COM-202: Signal Processing

Chapter 5.a: DFT, DFS, and DTFT

Overview

- DFT: recap and intuition
- the inverse DFT as a synthesis tool
- the DFT and the DFS
- the DTFT

The DFT: recap

The DFT algorithm

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

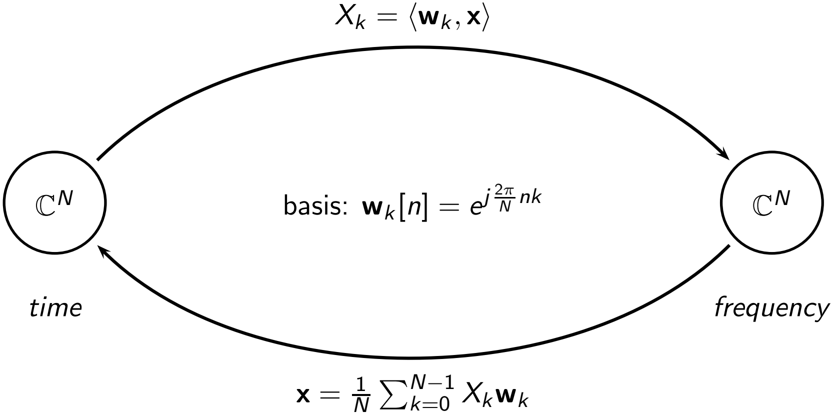
N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *time domain*

The DFT as an orthogonal change of basis



Why is vector space useful

- change of basis as a change of perspective
- orthogonal basis guarantees separation of information
- inner product is a measure of similarity:

$$X_k = \langle \mathbf{w}_k, \mathbf{x} \rangle = C_k e^{j\phi_k}$$

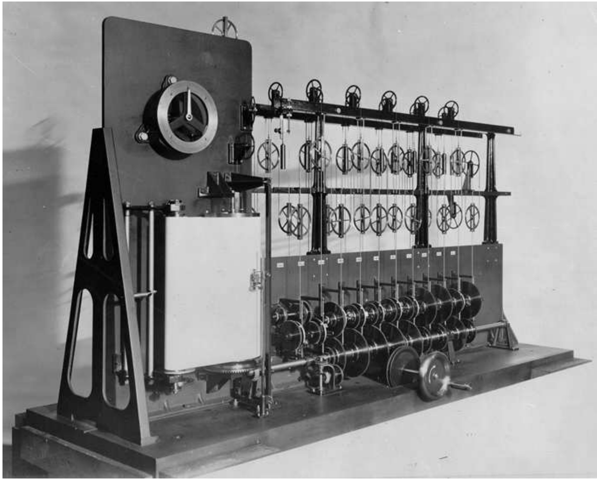
  - $C_k = |X_k|$: how "similar" the signal $x[n]$ is to an oscillation of frequency $\omega_k = \frac{2\pi}{N}k$
  - $\phi_k = \angle X_k$: phase shift that best "aligns" $x[n]$ to the oscillation at frequency ω_k

the inverse DFT as a synthesis tool

Wonderful website

http://jackschaedler.github.io/circles-sines-signals

The machine before DSP



tide-predicting machine (originally invented by Lord Kelvin)

Running the machine too long...

$$x[n+N] = x[n]$$

output signal is N-periodic!

Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

produces an N-point signal in the time domain

the analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

produces an N-point signal in the frequency domain

Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n \in \mathbb{Z}$$

produces an N-periodic signal in the time domain

the analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

produces an N-point signal in the frequency domain

Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n \in \mathbb{Z}$$

produces an N-periodic signal in the time domain

the analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k \in \mathbb{Z}$$

produces an N-periodic signal in the frequency domain

Discrete Fourier Series (DFS)

$$\text{DFS} = \text{DFT with periodicity explicit}$$

- the DFS maps an N-periodic signal onto an N-periodic sequence of Fourier coefficients
- the inverse DFS maps an N-periodic sequence of Fourier coefficients a set onto an N-periodic signal
- the DFS of an N-periodic signal is mathematically equivalent to the DFT of one period

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N-periodic sequence $\tilde{x}[n]$ with DFS $\tilde{X}[k]$:

- $\tilde{x}_M[n] = \tilde{x}[n - M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\{\tilde{x}_M[n]\}[k] = \boxed{e^{-j\frac{2\pi}{N}Mk}}\,\tilde{X}[k]$ (easy derivation)

a shift in time becomes a *linear phase* factor in frequency

Finite-length time shifts revisited

From the other side: for an N-periodic sequence $\tilde{x}[n]$ with DFS $\tilde{X}[k]$

- define $\tilde{X}_M[k] = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$
- IDFS $\left\{\tilde{X}_M[n]\right\}[n] = \tilde{x}[n-M]$

Finite-length time shifts revisited

For an N-point signal $x[n]$ with DFT $X[k]$:

- $x[n - M]$ is *not* well-defined
- $X_M[k] = e^{-j\frac{2\pi}{N}Mk}X[k]$ is well-defined
- what is the IDFT of $X_M[k]$?

Finite-length time shifts revisited

$$\begin{aligned}
\hat{x}[n] &= \text{IDFT}\left\{ e^{-j\frac{2\pi}{N}Mk} X[k] \right\}[n] \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{-j\frac{2\pi}{N}Mk}\, e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi}{N}mk} \right) e^{-j\frac{2\pi}{N}Mk}\, e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k}
\end{aligned}$$

Always remember the orthogonality of the roots of unity!

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}pk} = \begin{cases} N & \text{if } p \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

Finite-length time shifts revisited

$$\hat{x}[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k}$$

For what values of $0 \leq m < N$ is $(n - M - m)$ a multiple of N?

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} = \begin{cases} N & \text{for } m = (n-M) \mod N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \hat{x}[n] &= \frac{1}{N}\sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} \\ &= x[(n-M) \mod N] \end{aligned}$$

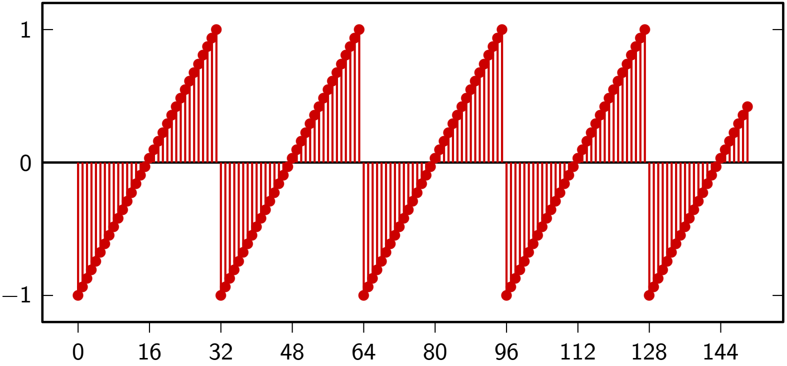
shifts for finite-length signals are "naturally" circular

the Fourier transform for periodic signals

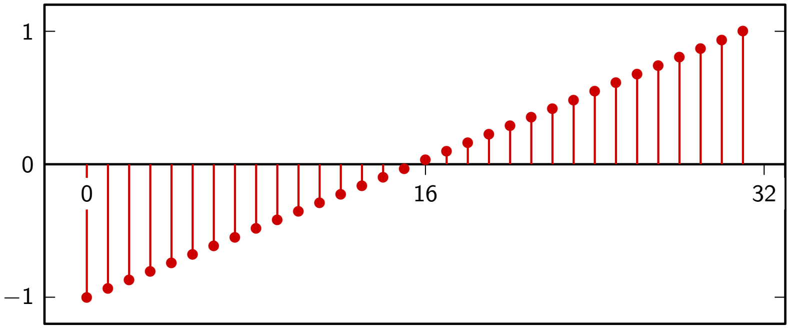
Periodic sequences: a bridge to infinite-length signals

- N-periodic sequence: N degrees of freedom
- DFS: only N Fourier coefficients capture all the information

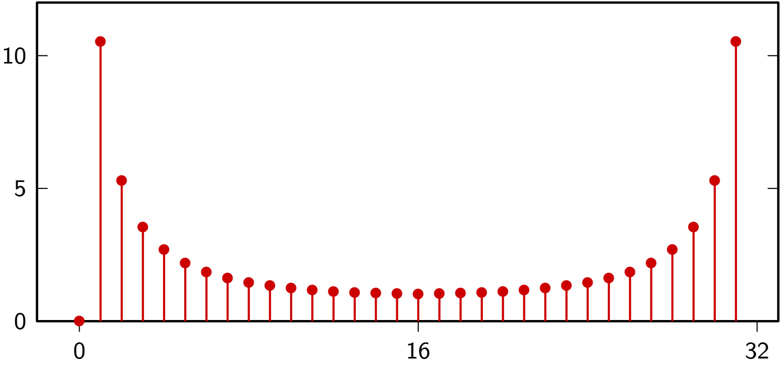
Example: sawtooth signal of period 32



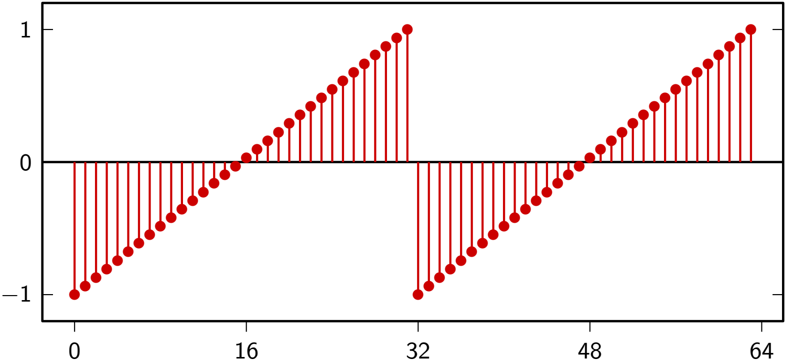
Let's compute the DFT of one period (32 points)



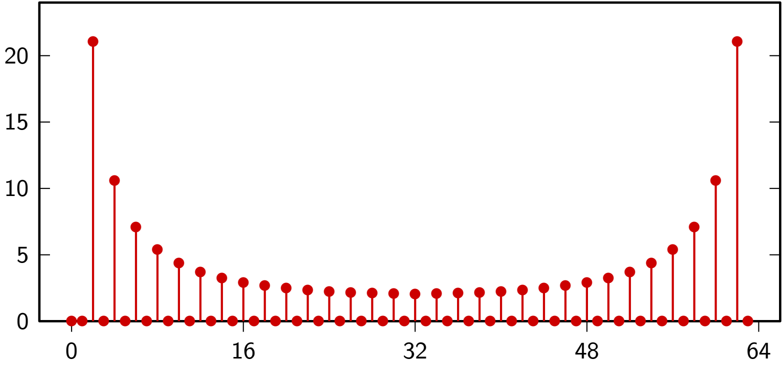
DFT of one period



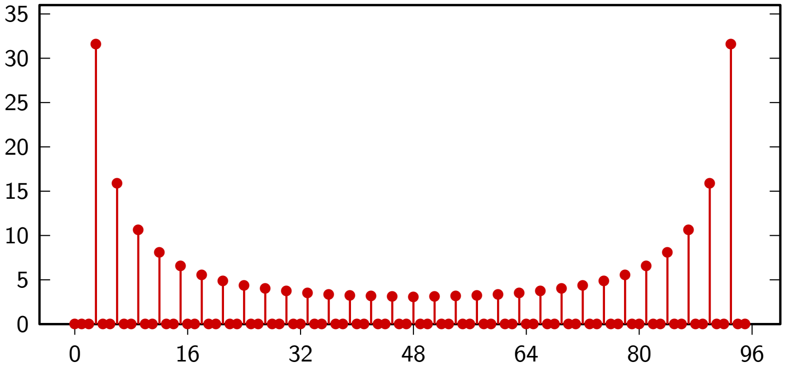
What if we take the DFT of two periods?



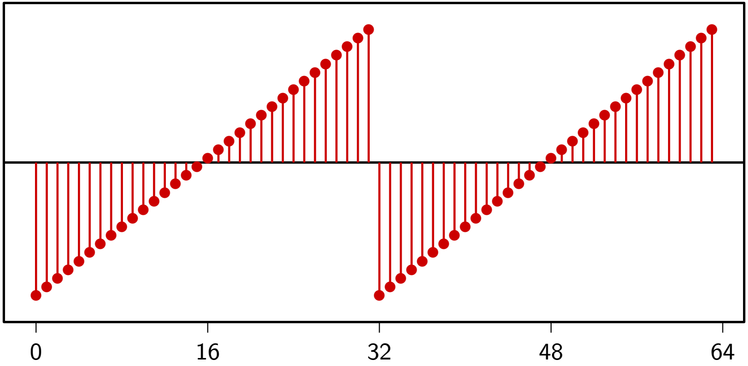
64-point DFT of two periods



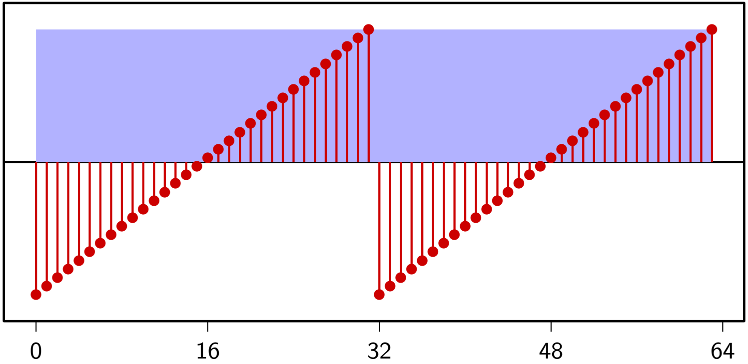
96-point DFT of three periods



DFT of two periods: intuition

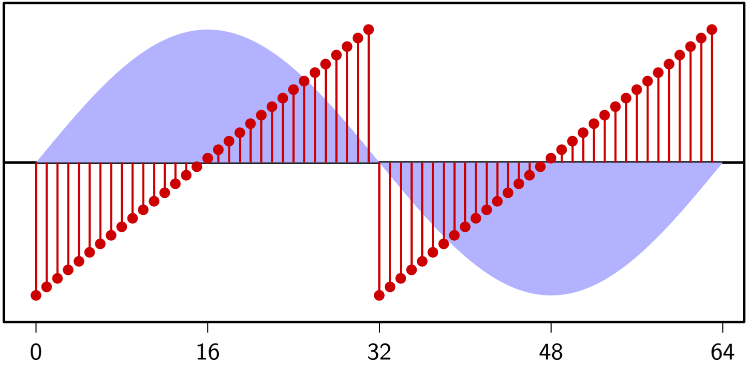


DFT of two periods: intuition



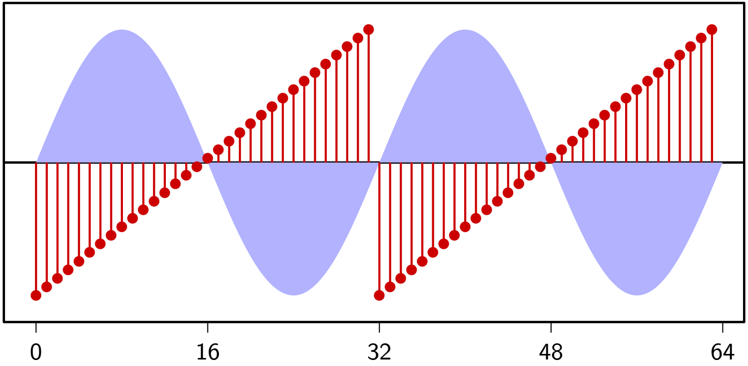
$k = 0$

DFT of two periods: intuition



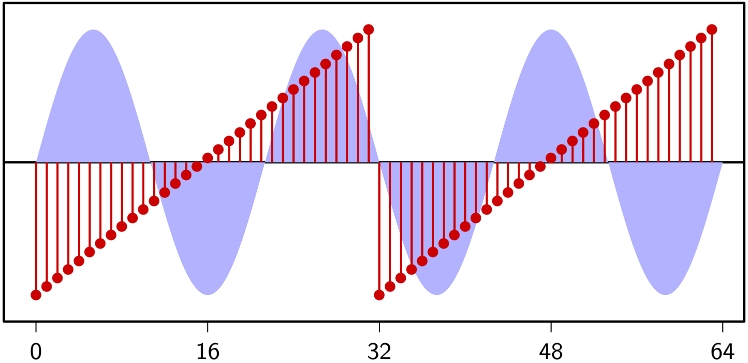
$k = 1$

DFT of two periods: intuition



$k = 2$

DFT of two periods: intuition



$k = 3$

DFT of L periods

ingredients:

- finite-length signal $x[n]$, $n = 0, 1, \ldots, N-1$
- N-periodic signal: $\tilde{x}[n] = x[n \mod N]$
- obviously $\tilde{x}[n] = \tilde{x}[n + pN]$ for all $p \in \mathbb{Z}$

DFT of L periods

$$X_L[k] = \sum_{n=0}^{LN-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} \qquad k = 0, 1, 2 \ldots, LN-1$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n + pN] e^{-j\frac{2\pi}{LN}(n+pN)k}$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{LN}nk}$$

We've seen this before

$$\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} = \begin{cases} L & \text{if } k \text{ multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

DFT of L periods

if k is a multiple of L then k/L is an integer, so:

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}n\frac{k}{L}} = X[k/L]$$

DFT of L periods

$$X_L[k] = \begin{cases} L\,X[k/L] & \text{if } k = 0, L, 2L, 3L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

DFT and DFS

- again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period
- to stress the periodicity of the underlying signal, we use the term DFS

The situation so far

Fourier representation for signal classes:

- N-point finite-length: DFT
- N-point periodic: DFS
- infinite length: ?

the Fourier transform for infinite-length signals

DFT of increasingly long signals

- Start with the DFT. What happens when $N \to \infty$?
- $\frac{2\pi}{N}k$ becomes denser in $[0, 2\pi]$...
- In the limit $\frac{2\pi}{N}k \to \omega$:

$$\sum_n x[n]\, e^{-j\omega n} \quad \omega \in \mathbb{R}$$

- and it still looks like an inner product in $\mathbb{C}^\infty$: $\langle e^{j\omega n}, x[n] \rangle$

Discrete-Time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- inversion (when $X(\omega)$ exists):

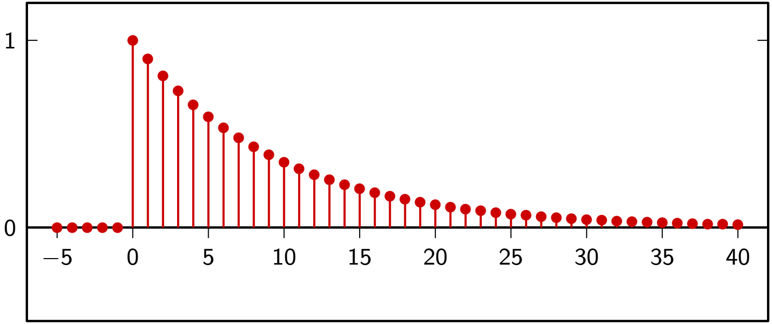
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

DTFT periodicity and notation

- $e^{j\omega n} = e^{j(\omega+2k\pi)n} \quad \forall k \in \mathbb{N}$
- $X(\omega)$ is 2π-periodic
- by convention, $X(\omega)$ is represented over $[-\pi, \pi]$
- to stress periodicity (and for other reasons) another common notation is to write *(See exercise 5 on the homework)*

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$x[n] = \alpha^n u[n], \quad |\alpha| < 1$



DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n}$$

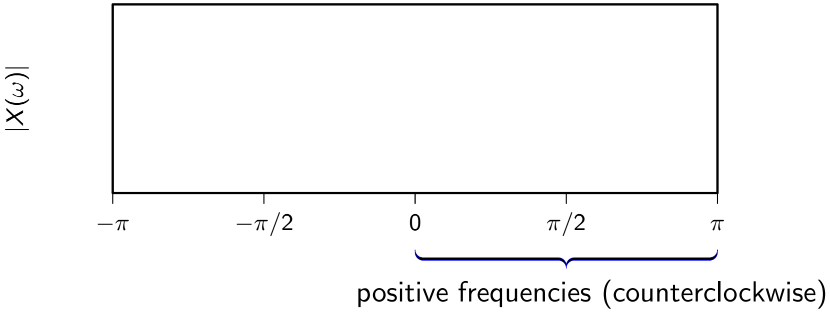
$$= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

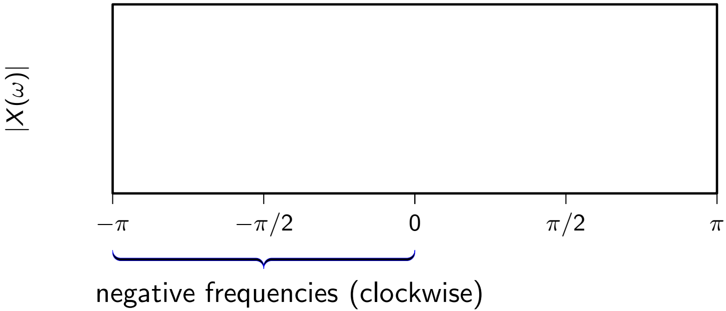
DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$|X(\omega)|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos \omega}$$

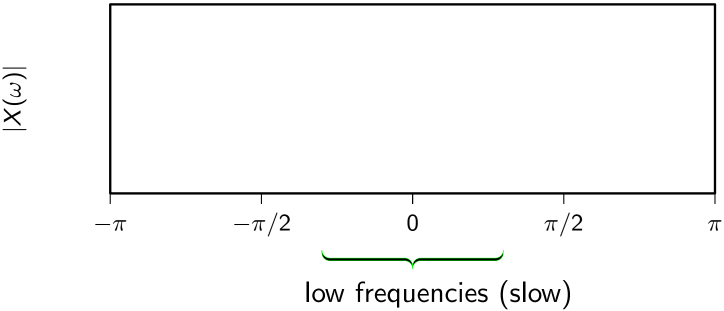
Plotting the DTFT



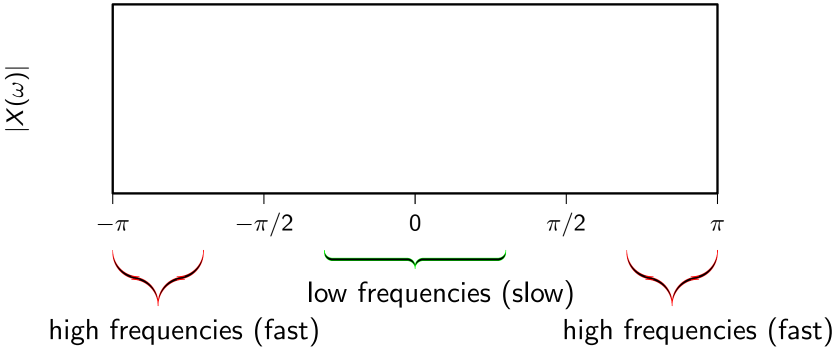
Plotting the DTFT



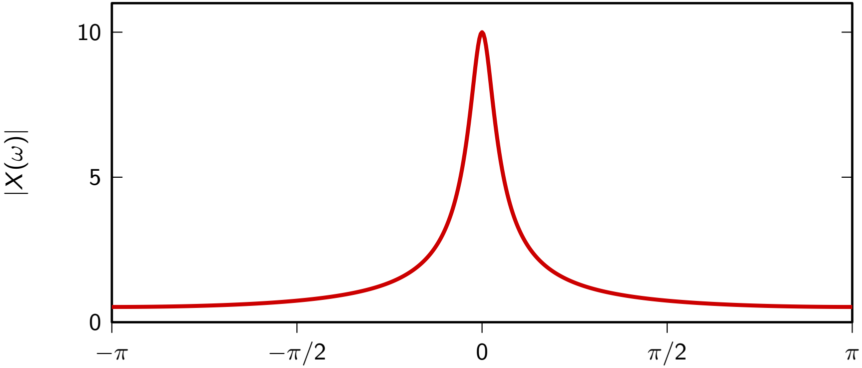
Plotting the DTFT



Plotting the DTFT

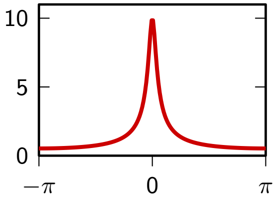


DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

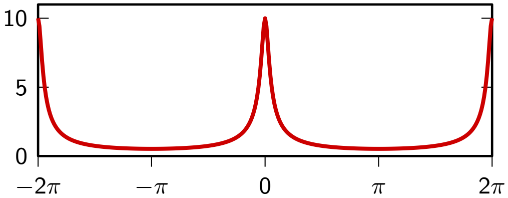


$\alpha = 0.9$

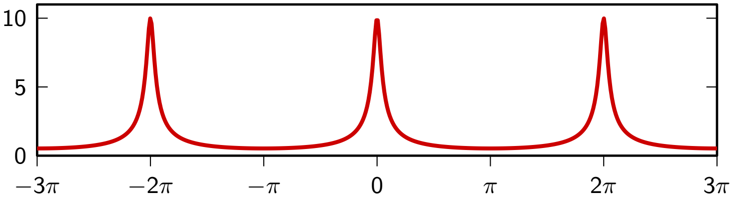
Remember the periodicity!



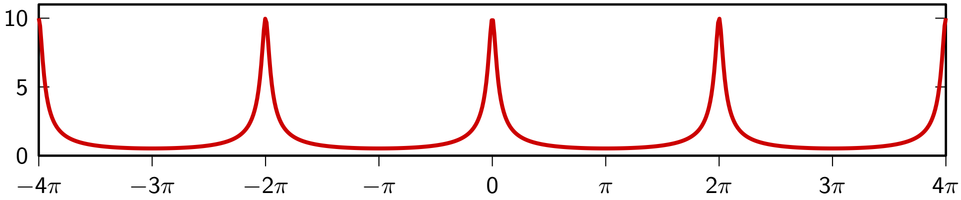
Remember the periodicity!



Remember the periodicity!



Remember the periodicity!



DTFT intuition and properties

Overview:

- DTFT Existence
- Properties
- DTFT as basis expansion

Discrete-Time Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- when does it exist?
- is it a change of basis?

Existence easy for absolutely summable sequences

$$
\begin{aligned}
|X(\omega)| &= \left|\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\right| \\
&\le \sum_{n=-\infty}^{\infty} \left|x[n]\, e^{-j\omega n}\right| \\
&= \sum_{n=-\infty}^{\infty} |x[n]| \\
&< \infty
\end{aligned}
$$

Inversion easy for absolutely summable sequences

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right) e^{j\omega n} d\omega \\ &= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega \\ &= x[n]\end{aligned}$$

Inversion easy for absolutely summable sequences

$$\int_{-\pi}^{\pi} \frac{e^{j\omega m}}{2\pi} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = 1 \qquad \text{for } m = 0$$

$$\begin{aligned} &= \frac{1}{2\pi} \frac{1}{jm} \left. e^{j\omega m} \right|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \frac{1}{jm} \left(e^{j\pi m} - e^{-j\pi m} \right) = 0 \qquad \text{otherwise} \end{aligned}$$

DTFT as a change of basis

Non-rigorous analogies:

- formally the DTFT looks just like an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- the “basis” is an infinite, uncountable set: $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$
- something “breaks down”: we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

The actual deal

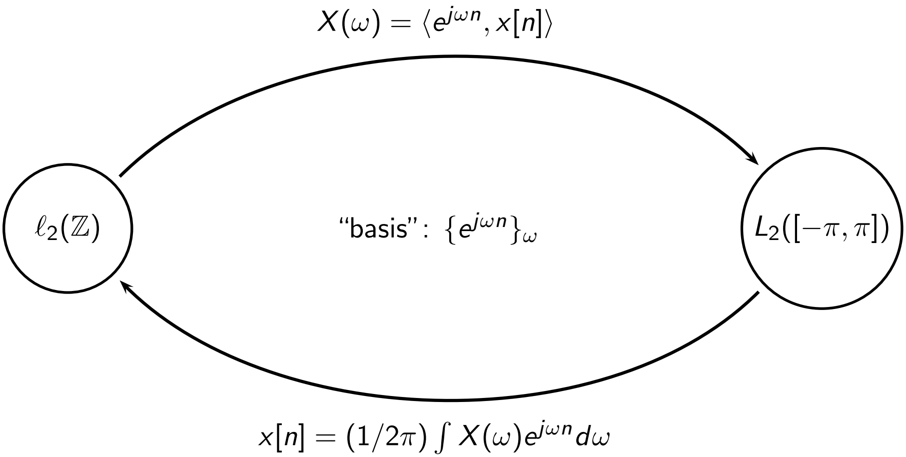
A mathematically precise interpretation

- we *start* in $L_2([-\pi, \pi])$
- the countable set $\{e^{-j\omega n}\}_{n\in\mathbb{Z}}$ is an orthogonal basis for $L_2([-\pi, \pi])$
- basis expansion coefficients are the inner products

$$\langle e^{-j\omega n}, X(\omega)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega = x[n]$$

- any element of $L_2([-\pi, \pi])$ is equivalent to a square summable sequence $x[n] \in \ell_2(\mathbb{Z})$

DTFT



$X(\omega) = \langle e^{j\omega n}, x[n] \rangle$
$\ell_2(\mathbb{Z})$
"basis": $\{e^{j\omega n}\}_\omega$
$L_2([-\pi, \pi])$
$x[n] = (1/2\pi) \int X(\omega) e^{j\omega n} d\omega$

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(\omega) + \beta Y(\omega)$$

- time shift

$$\text{DTFT}\{x[n - M]\} = e^{-j\omega M} X(\omega)$$

- modulation (dual)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(\omega - \omega_0)$$

DTFT properties

- time reversal

$$\text{DTFT}\{x[-n]\} = X(-\omega)$$

- conjugation

$$\text{DTFT}\{x^*[n]\} = X^*(-\omega)$$

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \iff X(\omega) = X(-\omega)$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \iff X(\omega) = X^*(-\omega)$$

- in other words: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \iff |X(\omega)| = |X(-\omega)|$$

- finally, if $x[n]$ is real and symmetric, the DTFT is also real and symmetric!