The mathematical truth (to clear my conscience)

- DTFT is an (invertible) mapping from $L_2([-\pi, \pi])$ to $\ell_2(\mathbb{Z})$
- the *countable* set of 2π-periodic functions $\{e^{-j\omega n}\}_n$ is an orthogonal basis for $L_2([-\pi, \pi])$:

$$\langle e^{-j\omega n}, e^{-j\omega m}\rangle = \int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega = 2\pi\delta[n-m]$$

- the *inverse* DTFT is a basis expansion; the analysis coefficients are the time-domain values:

$$x[n] \propto \langle e^{-j\omega n}, X(e^{j\omega n})\rangle$$

A more intuitive approach

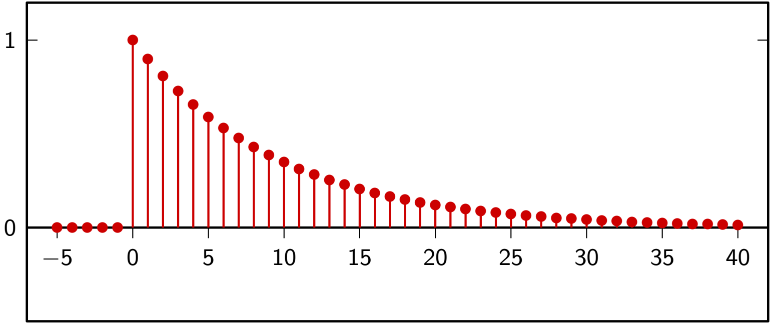
Consider the DFT when N grows very large:

- N basis vectors with frequencies $\frac{2\pi}{N}k,\ k = 0, 1, \ldots, N-1$
- $\left\{\frac{2\pi}{N}k\right\}_k$ becomes denser in $[0, 2\pi]$...
- In the limit $\frac{2\pi}{N}k \rightarrow \omega$:

$$X(\omega) = \sum_n x[n]\, e^{-j\omega n} \quad \omega \in \mathbb{R}$$

Example: decaying exponential

$$x[n] = a^n\, u[n], \quad |a| < 1$$



Example: decaying exponential

Compute the DFT of the length-N signal $x[n] = a^n$, $n = 0, 1, \ldots, N-1$:

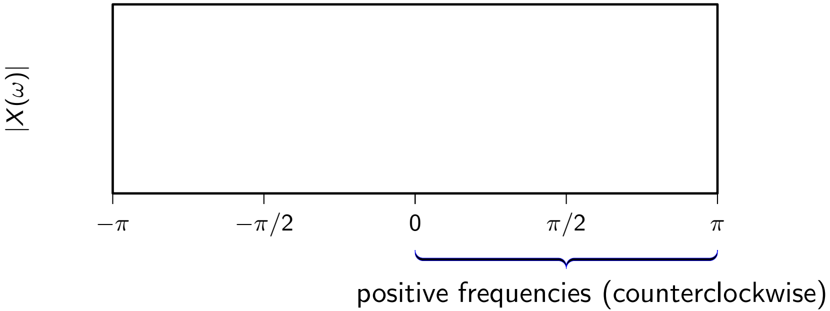
$$X[k] = \sum_{n=0}^{N-1} a^n \, e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} (a \, e^{-j\frac{2\pi}{N}k})^n$$

$$= \frac{1 - (a \, e^{-j\frac{2\pi}{N}k})^N}{1 - a \, e^{-j\frac{2\pi}{N}k}} = \frac{1 - a^N}{1 - a \, e^{-j\frac{2\pi}{N}k}}$$

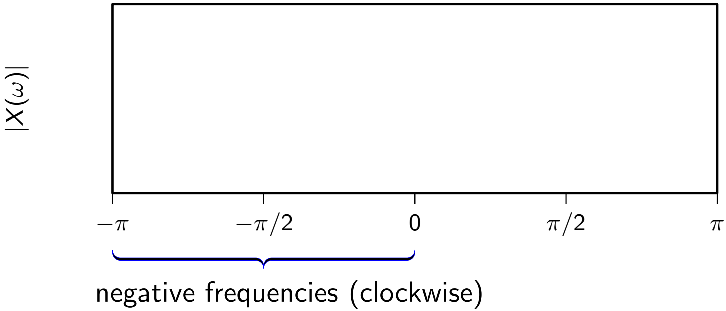
As $N \to \infty$, $(2\pi/N)k \to \omega$:

$$\lim_{N\to\infty} \frac{1 - a^N}{1 - a \, e^{-j\frac{2\pi}{N}k}} = \frac{1}{1 - a \, e^{-j\omega}} = X(\omega)$$

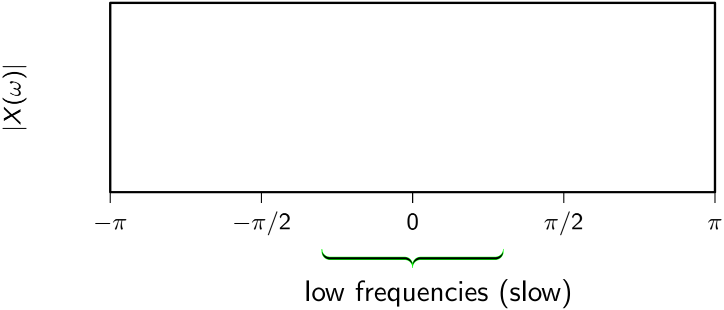
Plotting the DTFT



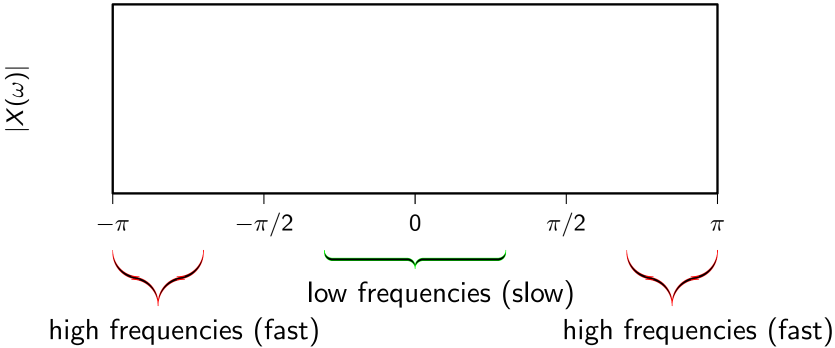
Plotting the DTFT



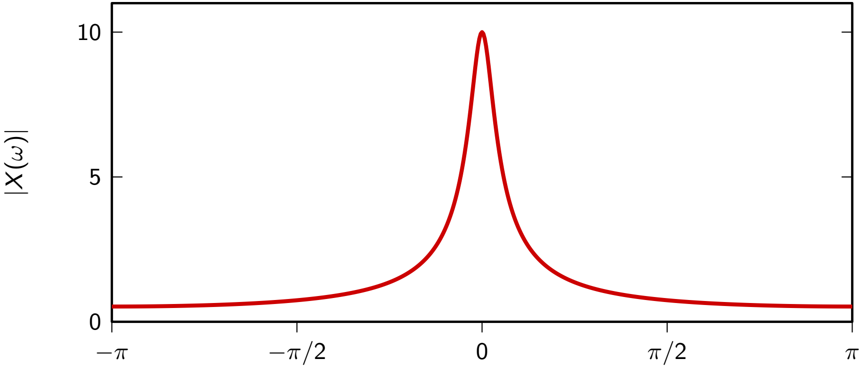
Plotting the DTFT



Plotting the DTFT



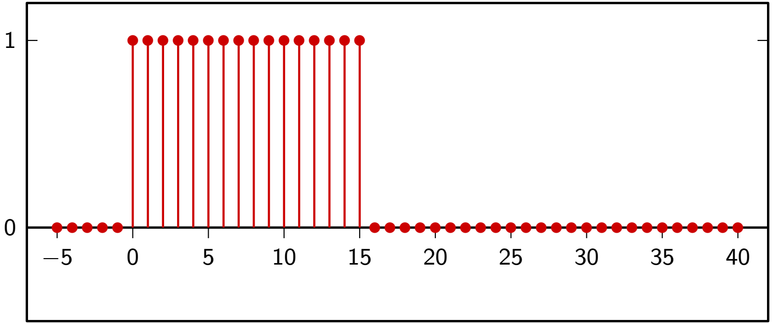
DTFT of $x[n] = a^n u[n], \quad |a| < 1$



$a = 0.9$

Example: rectangular sequence

$$x[n] = \begin{cases} 1 & 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$



Example: rectangular sequence

Compute the DFT of a finite-length rectangular sequence:

$$X[k] = \sum_{n=0}^{N-1} x[n], e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{M-1} e^{-j\frac{2\pi}{N}nk}$$

$$\ldots \qquad \textit{(see lecture 6)}$$

$$= \frac{\sin\left(\frac{\pi}{N}k \cdot M\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}k\cdot(M-1)}$$

As $N \to \infty$, $(2\pi/N)k \to \omega$:

$$X(\omega) = \frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(M-1)}$$

DFT vs DTFT

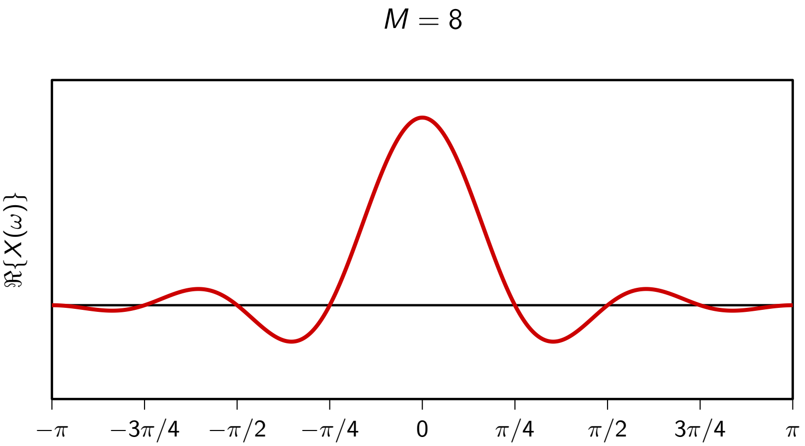
DFT of size-M rectangular signal in $\mathbb{C}^N$:

$$X[k] = \frac{\sin\left(\frac{\pi}{N} M k\right)}{\sin\left(\frac{\pi}{N} k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

DTFT of size-M rectangular signal in $\ell_2(\mathbb{Z})$:

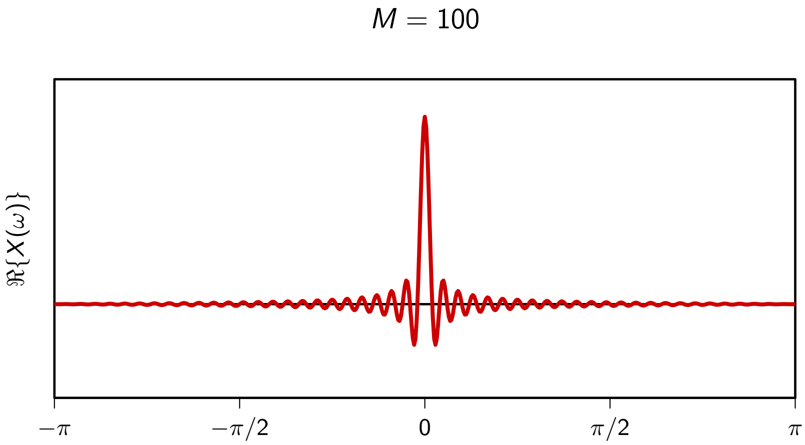
$$X(\omega) = \frac{\sin\left(\frac{\omega}{2} M\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(M-1)}$$

DTFT of rectangular signal (real part)

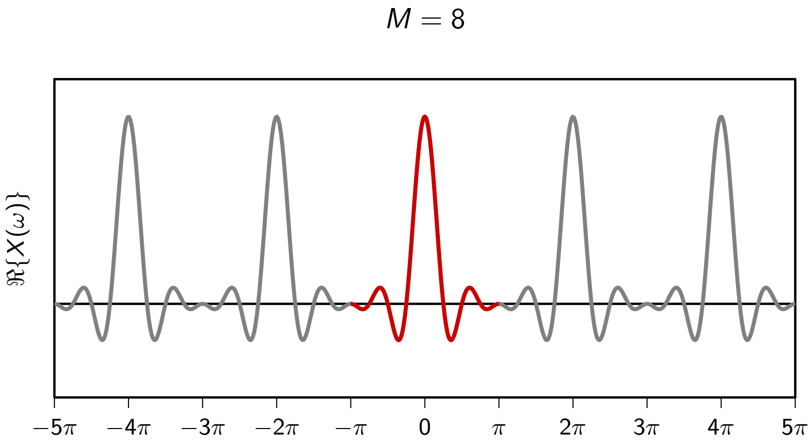


note that $X(\omega) = 0$ for $\omega = (2\pi/M)k,\ k \in \mathbb{Z}/\{0\}$

DTFT of rectangular signal (real part)



Never forget the 2π-periodicity!



DTFT properties

The Discrete-Time Fourier Transform

Analysis formula:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad \omega \in [-\pi, \pi]$$

Synthesis formula:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(\omega) + \beta Y(\omega)$$

- time shift

$$\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(\omega)$$

- frequency shift (aka *modulation*)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$$

DTFT properties

- time reversal

$$\text{DTFT}\{x[-n]\} = X(-\omega)$$

- conjugation

$$\text{DTFT}\{x^*[n]\} = X^*(-\omega)$$

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \iff X(\omega) = X(-\omega)$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \iff X(\omega) = X^*(-\omega)$$

As a consequence:

- if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \iff |X(\omega)| = |X(-\omega)|$$

- if $x[n]$ is real *and* symmetric, the DTFT is also real and symmetric

the DTFT formalism for non ℓ_2 sequences

The DTFT as the limit of the DFT

Some key results carry over from finite-length to infinite-length:

- $\text{DFT}\left\{\delta[n]\right\} = 1$
- $\text{DTFT}\left\{\delta[n]\right\} = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$

The DTFT as the limit of the DFT

... but other things do not:

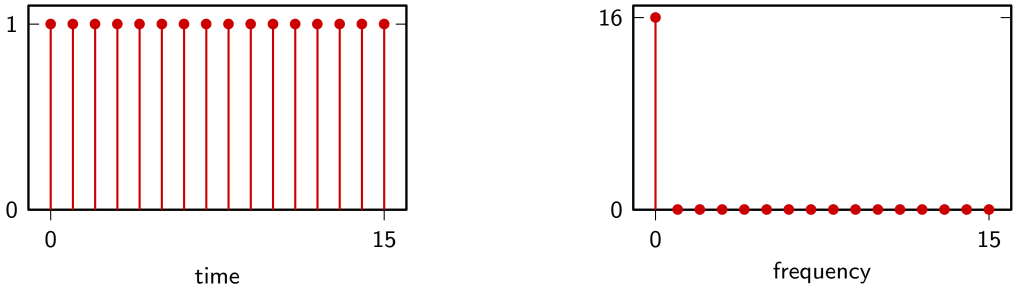
- $\text{DFT}\{1\} = \delta[n]$
- $\text{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$

- problem: too many interesting sequences are *not* square summable!

DTFT vs DFT

Remember the DFT of the constant signal $x[n] = 1$:

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = N\delta[k]$$

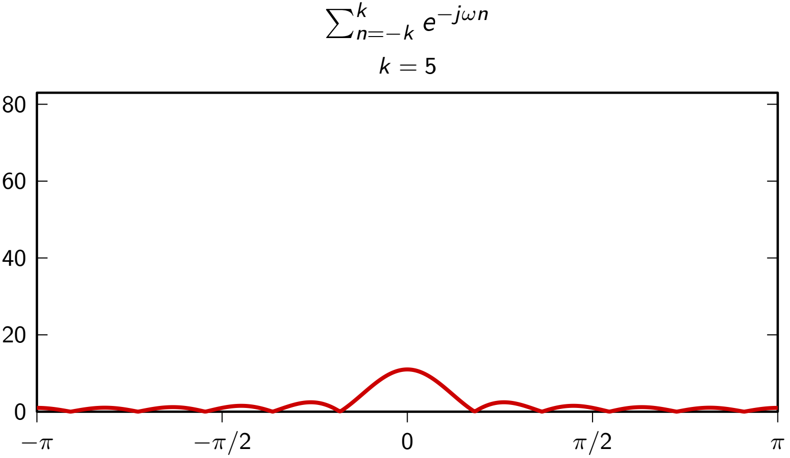


DTFT of $x[n] = 1$

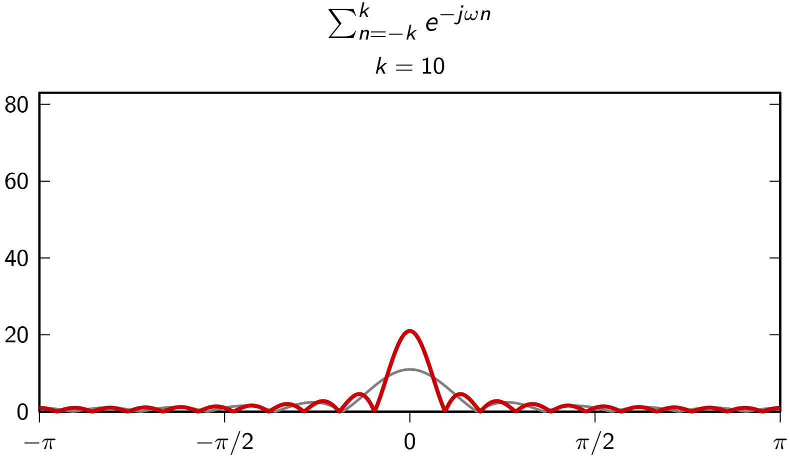
We would like something along the lines of

$$\mathrm{DTFT}\{1\} = C(\omega) = \begin{cases} 0 & \text{for } \omega \neq 0 \\ \text{nonzero} & \text{for } \omega = 0 \end{cases}$$

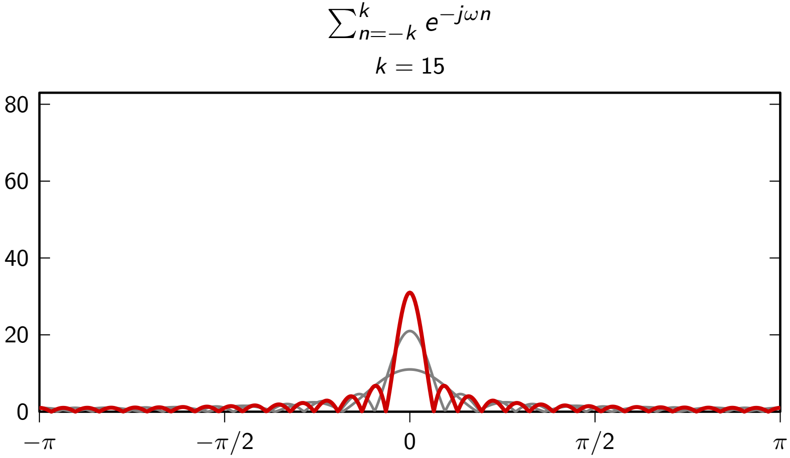
DTFT of $x[n] = 1$: partial DTFT sums



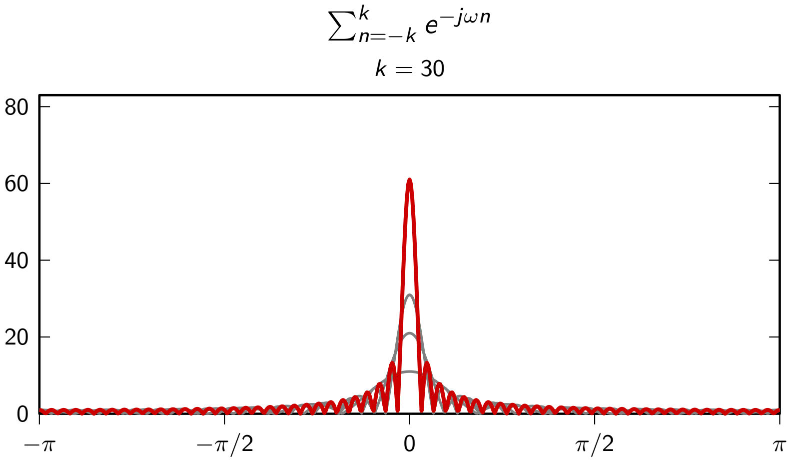
DTFT of $x[n] = 1$: partial DTFT sums



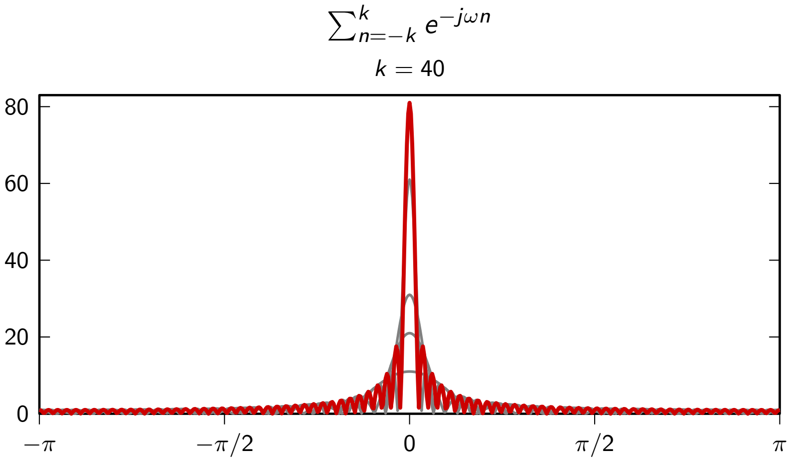
DTFT of $x[n] = 1$: partial DTFT sums



DTFT of $x[n] = 1$: partial DTFT sums



DTFT of $x[n] = 1$: partial DTFT sums



DTFT of $x[n] = 1$

it would appear that we need something like

$$C(\omega) = \begin{cases} 0 & \text{for } \omega \neq 0 \\ \infty & \text{for } \omega = 0 \end{cases}$$

DTFT of $x[n] = 1$

but we also should have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} C(\omega)\, e^{j\omega}\, d\omega = 1$$

Clearly $C(\omega)$ is not a "normal" function...

The Dirac delta

Theory of functionals in 2 slides

functional: linear operator that acts on *functions*

Examples:

- average over $[-A/2, A/2]$: $\mathcal{M}_A\{f\} = (1/A)\int_{-A/2}^{A/2} f(x)dx$
- n-th moment: $\mathcal{B}_n\{f\} = \int_{-\infty}^{\infty} x^n f(x)dx$
- value in zero: $\mathcal{D}\{f\} = f(0)$

Python example (this slide doesn't count)

```
def my_function(x):
    return x * x + 1

def my_functional(f):
    return f(2)

> print(my_function(0))
> print(my_functional(my_function))
> print(my_functional(sqrt))
```

Output:

```
1
5
1.4142135623730951
```

Theory of functionals in 2 slides

The action of functionals can often be expressed as an inner product between the input function and a function known as the functional's *kernel*:

- average over $[-A/2, A/2]$: $\mathcal{M}_A\{f\} = (1/A)\int_{-A/2}^{A/2} f(x)dx$

  $$\mathcal{M}_A\{f\} = \int_{-\infty}^{\infty} m_A(x)f(x)dx, \quad m_A(x) = (1/A)\,\mathrm{rect}(Ax)$$

- n-th moment: $\mathcal{B}_n\{f\} = \int_{-\infty}^{\infty} x^n f(x)dx$

  $$\mathcal{B}_n\{f\} = \int_{-\infty}^{\infty} b_n(x)f(x)dx, \quad b_n(x) = x^n$$

- value in zero: $\mathcal{D}\{f\} = f(0)$

  $$\mathcal{D}\{f\} = \int_{-\infty}^{\infty} d(x)f(x)dx, \quad d(x) = ??$$

The Dirac delta functional

the Dirac delta $\delta(t)$ is defined as the "kernel" that implements $\mathcal{D}$:

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

for all continuous $f : \mathbb{R} \to \mathbb{C}$.

The Dirac delta functional

the Dirac delta can be shifted anywhere

$$\int_{-\infty}^{\infty} \delta(t-s) f(t) dt = f(s)$$

About the integration limits

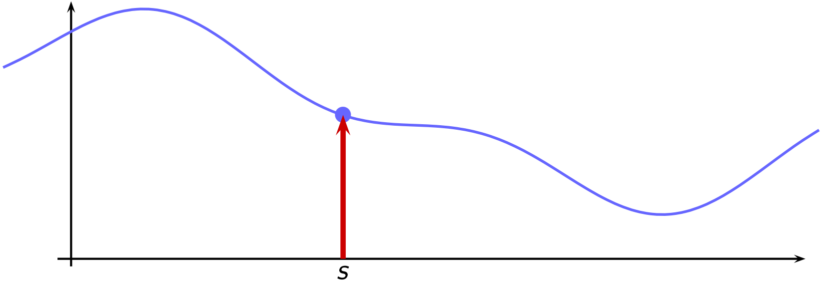
the integral limits only need to include the Dirac delta's location:

$$\int_{-\infty}^{\infty} \delta(t-s)f(t)dt = \int_{s-\epsilon_a}^{s+\epsilon_b} \delta(t-s)f(t)dt = f(s) \quad \forall \epsilon_a, \epsilon_b > 0$$

conversely

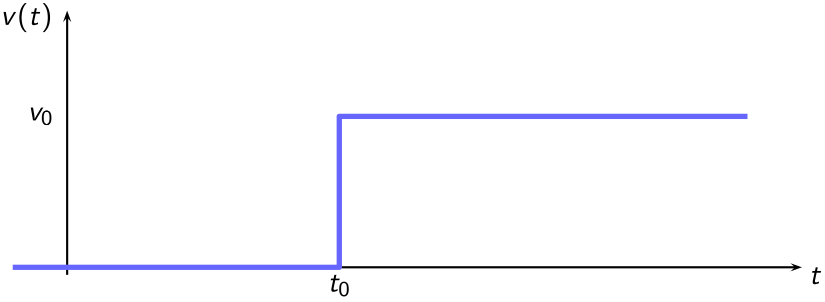
$$\int_I \delta(t-s)f(t)dt = 0 \quad \text{if } s \notin I$$

The Dirac delta functional



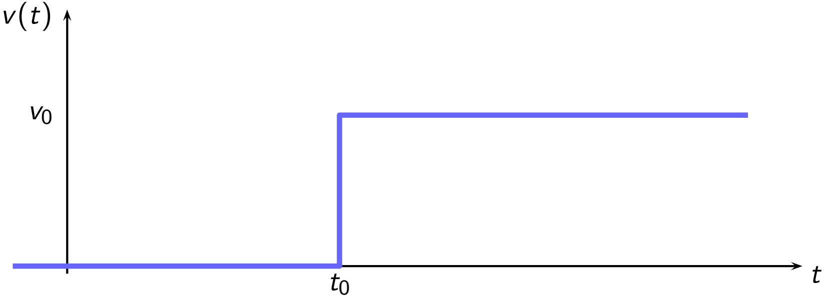
$$\int_{-\infty}^{\infty} \delta(t-s)f(t)dt = f(s)$$

The Dirac delta functional in physics



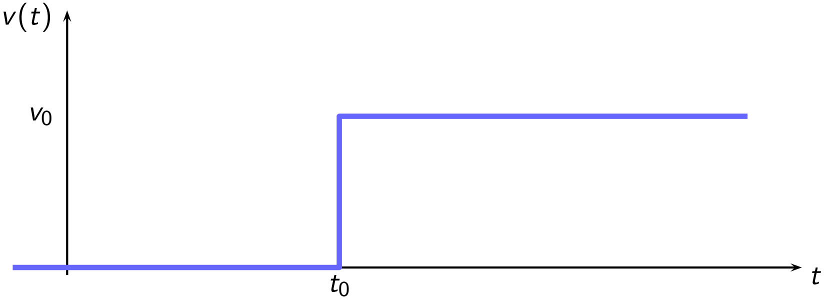
$$F(t) = ma(t) = m\frac{\partial v(t)}{\partial t}$$

The Dirac delta functional in physics



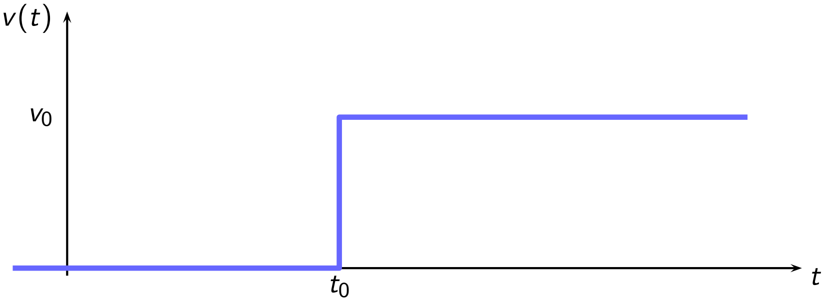
$$a(t_0) = \infty?$$

The Dirac delta functional in physics



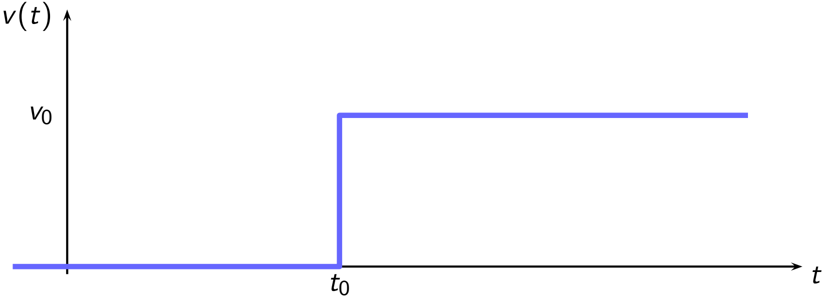
from the other side: $v(t) = \int_{-\infty}^{t} a(\tau) d\tau = \begin{cases} 0 & \text{for } t < t_0 \\ v_0 & \text{for } t > t_0 \end{cases}$

The Dirac delta functional in physics



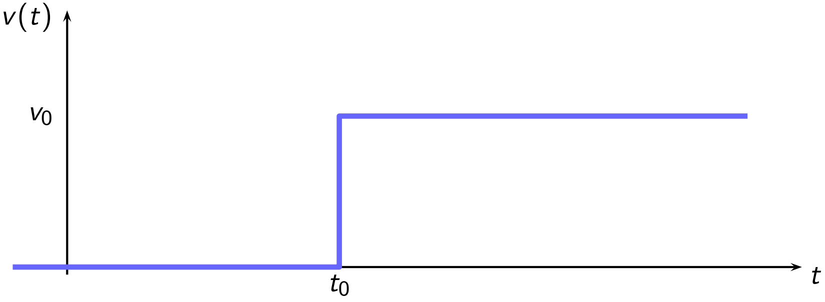
from the other side: $v(t) = \int_{-\infty}^{t} v_0 \delta(\tau - t_0) d\tau$

The Dirac delta functional in physics



$$a(t) = v_0 \delta(t - t_0)$$

The Dirac delta functional in physics



$$F(t) \propto \delta(t - t_0) \approx \begin{cases} \infty & \text{for } t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

And here we go again...

again, it would appear that we need something like

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

Intuition

consider a family of *localizing* functions $r_k(t)$ with $k \in \mathbb{N}$ and $t \in \mathbb{R}$ where:

- support inversely proportional to k
- constant area

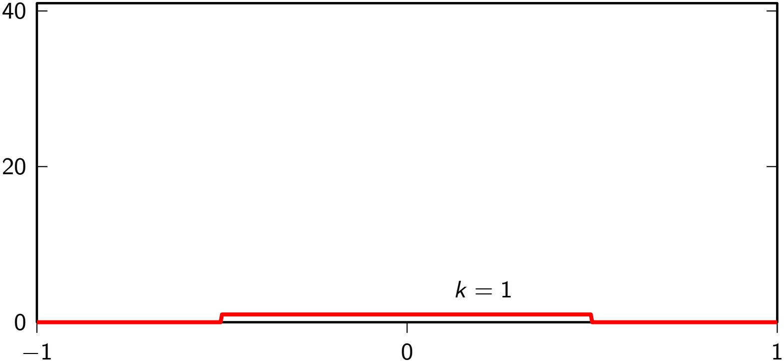
Intuition

$$\text{rect}(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

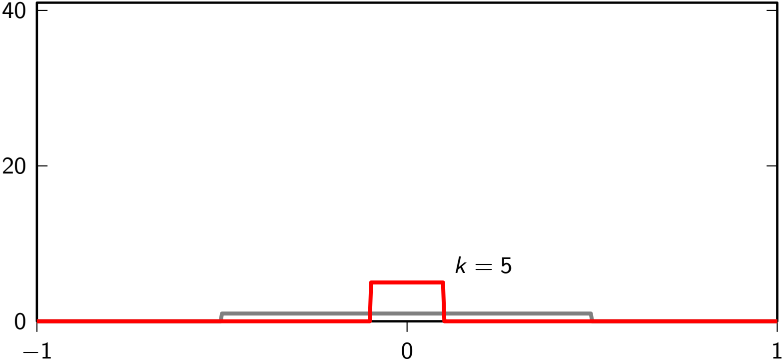
we can build a localizing family as $r_k(t) = k\,\text{rect}(kt)$:

- nonzero over $[-1/(2k), 1/(2k)]$, i.e. support is $1/k$
- area is 1

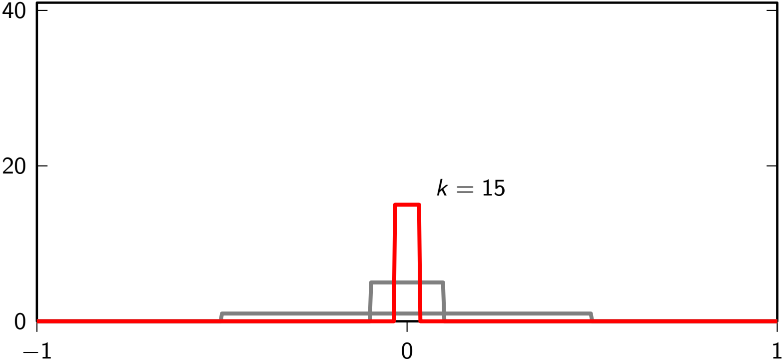
The family $r_k(t) = k\,\mathrm{rect}(kt)$



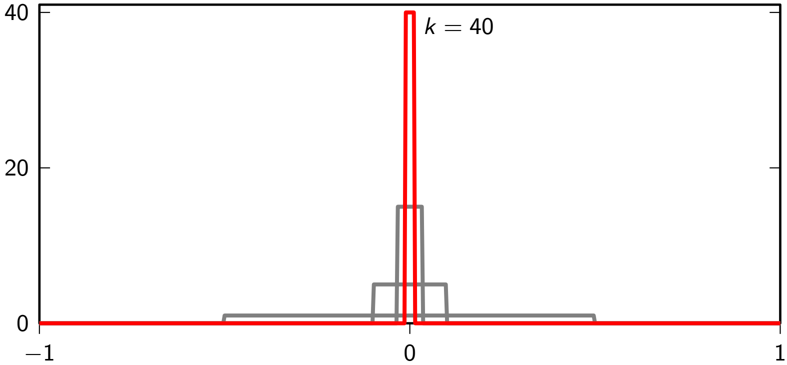
The family $r_k(t) = k\,\mathrm{rect}(kt)$



The family $r_k(t) = k\,\mathrm{rect}(kt)$



The family $r_k(t) = k\,\mathrm{rect}(kt)$

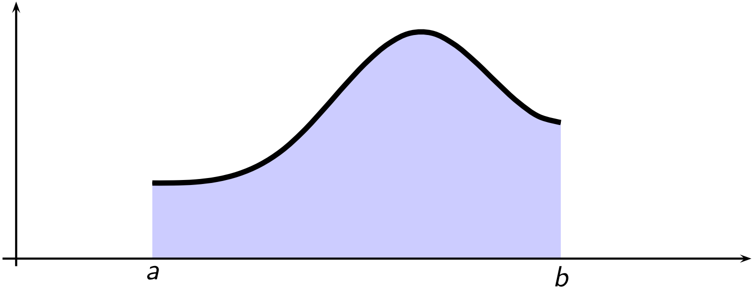


Remember the Mean Value Theorem?

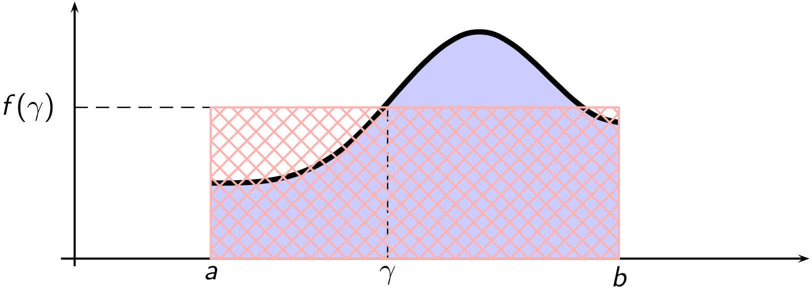
for any continuous function over the interval $[a, b]$ there exists $\gamma \in [a, b]$ s.t.

$$\int_a^b f(t)dt = (b - a) f(\gamma)$$

The Mean Value Theorem



The Mean Value Theorem



Extracting a point value

for our family of localizing functions:

$$\int_{-\infty}^{\infty} r_k(t) f(t) dt = k \int_{-1/(2k)}^{1/(2k)} f(t) dt$$

$$= f(\gamma)|_{\gamma \in [-1/(2k), 1/(2k)]}$$

and so:

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} r_k(t) f(t) dt = f(0)$$

The Dirac delta functional

The delta functional is a shorthand. Instead of writing

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} r_k(t-s)f(t)dt$$

we write

$$\int_{-\infty}^{\infty} \delta(t-s)f(t)dt.$$

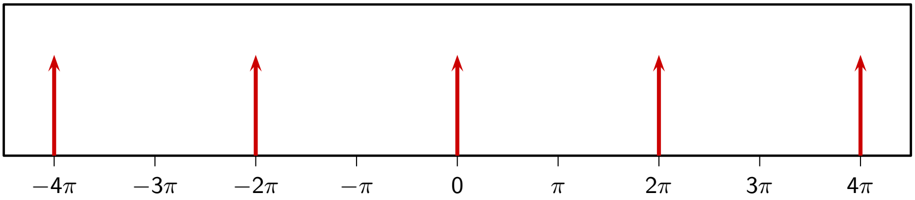
as if $\lim_{k\to\infty} r_k(t) = \delta(t)$,

The “pulse train”

little technical detail: to bring the Dirac delta to the space where DTFTs live, we need to periodize and scale:

$$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Graphical representation



Now let the show begin!

$$\begin{aligned}\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\delta}(\omega)e^{j\omega n}d\omega \\ &= \int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega \\ &= e^{j\omega n}\big|_{\omega=0} \\ &= 1\end{aligned}$$

In other words

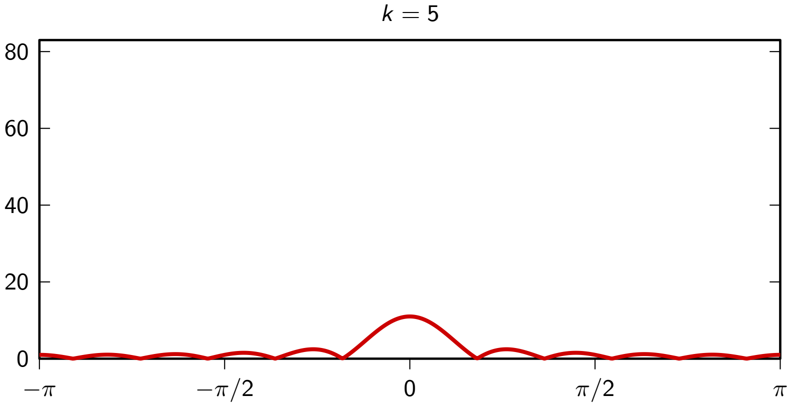
$$\mathsf{DTFT}\left\{1\right\} = \tilde{\delta}(\omega)$$

Does it make sense?

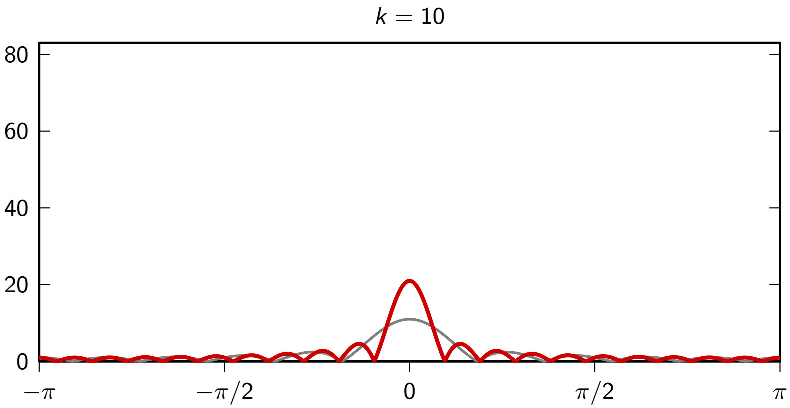
Partial DTFT sum:

$$S_k(\omega) = \sum_{n=-k}^{k} e^{-j\omega n}$$

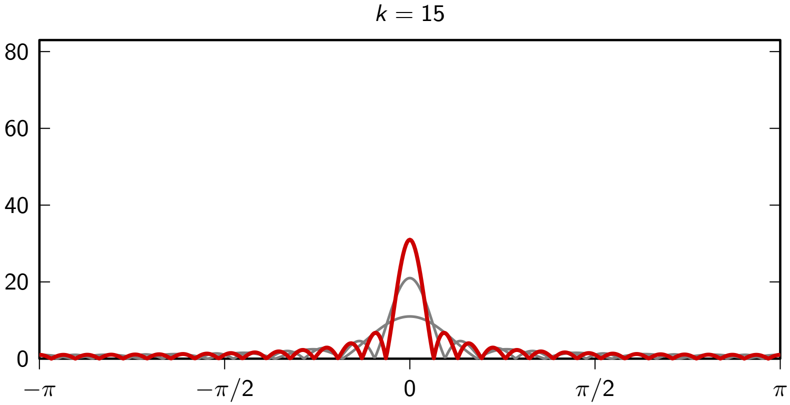
Plotting $|S_k(\omega)|$



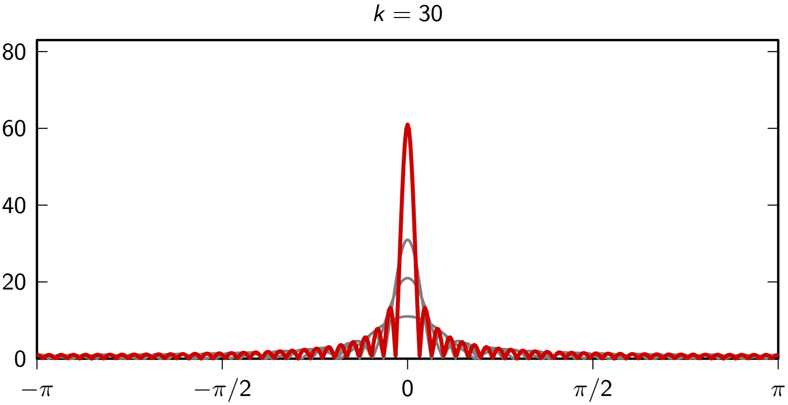
Plotting $|S_k(\omega)|$



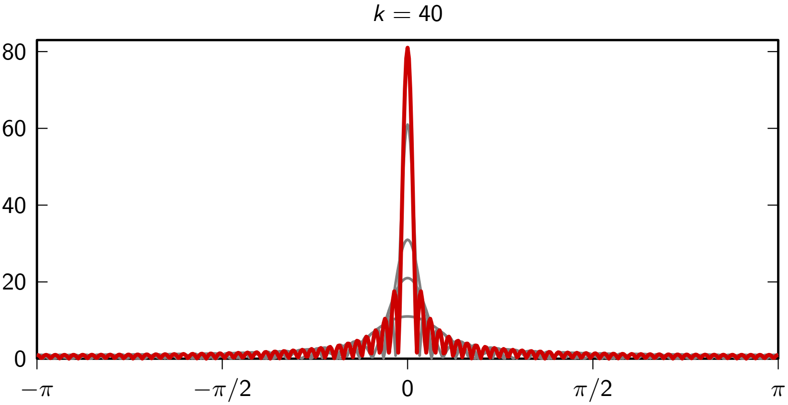
Plotting $|S_k(\omega)|$



Plotting $|S_k(\omega)|$



Plotting $|S_k(\omega)|$



Does it make sense?

Partial DTFT sums look like a family of localizing functions:

$$S_k(\omega) \to \tilde{\delta}(\omega)$$

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega - \omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\{e^{j\omega_0 n}\} = \tilde{\delta}(\omega - \omega_0)$
- $\text{DTFT}\{\cos \omega_0 n\} = [\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)]/2$
- $\text{DTFT}\{\sin \omega_0 n\} = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]/2$

Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved

relationships between transforms

Embedding finite-length signals into infinite sequences

Consider a length-N signal $x[n]$, with DFT $X[k]$.
We can turn this into an infinite sequence in two ways:

- periodic extension: $\tilde{x}[n] = x[n \mod N]$
- finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$

how does $X[k]$ relate to the DTFTs of the embedded signals?

DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\begin{aligned}
\tilde{X}(\omega) &= \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)
\end{aligned}$$

We've seen this before

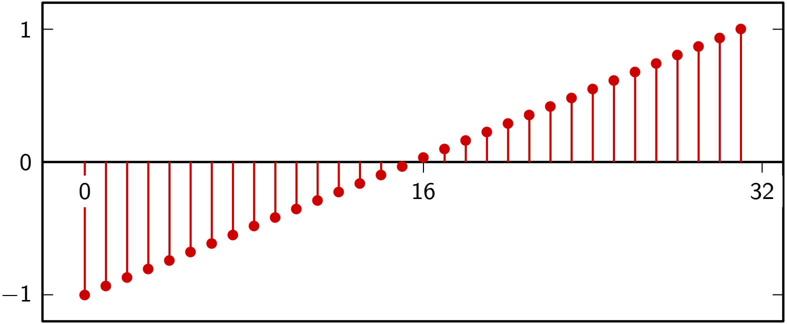
$$\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} = \text{DTFT}\left\{e^{j\frac{2\pi}{N}nk}\right\}$$

$$= \tilde{\delta}\left(\omega - \frac{2\pi}{N}k\right)$$

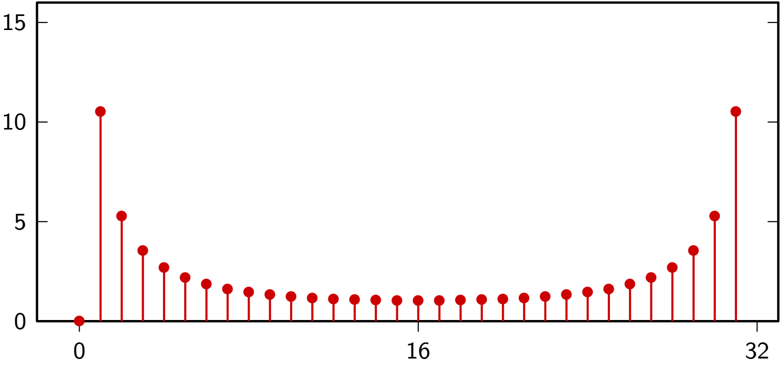
DTFT of periodic signals

$$\tilde{X}(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \tilde{\delta} \left(\omega - \frac{2\pi}{N} k \right)$$

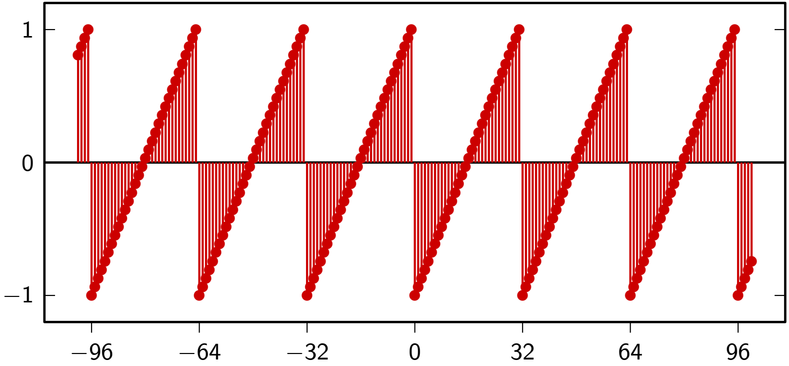
32-tap sawtooth



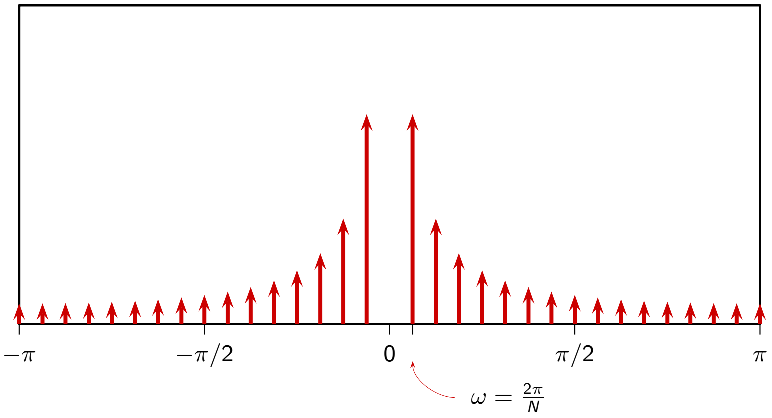
DFT of 32-tap sawtooth



32-periodic sawtooth



DTFT of periodic extension



DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\bar{X}(\omega) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \sum_{k=0}^{N-1} X[k]
\end{aligned}$$

DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \bar{X}(\omega) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X[k] \end{aligned}$$

DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\bar{X}(\omega) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} \left(\frac{1}{N} e^{j\frac{2\pi}{N}kn} \right) e^{-j\omega n}
\end{aligned}$$

DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\bar{X}(\omega) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \sum_{k=0}^{N-1} X[k] \boxed{\sum_{n=0}^{N-1} \left(\frac{1}{N} e^{j\frac{2\pi}{N}kn} \right) e^{-j\omega n}}
\end{aligned}$$

We've seen this before

DTFT of a rectangular sequence of length N (scaled by $1/N$):

$$r_N[n] = \begin{cases} 1/N & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} R_N(\omega) &= \sum_{n=0}^{N-1} \frac{1}{N} e^{-j\omega n} \\ &= \frac{1}{N} \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)} \end{aligned}$$

Frequency shift property

$$\text{DTFT}\left\{e^{j\omega_0 n}x[n]\right\} = X(e^{j(\omega-\omega_0)})$$

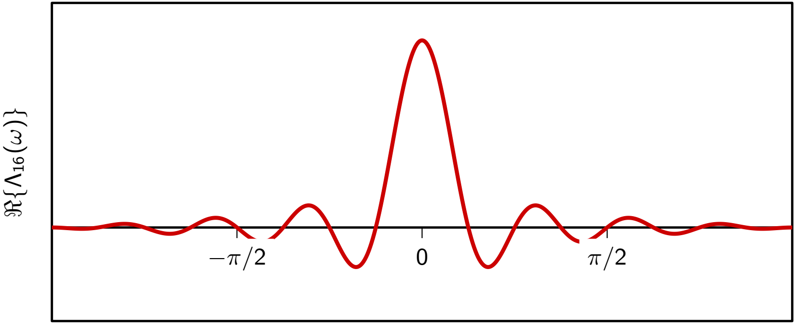
$$\boxed{\sum_{n=0}^{N-1}\left(\frac{1}{N}e^{j\frac{2\pi}{N}kn}\right)e^{-j\omega n}} = \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}kn} r_N[n] e^{-j\omega n}$$

$$= \text{DTFT}\left\{e^{j\frac{2\pi}{N}kn} r_N[n]\right\}$$

$$= R_N(e^{j(\omega-\frac{2\pi}{N}k)})$$

$$\equiv \Lambda_N\left(\omega - \frac{2\pi}{N}k\right)$$

Interpolating function $\Lambda_{16}(\omega)$



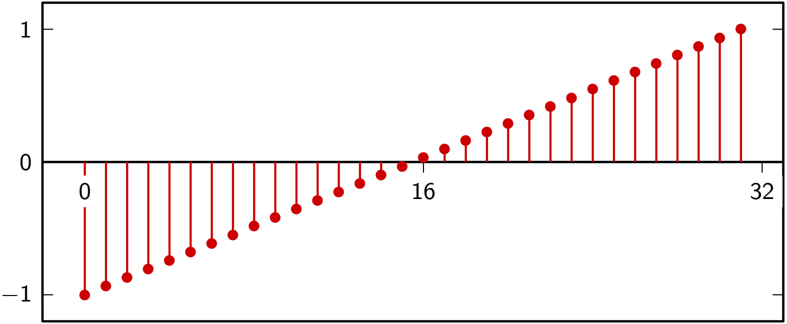
again, $\Lambda(\omega) = 0$ for $\omega = (2\pi/N)k,\ k \in \mathbb{Z}/\{0\}$

DTFT of finite-support signals

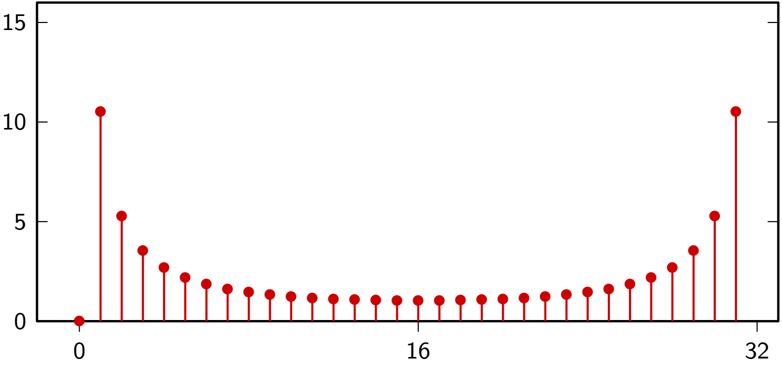
$$\begin{aligned}\bar{X}(\omega) &= \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} \left(\frac{1}{N} e^{j\frac{2\pi}{N}kn} \right) e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X[k] \Lambda_N \left(\omega - \frac{2\pi}{N} k \right)\end{aligned}$$

the DTFT is the smooth interpolation of the original DFT values

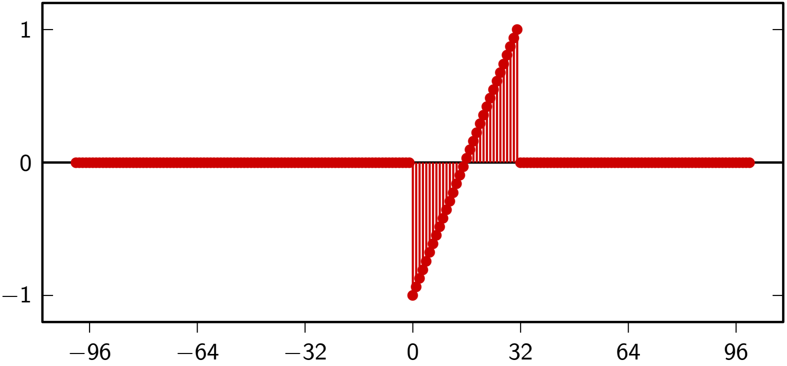
32-tap sawtooth



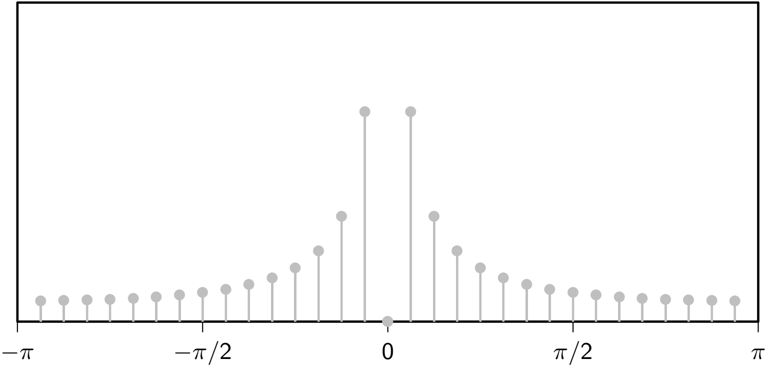
DFT of 32-tap sawtooth



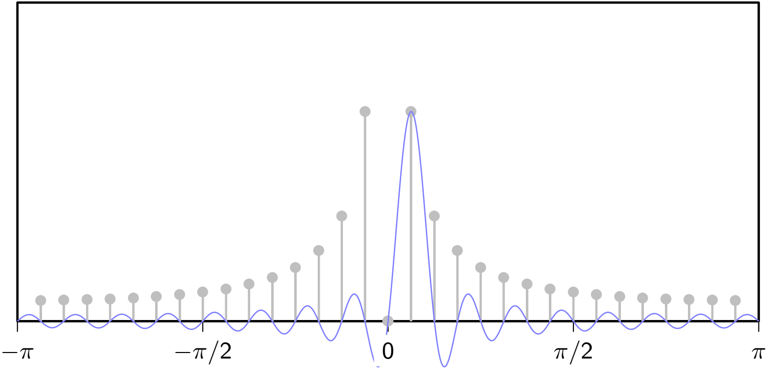
Sawtooth: finite support extension



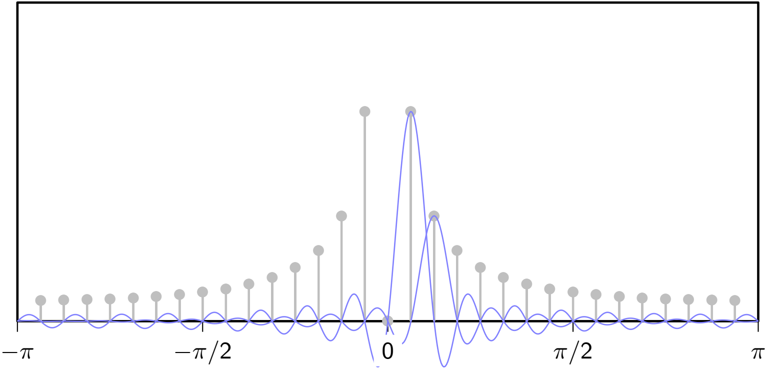
DTFT of finite support extension (sketch)



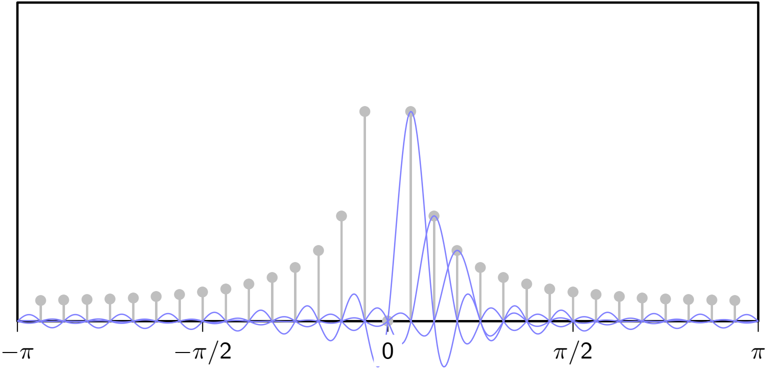
DTFT of finite support extension (sketch)



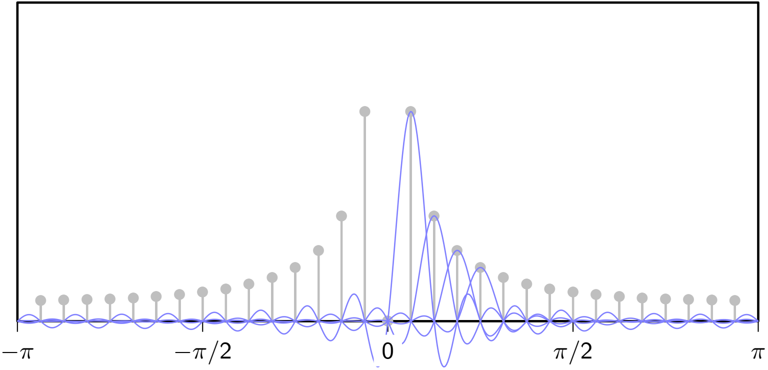
DTFT of finite support extension (sketch)



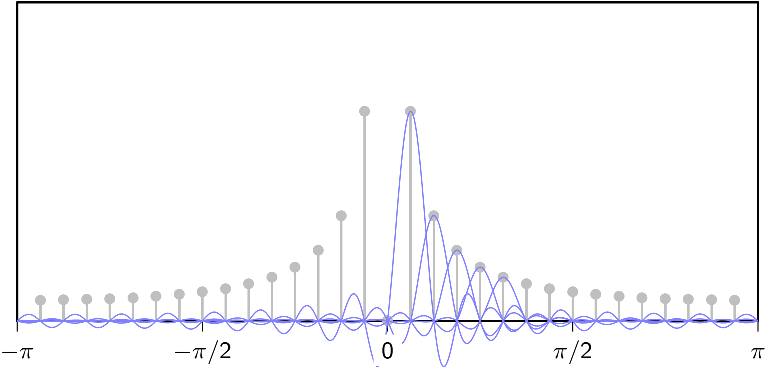
DTFT of finite support extension (sketch)



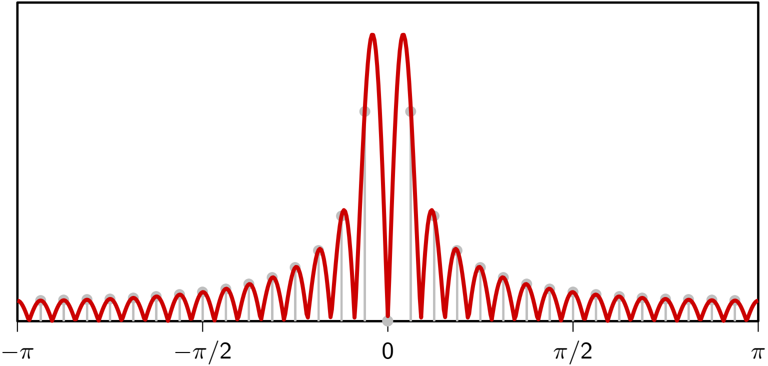
DTFT of finite support extension (sketch)



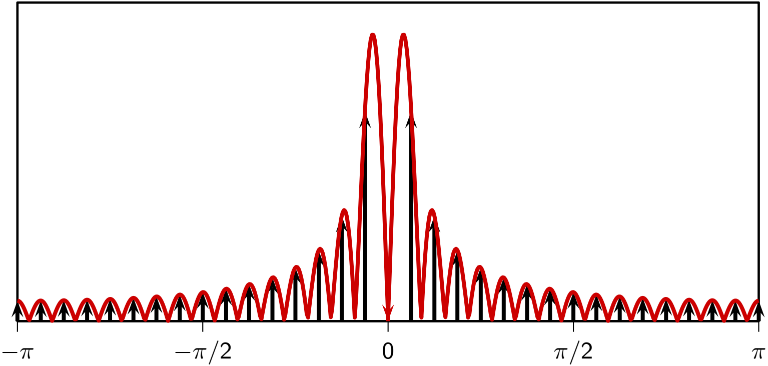
DTFT of finite support extension (sketch)



DTFT of finite support extension



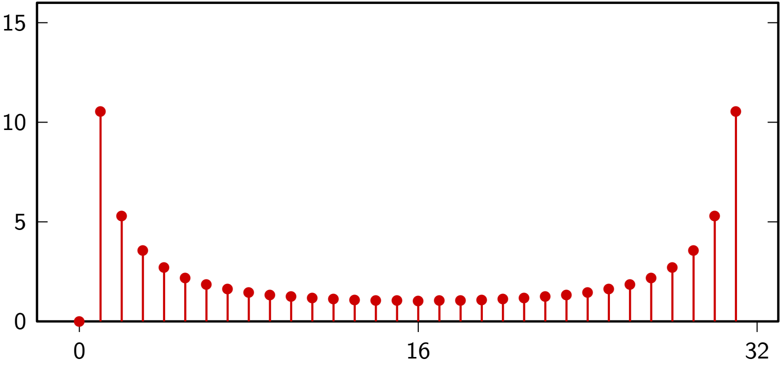
As a comparison...



About zero-padding

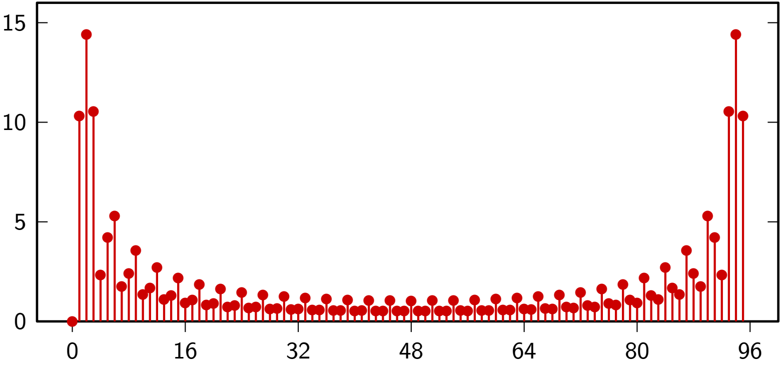
When computing the DFT numerically
one may “pad” the data vector with zeros to obtain “nicer” plots

DFT of 32-tap sawtooth



$$\mathbf{x} = [x_0 \; x_1 \; \ldots \; x_{31}]$$

DFT of 32-tap sawtooth, zero-padded to 96 points



$$\mathbf{x} = [x_0\; x_1\; \ldots\; x_{31}\; 0\; \ldots\; 0]$$

About zero-padding

- zero padding does not add information
- a zero-padded DFT is simply a sampled DTFT of the finite-support extension

All zero-padded versions come from the same DTFT

Consider the finite-support extension of the original signal:

$$\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

Any zero-padded version is simply a truncated finite-support extension:

$$x_M[n] = \bar{x}[n], \qquad n = 0, 1, \ldots, M-1 \quad (M \geq N)$$

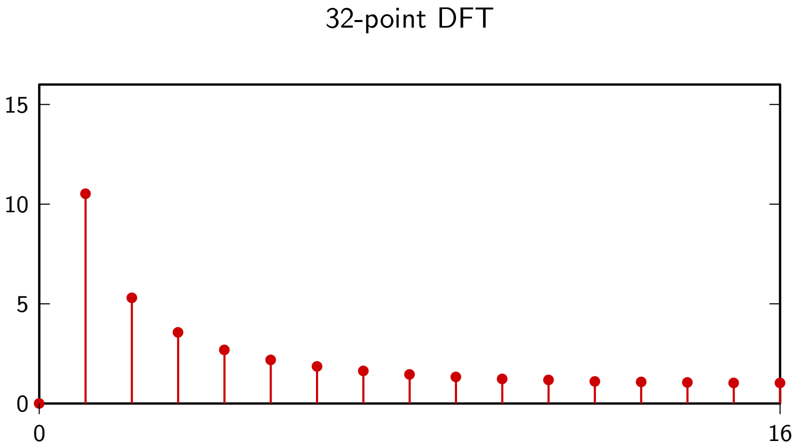
M-point DTFT with zero-padding

$$X_M[h] = \sum_{n=0}^{M-1} x_M[n] e^{-j\frac{2\pi}{M}nh}$$

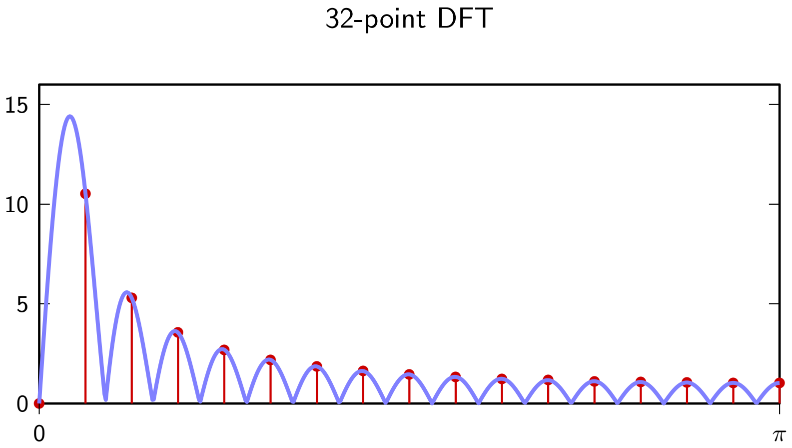
$$= \left. \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} \right|_{\omega=\frac{2\pi}{M}h}$$

$$= \bar{X}(\omega)\big|_{\omega=\frac{2\pi}{M}h}$$

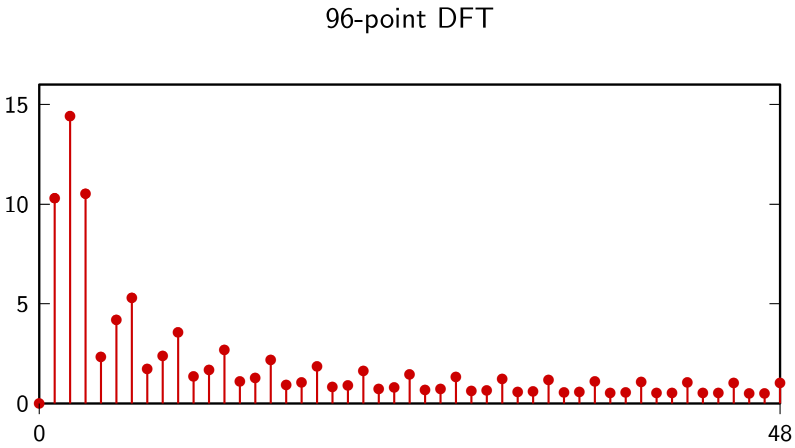
DFT of 32-tap sawtooth, zero-padded



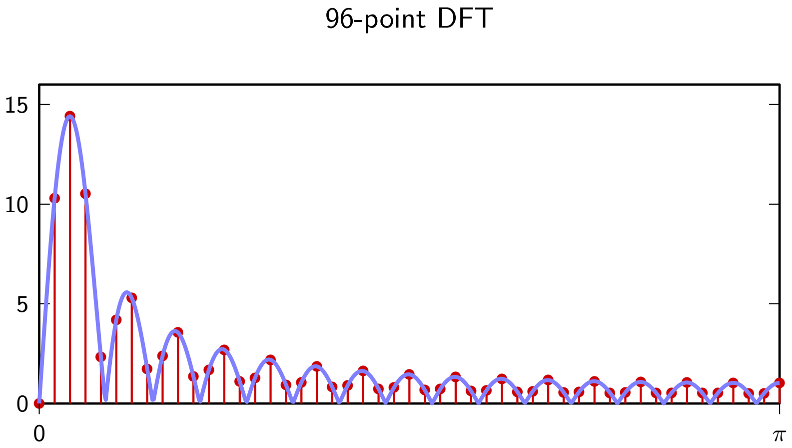
DFT of 32-tap sawtooth, zero-padded



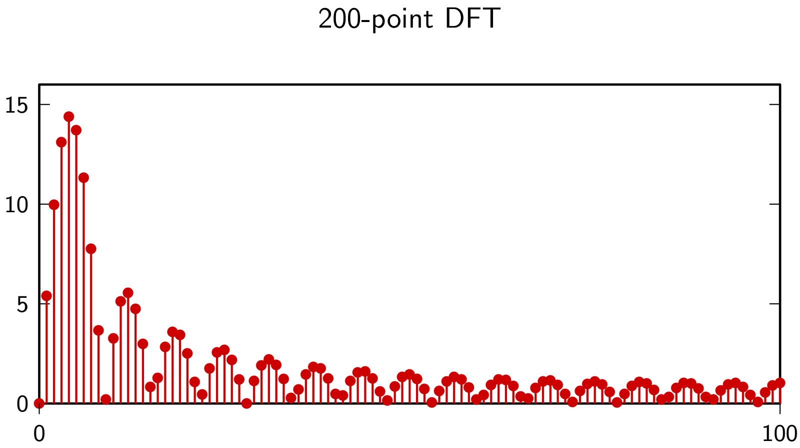
DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded