Chapter 4

The Discrete-Time Fourier Transform

The DFT provides us with a frequency-domain representation of signals that contain a finite amount of data; periodic signals, which are also uniquely described by a finite number of data samples, use a formally identical frequency representation that, for clarity, we call the DFS. In both cases, the transform is a straightforward change of basis in $\mathbb{C}^N$ and, as such, it can be described algorithmically as a matrix-vector multiplication involving a finite number of operations.

We now consider the problem of obtaining a frequency-domain representation for aperiodic, infinite-length sequences. Such signals are an idealized mathematical concept, of course, but they are indispensable if we want to prove theoretical results that hold independently of the size of the data set and, most importantly, that are still true in the limit case of an infinite number of samples. Obviously we would very much like for this new transform to be a simple extension of the DFT, that is, we would like it to retain its intuitive interpretation as a measure of similarity (that is, an inner product) between an input signal and a set of oscillatory components. While this is ultimately the case, from the mathematical point of view unfortunately something "breaks down" and, as we move from finite to infinite lengths, we won't be able to obtain a Fourier transform that acts as an endomorphism on the space of discrete-time sequences; namely, while the frequency representation of a time-domain signal in $\mathbb{C}^N$ is itself a vector in $\mathbb{C}^N$, the Fourier transform of a sequence will not be a sequence but a function of a real variable. Although the exact reasons for this are quite technical and beyond the scope of this book, we will try to provide some intuition in the next pages.

4.1 Definition

The frequency-domain representation of an infinite-length, discrete-time signal $\mathbf{x}$ is called the Discrete-Time Fourier Transform (DTFT) and it is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}. \tag{4.1}$$

Formally, the DTFT can be seen as an operator that maps discrete-time sequences to complex-valued functions of the frequency variable $\omega \in \mathbb{R}$; since the argument ω only appears in the phase of the complex exponentials in the sum, a DTFT is always 2π-periodic. To stress out this property[1] many authors use the notation $X(e^{j\omega})$ to indicate a DTFT, and we may do so as well if the context calls for it. Finally, when no confusion can arise, we will also use the compact vector notation

$$\mathbf{x} \stackrel{\text{DTFT}}{\longleftrightarrow} \mathbf{X}.$$

Existence. The DTFT is guaranteed to exist if the sum in (4.1) converges, that is, if the partial sum

$$X_N(\omega) = \sum_{n=-N}^{N} x[n]\, e^{-j\omega n} \tag{4.2}$$

is finite for all values of ω as N grows to infinity. Convergence is very easy to prove for *absolutely summable* sequences, that is for sequences satisfying

$$\lim_{N\to\infty} = \sum_{n=-N}^{N} \left|x[n]\right| < \infty \tag{4.3}$$

since, according to the triangle inequality,

$$\left|X_N(\omega)\right| \leq \sum_{n=-N}^{N} \left|x[n]\, e^{-j\omega n}\right| = \sum_{n=-N}^{N} \left|x[n]\right|. \tag{4.4}$$

For this class of sequences it can also be proved that the convergence of $X_N(\omega)$ to $X(\omega)$ is uniform and that $X(\omega)$ is continuous. While absolute summability is a sufficient condition for the existence of the DTFT, it can be shown that the sum in (4.2) exists also for all sequences in $\ell_2(\mathbb{Z})$, the space of *square-summable* (i.e., finite-energy) sequences. For square-summable signals that are not absolutely summable, however, the convergence of (4.2) is no longer uniform but takes place only in the mean-square sense, i.e.

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left|X_N(\omega) - X(\omega)\right|^2 d\omega = 0. \tag{4.5}$$

This type of convergence implies that, while the total energy of the difference between functions goes to zero, the functions may differ in value over a countable set of points.[2] This also implies that, in this case, $X(\omega)$ is no longer guaranteed to be continuous.

[1] And also to provide a convenient notational framework which unifies the DTFT and the z-transform, as we will see in Chapter **??**.

[2] A particular manifestation of this behavior is called the *Gibbs phenomenon*, which has important consequences in the problem of filter design, as we will study in Chapter **??**.

Inversion. The DTFT, when it exists, can be inverted via the integration

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega; \tag{4.6}$$

this can be easily verified by substituting (4.1) into (4.6) and recalling that

$$\int_{-\pi}^{\pi} e^{j\omega(n-k)} = 2\pi\, \delta[n-k].$$

In fact, due to the 2π-periodicity of the DTFT, the integral in (4.6) can be computed over *any* 2π-wide interval on the real line; by convention, though, the DTFT is generally represented over the $[-\pi, \pi]$ interval.

4.2 Intuitive interpretations

As we said, we would like the DTFT to emerge as a natural extension of the DFT, whose simplicity and whose properties we are now familiar and happy with. There are at least three rather intuitive paths that lead from the DFT to the DTFT, although (be forewarned) all of them will rely at some point on a certain amount of willful carelessness with mathematical precision.

4.2.1 The DTFT as the limit of a DFT.

The simplest approach to the DTFT considers an infinite sequence as the limit of a finite-length signal whose length grows larger and larger. In the analysis formula for an N-point DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, \ldots, N-1,$$

the fundamental frequency for the space is $\omega_N = 2\pi/N$. As N grows large, ω_N becomes infinitesimally small and so its multiples $k\omega_N$ will become infinitely dense in the $[0, 2\pi]$ interval, much like a real-valued variable would. By replacing $2k\pi/N$ by ω and by shifting the reference interval to $[-\pi, \pi]$, the metamorphosis is complete. The same limiting argument can be carried out for the synthesis formula, as in Exercise **??**.

4.2.2 The DTFT as the limit of a DFS.

An aperiodic sequence can also be interpreted as the limit of a periodic signal whose period grows to infinity. Pick an absolutely summable sequence $\mathbf{x}$ and build the N-periodic signal

$$\tilde{x}_N[n] = \sum_{p=-\infty}^{\infty} x[n + pN] \tag{4.7}$$

with $N > 0$; since $\mathbf{x}$ is absolutely summable, the above expression will be finite for all n (see Example **??**). As N grows larger, the copies that make up the periodic signal will be spaced further and further apart and, in the limit,

$$\lim_{N\to\infty} \tilde{\mathbf{x}}_N = \mathbf{x}.$$

For all finite values of N, the natural frequency representation for $\tilde{\mathbf{x}}_N$ is its DFS:

$$\tilde{X}_N[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}nk} = \sum_{p=-\infty}^{\infty} \left(\sum_{n=0}^{N-1} x[n+pN]\, e^{-j\frac{2\pi}{N}(n+pN)k} \right); \tag{4.8}$$

in the derivation, we have used the definition of $\tilde{\mathbf{x}}_N$ and exploited the fact that $e^{-j(2\pi/N)nk} = e^{-j(2\pi/N)(n+pN)k}$. Now, for every value of p in the outer sum, the value $(n+pN)$ in the inner sum ranges from pN to $pN+N-1$ so that the double sum can be simplified to

$$\tilde{X}_N[k] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\frac{2\pi}{N}kn}. \tag{4.9}$$

By defining the following *function* of a real-valued variable ω

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \tag{4.10}$$

we can express the DFS of $\tilde{\mathbf{x}}_N$ as the set of N regularly spaced samples[3]

$$\tilde{X}_N[k] = X\left(\frac{2\pi}{N}k\right);$$

this is shown in Figure 4.1 for different values of N. As N grows large, the set of samples will grow denser in the $[0, 2\pi]$ interval; since, in the limit, $\tilde{\mathbf{x}}_N$ tends to $\mathbf{x}$, it appears that the frequency-domain representation for $\mathbf{x}$ is indeed the 2π-periodic function $X(\omega)$.

4.2.3 The DTFT as a formal change of basis.

The analysis formula in (4.1) has the same formal structure as the inner product between the infinite sequence $\mathbf{x}$ and a complex exponential at frequency ω; according to our usual interpretation, every DFT value thus measures the similarity between the input sequence $\mathbf{x}$ and an infinitely long oscillation at frequency ω. It's tempting at this point to simply interpret the DTFT as another change of basis, this time in $\ell_2(\mathbb{Z})$, where the new "basis vectors," indexed by $\omega \in \mathbb{R}$, are the set $\left\{e^{j\omega n}\right\}_\omega$ and where the expansion coefficients are given by

$$X(\omega) = \langle e^{j\omega n}, \mathbf{x} \rangle. \tag{4.11}$$

[3]Once again, it is instructive to notice the time-frequency duality pattern: in Section **??** we showed how a maximally compact time-domain signal possesses a maximally wide spectrum, and vice-versa; here we see that increasing the period spacing in time decreases the spacing of samples in frequency.

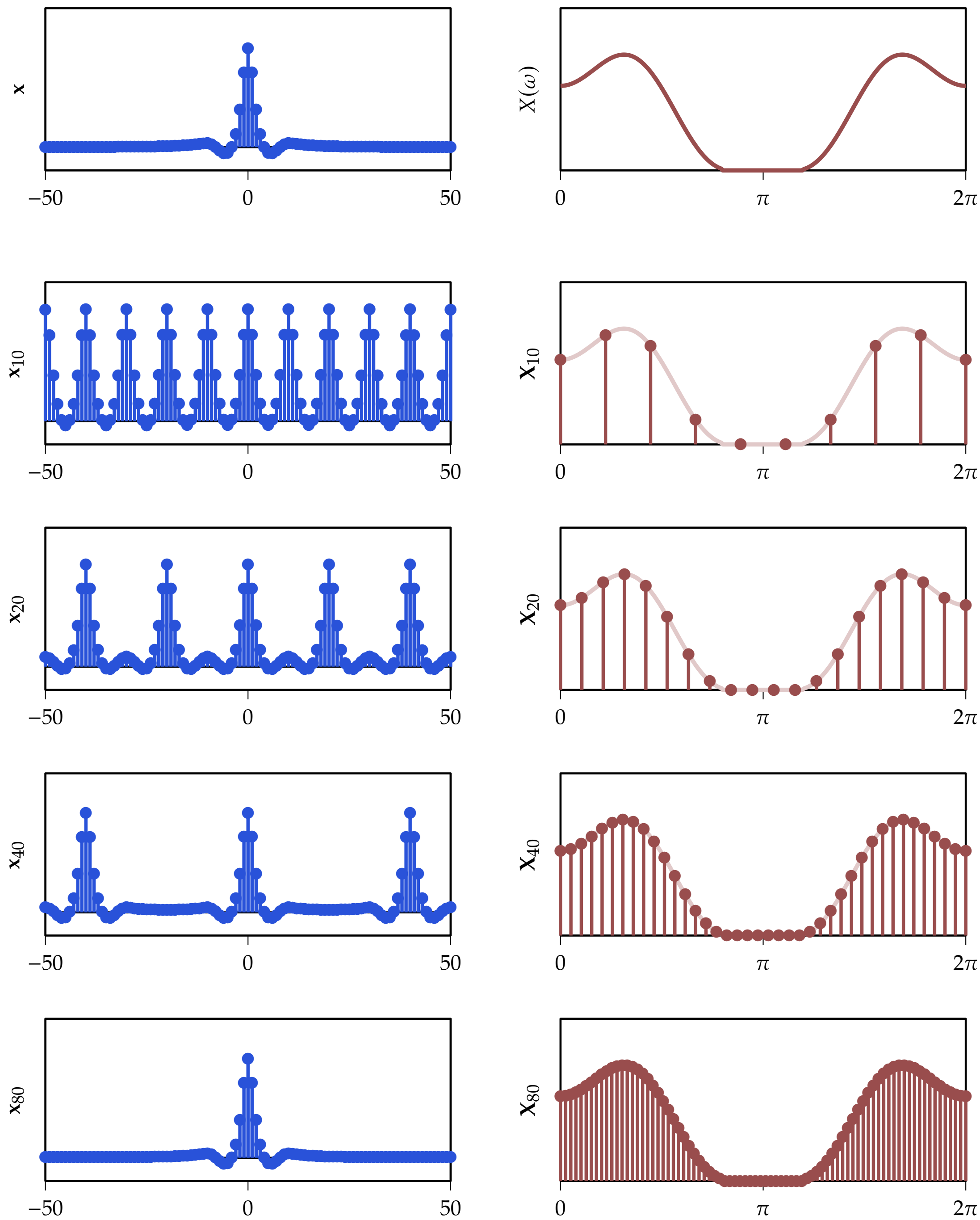

Figure 4.1: *top row: original infinite-length, absolutely summable signal (left) and the function $X(\omega)$ defined in (4.10); rows 2-5: periodized signal $\mathbf{x}_N$ and its DFS for increasing values of N; all DFS values are samples of $X(\omega)$ at multiples of $2\pi/N$.*

While appealing, unfortunately this interpretation is not mathematically correct because those "basis vectors" are not even elements of $\ell_2(\mathbb{Z})$ (their energy is infinite); additionally, while a basis can indeed contain an infinite number of elements, they should form at least a *countable* set, that is, they should be indexable by an integer variable. And yet, the DTFT *is* in fact a change of basis — the only trick is that we have to look it *backwards*, from the frequency domain to the time domain, as we will show in the next section.

4.3 The DTFT as a basis expansion

In Section **??** we showed how the elements of a vector space can be standard functions (rather than Euclidean tuples) and the set of square-integrable functions over the $[-\pi, \pi]$ interval is a Hilbert space known as $L_2([-\pi, \pi])$. A vector $\mathbf{F} \in L_2([-\pi, \pi])$ is thus a complex-valued function for which the integral

$$\int_{-\pi}^{\pi} |F(\omega)|^2 \, d\omega < \infty.$$

The usual definition for the inner product in this space is

$$\langle \mathbf{F}, \mathbf{G} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F^*(\omega) G(\omega) d\omega \tag{4.12}$$

and with this it can be shown that the set

$$\left\{ e^{-j\omega n} \right\}_{n \in \mathbb{Z}} \tag{4.13}$$

forms an orthogonal basis[4]. Note that the set of basis functions is countable (it is indexed by an integer) and that each element is obviously square-integrable over the finite interval $[-\pi, \pi]$.

Now, since the DTFT is 2π-periodic, it is completely defined by its values over any 2π-wide interval and we can certainly choose this interval to be $[-\pi, \pi]$. And so, a DTFT that is square-integrable over $[-\pi, \pi]$ can be considered a bona-fide element of $L_2([-\pi, \pi])$. With this in mind, the DTFT inversion formula in (4.6) reads as the inner product between an element of $L_2([-\pi, \pi])$ and the n-th basis vector for the space. In other words, any square-integrable DTFT can be decomposed into a linear combination of vectors from (4.13) and the coefficients of the expansion form the discrete-time sequence

$$x[n] = \langle e^{-j\omega n}, X(\omega) \rangle. \tag{4.14}$$

While this inverted viewpoint may seem just a bit of mathematical sophistry, it actually allows us to legitimately invoke the properties of Hilbert space for the DTFT and, in particular, the conservation of energy guaranteed by Parseval's theorem:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega. \tag{4.15}$$

[4]While orthogonality is elementary to show, the really tricky part is proving completeness. Please refer to [**?**] for details.

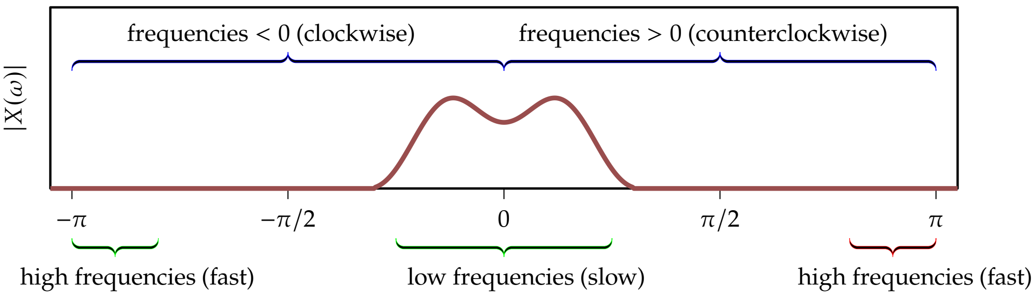

Figure 4.2: *Reading a DTFT plot.*

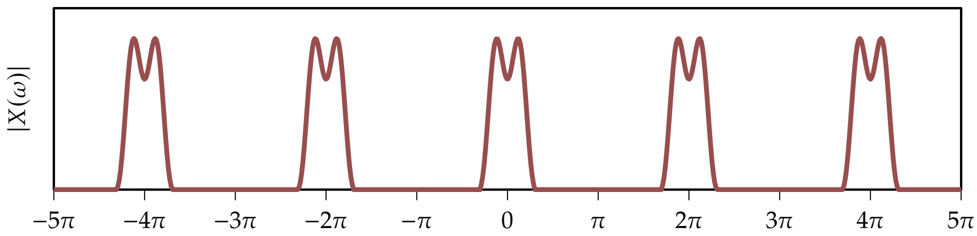

Figure 4.3: *Every DTFT is a 2π-periodic function.*

This means that the square magnitude of the DTFT encodes the signal's distribution of energy in the frequency domain, just as in the case of the DFT; and this also guarantees that, given a DTFT in $L_2([-\pi, \pi])$, the associated time-domain sequence will necessarily be in $\ell_2(\mathbb{Z})$.

4.4 Examples

The DTFT of a discrete-time signal is a complex-valued 2π-periodic function of a real-valued frequency variable. Just as we discussed in Section **??**, DTFT plots display separately the magnitude and the phase whose physical significance remains unchanged with respect to the DFT. In this case however the range of choice for the frequency variable is the zero-centered interval $[-\pi, \pi]$, which clearly shows the contribution of positive (counterclockwise) and negative (clockwise) frequencies as shown in Figure 4.2.

It is important to remember that the DTFT is always 2π-periodic and therefore, if the plot encompasses a larger range of frequencies, both magnitude and phase repeat *ad infinitum*, as shown in Figure 4.3.

We will now look at the DTFT of some elementary signals.

Impulse. The DTFT of a discrete-time delta is a complex exponential:

$$\text{DTFT}\{\boldsymbol{\delta}_m\} = \sum_{n=-\infty}^{\infty} \delta[n-m]e^{-j\omega n} = e^{-j\omega m};$$

the magnitude of the transform is constant and equal to one, whereas the phase is linear and proportional to m. Again, as we saw for the DFT, the Fourier transform the most "concentrated" signal in time is nonzero over its entire freqency support.

Rectangular signal. An infinite-length rectangular sequence is a signal that is nonzero only over a finite number of contiguous time indexes:

$$x[n] = \begin{cases} 1 & 0 \le n < M \\ 0 & \text{otherwise} \end{cases}$$

Its DTFT is:

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{M-1} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} \\ &= \frac{e^{-j\omega M/2}\left(e^{j\omega M/2}\right) - e^{-j\omega M/2})}{e^{j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \\ &= \frac{\sin\left(\omega M/2\right)}{\sin(\omega/2)} e^{-j\omega(M-1)/2}. \end{aligned}$$

Again, we have manipulated the expression for the transform into a product of a real-valued term and a pure phase term; this allows us to easily plot the DFT as in Figure 4.4; note how the magnitude is zero at all nonzero multiples of $2\pi/M$ and how the sign changes in the magnitude at those points determine an additional shift of $\pm\pi$ in the phase.

Exponential decay. Consider the exponential decay defined in section **??**,

$$x[n] = a^n u[n];$$

for $|a| < 1$ the sequence is absolutely summable and its DTFT is easily computed as

$$X(\omega) = \frac{1}{1-ae^{-j\omega}}. \tag{4.16}$$

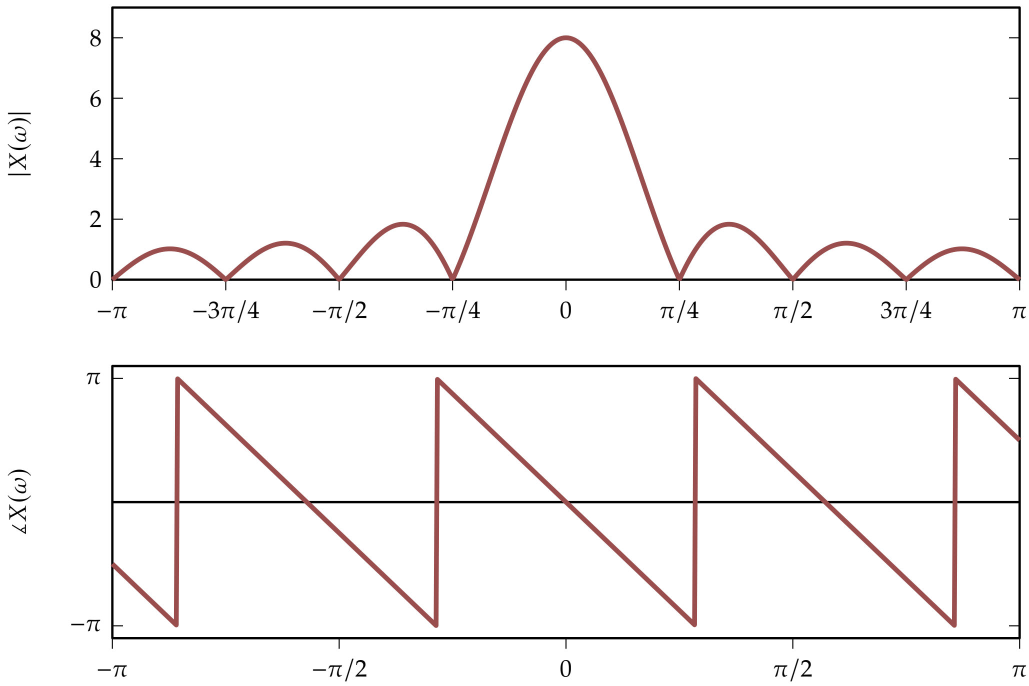

Figure 4.4: *DTFT of an 8.0 -point rectangular signal.*

Magnitude and phase are, respectively,

$$\left|X(\omega)\right|^2 = \frac{1}{1 + |a|^2 - 2a\cos(\omega)} \tag{4.17}$$

$$\measuredangle X(\omega) = \arctan\left[\frac{-a\sin(\omega)}{1 - a\cos(\omega)}\right] \tag{4.18}$$

and they are plotted in Figure 4.5.

4.5 Properties of the DTFT

In this section we will list, without formal proof, a series of elementary properties for the DTFT, which you are encouraged to verify using the definition of the transform.

Linearity. The DTFT is a linear operator:

$$ax[n] + by[n] \overset{\text{DTFT}}{\longleftrightarrow} aX(\omega) + bY(\omega) \tag{4.19}$$

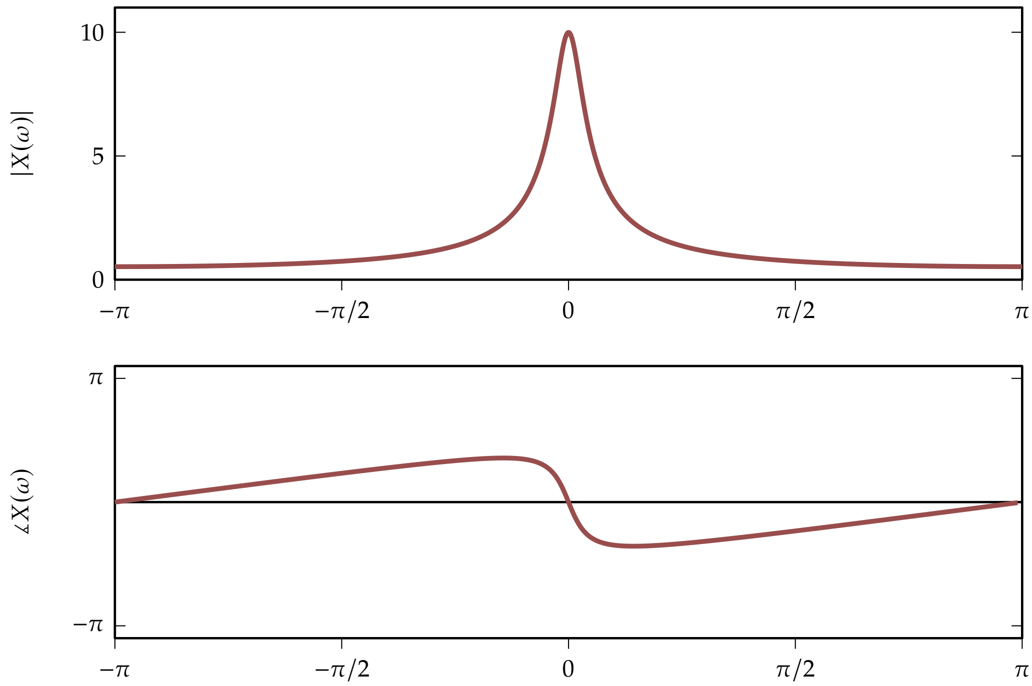

Figure 4.5: DTFT of the exponential decay for $a = 0.9$.

Time shift. The shift of a signal in time correspond to multiplication by a linear phase factor in frequency:

$$x[n-n_0] \stackrel{\text{DTFT}}{\longleftrightarrow} e^{-j\omega n_0}\, X(\omega). \tag{4.20}$$

Frequency shift. Multiplication in the time domain by a linear phase term corresponds to a shift in frequency:

$$x[n]\, e^{j\omega_0 n} \stackrel{\text{DTFT}}{\longleftrightarrow} X(\omega-\omega_0) \tag{4.21}$$

and, by linearity,

$$x[n]\, \cos(\omega_0 n) \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right].$$

Time and frequency reversal. Reversing a signal in time reverses its transform:

$$x[-n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(-\omega) \tag{4.22}$$

Conjugating the time-domain signal results in both conjugation and reversal of the DTFT:

$$x^*[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X^*(-\omega) \tag{4.23}$$

Symmetries. In the case of infinite-length sequences and periodic functions, symmetries (and antisymmetries) are intended around zero and have a straightforward definition. With this,

- the DTFT of a symmetric signal is symmetric
- the DTFT of a real-valued signal is Hermitian-symmetric
- the magnitude DTFT of a real-valued signal is symmetric.
- the DTFT of a real-valued symmetric signal is real-valued and symmetric.

4.6 The DTFT of power signals

The DTFT is mathematically well-defined for all finite-energy sequences and so every signal in $\ell_2(\mathbb{Z})$ is *uniquely* associated to a square-integrable function in $L_2([-\pi, \pi])$. As we mentioned in Section **??**, we are also interested in *power* signals such as a the unit sequence, the unit step, and sinusoidal oscillations; these signals are not square summable but their energy per unit of time remains finite. In this section we will describe how the DTFT formalism can be extended to power signals as well; this is achieved by means of the *Dirac delta*, a generalized function that, with adequate care, can be manipulated almost like a standard function. The resulting generalized spectrum of a power signal, which always contains one or more Dirac deltas, will have the same properties of a standard DTFT and will behave as expected with respect to frequency-domain operations such as filtering and modulation.

4.6.1 The Dirac delta

The Dirac delta is a quirky mathematical object that has gained widespread use in applied sciences as the model for idealized impulsive phenomena. While not a regular function, it is often used as one via the notation $\delta(t)$; it can be formally defined in terms of its *sifting property*, valid for any sufficiently well-behaved function f:

$$\int_{-\infty}^{+\infty} \delta(t-\tau) f(\tau) d\tau = f(t). \tag{4.24}$$

While no standard function possesses this property, we can interpret the Dirac delta as a shorthand for a limiting operation in which the integral is computed using increasingly "peaked" and increasingly "narrow" regular functions; informally, as we will see in the following examples, $\delta(t)$ can be described as a sort of function with unit area and that is zero everywhere except in the origin $t = 0$ where the amplitude grows to infinity[5].

Physical impulses. In physics textbooks, quite a few exercises in classical mechanics involve *"a particle with mass m, initially at rest, that begins to move at time $t_0 = 0$ with velocity v_0..."* If the particle starts to move, it means that a force $F(t)$ acted upon it, producing an acceleration $a(t) = F(t)/m$; the particle's velocity over time is the antiderivative of its acceleration and so

$$v(t) = \int_{-\infty}^{t} a(t)\,dt = \begin{cases} 0 & t < t_0 \\ v_0 & t \geq t_0 \end{cases}$$

(assuming velocity remains constant after t_0). The accelerating force, therefore, seems to be a function with infinitesimally small support but with finite area; if such a function existed, what would it look like? If we write out the expression for the particle's acceleration we have

$$a(t) = \lim_{\epsilon \to 0} \frac{v(t) - v(t-\epsilon)}{\epsilon} = \begin{cases} 0 & t < t_0 \\ 0 & t > t_0 \\ \displaystyle\lim_{\epsilon \to 0} \frac{v_0 - 0}{\epsilon} \stackrel{?}{=} \infty & t = t_0 \end{cases}$$

which we can interpret as a "function" that is zero everywhere except for a point of infinite amplitude in t_0: not a standard function for sure.

Mathematical impulses. Many scalar functional descriptors are defined as the integral of the product between the function and a given *kernel*[6]. For instance, the n-th moment of a function is defined as

$$m_n = \int_{-\infty}^{+\infty} t^n f(t)\,dt;$$

if $n = 0$ the kernel is $t^0 = 1$ and we obtain the function's total area, whereas for $n = 1$ the kernel is t and we obtain the function's center of mass. As another example, consider a family of real-valued kernels, parametrized by a positive number α, and defined as

$$r_\alpha(t) = \begin{cases} 1/\alpha & |t| \leq \alpha/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.25}$$

Each kernel is nonzero only over the symmetric interval $[-\alpha/2, \alpha/2]$ and all kernels have unit area independently of α; some examples are plotted in Figure 4.6. By integrating the

[5]As a passing but obligatory remark, please be mindful of the profound difference between the Dirac delta and the by now familiar discrete-time delta sequence $\delta[n]$, also known as the *Kronecker delta*. The latter is a perfectly normal, finite-energy sequence indexed by an integer variable, whereas the former is a generalized function of a real variable. In discrete-time signal processing, the Dirac delta only appears in the frequency domain and its argument is always a real-valued frequency variable ω.

[6]That is to say, the inner product between kernel and fucntion in the vector space where both objects live.

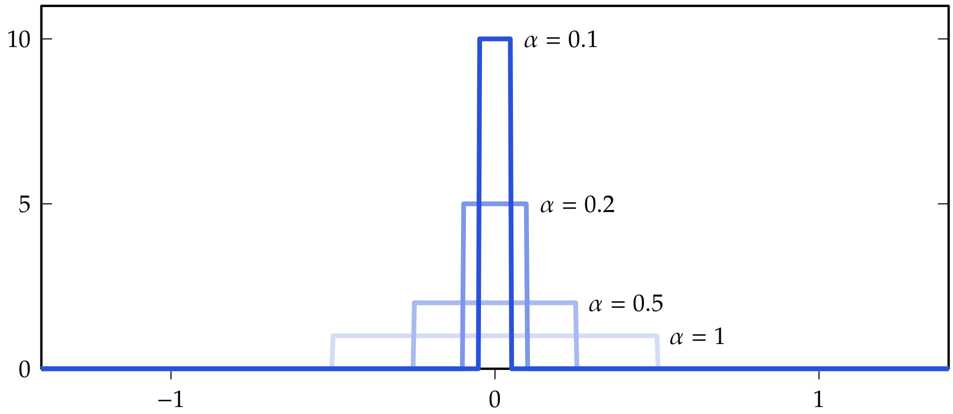

Figure 4.6: *Family of constant-area indicator functions* $r_\alpha(t)$.

product of $r_\alpha(t)$ with any well-behaved function we obtain the average value of f over $[-\alpha/2, \alpha/2]$:

$$\int_{-\infty}^{\infty} r_\alpha(t) f(t)\, dt = \frac{1}{\alpha} \int_{-\alpha/2}^{\alpha/2} f(t)\, dt$$

At the same time, the mean value theorem[7] guarantees that there always exists a value $|\gamma| \leq \alpha/2$ such that

$$\int_{-\alpha/2}^{\alpha/2} f(t)\, dt = \alpha\, f(\gamma). \tag{4.26}$$

If we now use smaller and smaller values for α, in the limit the range of possible values for γ gets "squeezed" to the single value zero so that in the end

$$\lim_{\alpha \to 0} \int_{-\infty}^{\infty} r_\alpha(t) f(t)\, dt = f(0).$$

With much mathematical daring we could move the limit inside the integral and pretend that there exists a "function" $\delta(t)$ such that

$$\lim_{\alpha \to 0} r_\alpha(t) = \delta(t);$$

this would lead to the expression

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

[7]In particular, we refer here to the mean value theorem for definite integrals: *if f is a continuous function over the interval $[a, b]$, then there exists $c \in [a, b]$ such that*

$$\int_a^b f(t)\, dt = (b - a) f(c).$$

namely, we'd have defined a kernel that, via integration, can extract the value of a function in $t = 0$. Again, $\delta(t)$ doesn't appear to be a standard function and, quite interestingly, it seems to possess the same properties as our earlier model for an instantaneous acceleration:

- $\delta(t) = 0$ for $t \neq 0$ and $\delta(t) \to \infty$ for $t = 0$
- the integral of $\delta(t)$ is equal to one if the integration interval includes $t = 0$, and zero otherwise

4.6.2 The generalized DTFT of the unit sequence

Consider a family of double-sided decaying exponentials $\mathbf{d}_a$, with $d_a[n] = a^{|n|}$ and $0 < a < 1$. Each sequence belongs to $\ell_2(\mathbb{Z})$ since

$$\sum_{n=-\infty}^{\infty} |a^{|n|}|^2 = \frac{1+a^2}{1-a^2}$$

and its DTFT is (see Exercise **??**)

$$D_a(\omega) = \frac{1-a^2}{1-2a\cos\omega + a^2}; \tag{4.27}$$

note that the area under $D_a(\omega)$ is independent of a and equal to 2π since

$$\int_{-\pi}^{\pi} D_a(\omega)d\omega = 2\pi\, d_a[0] = 2\pi.$$

Figure 4.7 shows a few DTFT pairs $\mathbf{d}_a \leftrightarrow \mathbf{D}_a$ for different values of a and it is obvious that, as a approaches one, the finite-energy sequences $\mathbf{d}_a$ tend to the infinite energy, constant sequence $\mathbf{1}$:

$$\lim_{a\to 1} \mathbf{d}_a = \mathbf{1};$$

at the same time, the corresponding Fourier transforms $\mathbf{D}_a$ see their their maximum value $D_a(0)$ grows towards infinity while their constant area becomes incresingly and more concentrated around $\omega = 0$:

$$\lim_{a\to 1} D_a(\omega) = \begin{cases} \lim_{a\to 1} \dfrac{1+a}{1-a} & \omega = 0 \\ \\ 0 & \omega \neq 0. \end{cases}$$

By analogy with the family of finite-support kernels $r_a(t)$ defined in the previous example, we could already posit that $\mathbf{D}_a$ will be some form of Dirac delta; but we can make this argument much stronger since, for any value of a, $\mathbf{d}_a \in \ell_2(\mathbb{Z})$ and $\mathbf{D}_a \in L_2([-\pi,\pi])$. If we pick any other finite-energy sequence $\mathbf{x}$ and compute its inner product in $\ell_2(\mathbb{Z})$ with $\mathbf{d}_a$,

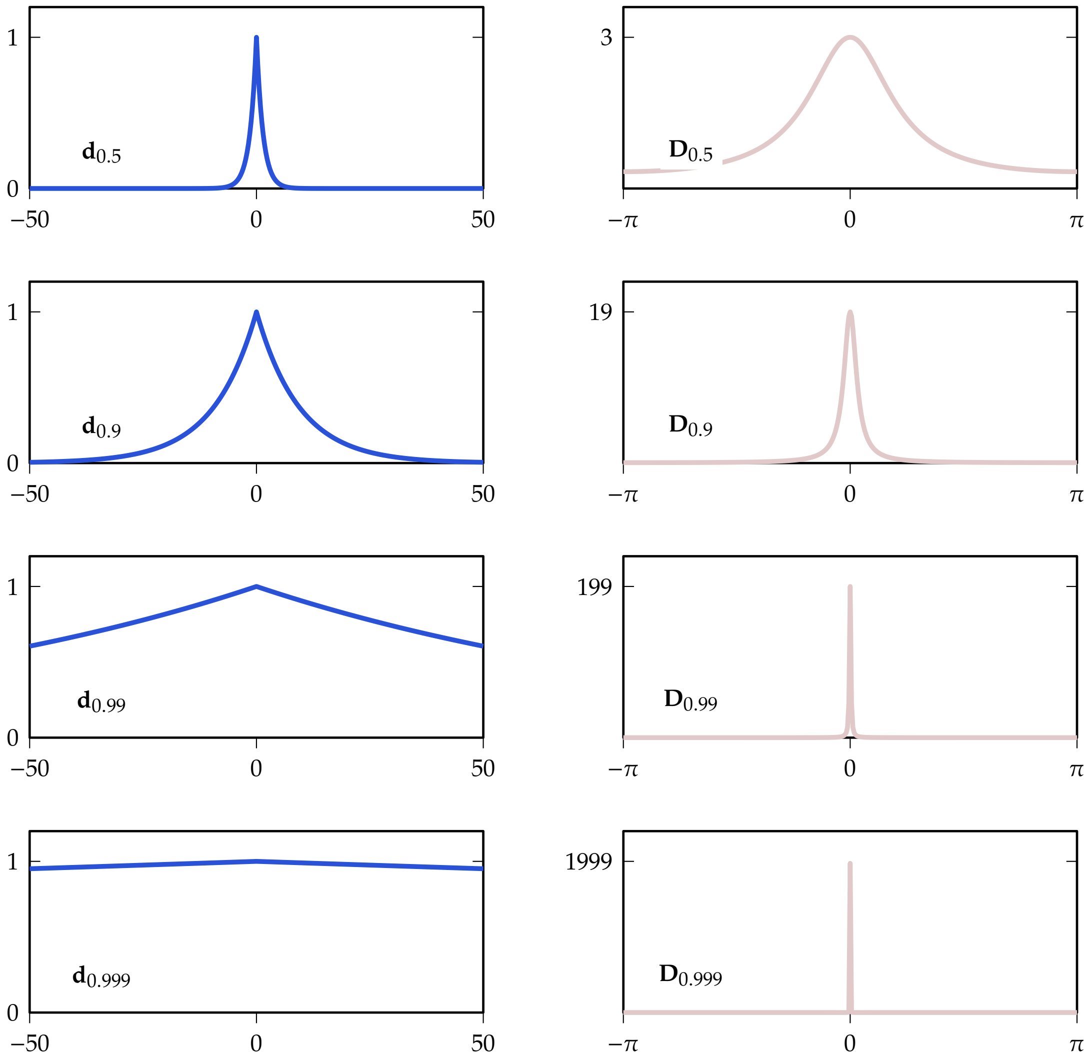

Figure 4.7: *Double-sided complex exponential sequences $d_a[n] = a^{|n|}$ and their DTFTs for increasing values of a.*

Parseval's theorem guarantees that the result will be preserved across the change of basis operated by the DTFT and will be equal to the inner product of the respective Fourier transforms in $L_2([-\pi, \pi])$:

$$\langle \mathbf{d}_a, \mathbf{a} \rangle = \langle \mathbf{D}_a, \mathbf{X} \rangle.$$

This can be written out explicitly as[8]

$$\sum_{n=-\infty}^{\infty} d_a[n]x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_a(\omega)X(\omega)d\omega$$

and if we now take the limit for $a \longrightarrow 1$ the left-hand side becomes

$$\lim_{a\to 1}\langle \mathbf{d}_a, \mathbf{x} \rangle = \sum_{n=-\infty}^{\infty} x[n] = X(0)$$

which implies

$$\lim_{a\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_a(\omega)X(\omega)d\omega = X(0).$$

With the same mathematical daring that we deployed earlier, we satisfy this equation by setting

$$\lim_{a\to 1} D_a(\omega) = 2\pi\delta(\omega)$$

which formally defines the DTFT of the unit signal as a scaled Dirac delta. To be precise, since DTFTs are by definition 2π-periodic, let's build a frequency-domain, 2π-periodic Dirac delta $\tilde{\boldsymbol{\delta}}$ as

$$\tilde{\delta}(\omega) = 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - 2k\pi); \tag{4.28}$$

with this we can finally write

$$\mathbf{1} \overset{\text{DTFT}}{\longleftrightarrow} \tilde{\boldsymbol{\delta}}.$$

And, to make this result a bit more intuitive, let's look at it from another angle, namely, as the limit of the DTFT of a truncated unit sequence. For any positive integer M, consider a sequence that is equal to one for $n \in [-M, M]$ and zero otherwise; the sequence has finite energy and its DTFT is

$$C_M(\omega) = \sum_{n=-M}^{M} e^{-j\omega n} = 1 + 2\sum_{n=1}^{M}\cos(n\omega);$$

[8]No need to conjugate the first operand since $\mathbf{d}_a$ is real and symmetric and therefore so is $\mathbf{D}_a$.

this function, also known as the Dirichlet kernel, has constant area 2π independently of M and maximum amplitude $2M+1$ in $\omega = 0$; some examples are plotted in magnitude in Figure 4.8. It's clear that, as M increases, the area under each curve concentrates more and more around the origin while the maximum value grows linearly with M; the set $\{\mathbf{C}_M\}_M$ is thus another family of increasingly narrow and increasingly peaked functions with constant area and, by analogy, we can deduce that the truncated DTFTs will to converge in the limit to a Dirac delta.

4.6.3 DTFTs of sinusoidal power signals

By accepting $\tilde{\boldsymbol{\delta}}$ as a bona fide Fourier transform, we can easily derive the generalized DTFT of sinusoidal signals by working backwards from the frequency domain; using the Dirac delta's sifting property (4.24), we can easily compute the IDTFT of a frequency-shifted periodic delta as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\delta}(\omega - \omega_0)\, e^{j\omega n} d\omega = e^{j\omega_0 n};$$

with this, we obtain the fundamental generalized DTFT pairs:

$$e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} \tilde{\delta}(\omega - \omega_0) \tag{4.29}$$

$$\cos(\omega_0 n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}[\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)] \tag{4.30}$$

$$\sin(\omega_0 n) \overset{\text{DTFT}}{\longleftrightarrow} -\frac{j}{2}[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)] \tag{4.31}$$

4.6.4 The DTFT of the unit step sequence

The unit step $\mathbf{u}$ is a signal that is most commonly used as a multiplicative mask in order to impose causality to the mathematical expression for a sequence; we have often written the one-sided decaying exponential, for example, as

$$h_a[n] = a^n\, u[n], \qquad |a| < 1. \tag{4.32}$$

But what is the DTFT of the unit step on its own? Obviously $\mathbf{u}$ is a power signal so its frequency representation will only exist in the generalized sense; to compute its expression we could begin by noticing that, as the parameter a approaches one, the exponential decay in (4.32) converges in the limit to the unit step. Since

$$H_a(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

it's tempting to simply apply the same limiting operation in the frequency domain to obtain

$$\lim_{a\to 1} H_a(\omega) = \frac{1}{1 - e^{-j\omega}} = S(\omega).$$

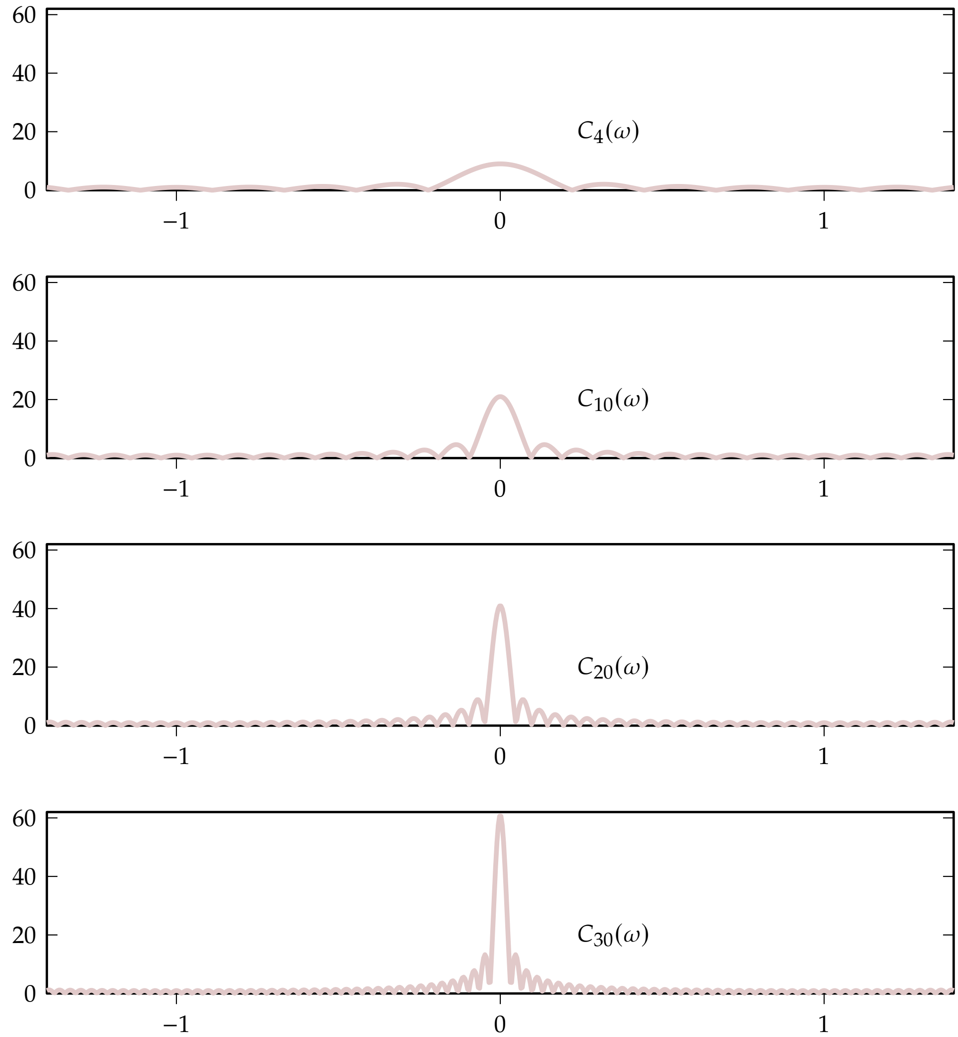

Figure 4.8: *The truncated DTFT of the unit sequence for increasing lengths.*

Obviously something is already off: there is no Dirac delta in this expression, even though we know $\mathbf{u}$ to be a power signal. Still, let's collect $e^{-j\omega/2}$ at numerator and denominator and write

$$S(\omega) = \frac{1}{2} - j\frac{1}{2}\cot\frac{\omega}{2}; \tag{4.33}$$

although divergent in $\omega = 0$, the function $S(\omega)$ seems sufficiently well-behaved so let's check if it is indeed the Fourier transform of the unit step. Using the same approach as before, for every valid DTFT pair $\mathbf{x} \leftrightarrow \mathbf{X}$ it should be

$$\langle \mathbf{u}, \mathbf{x} \rangle = \langle \mathbf{S}, \mathbf{X} \rangle$$

that is,

$$\sum_{n=0}^{\infty} x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S^*(\omega)X(\omega)d\omega. \tag{4.34}$$

Let's pick $\mathbf{x}$ real and symmetric, so that its DTFT will also be real and symmetric; the right-hand side becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S^*(\omega)X(\omega)d\omega = \frac{1}{2}x[0] + \frac{j}{4\pi}\int_{-\pi}^{\pi} \cot\frac{\omega}{2}X(\omega)\,d\omega.$$

In order to compute the last integral, given that the cotangent is divergent in zero, we could use Cauchy's principal value theorem; but since our chosen $X(\omega)$ is real and symmetric whereas $\cot(\omega/2)$ is real and antisymmetric, the integral of their product, if it exists, can only be equal to zero. Since we can easily find real symmetric sequences for which

$$\sum_{n=0}^{\infty} x[n] \neq \frac{1}{2}x[0]$$

we can conclude that $\mathbf{S}$ is *not* the Fourier transform of the unit step.

Disappointed but not yet discouraged, we may try another approach; let's take the unit step's first-order difference:

$$u[n] - u[n-1] = \delta[n]. \tag{4.35}$$

In the frequency domain this becomes

$$U(\omega) - e^{-j\omega}U(\omega) = 1$$

from which, once again, we obtain $U(\omega) = S(\omega)$ — which we know to be wrong! But this line of attack at least has the merit of leading us closer to the solution: the key observation is that the first-order difference in (4.35) remains valid even if an arbitrary constant offset is added to the unit step[9]; in other words, if we define

$$u_c[n] = u[n] + c, \qquad c \in \mathbb{R},$$

[9]Exactly in the same way as differentiation is unaffected by an additive constant: $\frac{\partial}{\partial t}[f(t) + c] = f'(t)$

we still have

$$u_c[n] - u_c[n-1] = \delta[n] \quad \forall c$$

so that $\mathbf{U}_c = \mathbf{S}$ for all c. But this means that an infinite number of distinct sequences have the same DTFT, which is in contradiction with the uniqueness of the Fourier transform (since the DTFT is a change of basis). So let's ask ourselves a different question instead: is it possible to find a value for the offset c so that $\mathbf{u}_c$ and $\mathbf{S}$ form a true DTFT pair? The answer comes once again from symmetry considerations: if $\text{DTFT}\{\mathbf{u}_c\} = \mathbf{S}$ then $\text{DTFT}\{\mathbf{u}_c + b\boldsymbol{\delta}\} = \mathbf{S} + b$ for any scalar b; if we pick $b = -1/2$ then $\mathbf{S} - 1/2$ becomes a purely imaginary DTFT and therefore the sequence $\mathbf{u}_c - (1/2)\boldsymbol{\delta}$ must be antisymmetric. This means that

$$u[n] + c - (1/2)\delta[n] = -(u[-n] + c - (1/2)\delta[n])$$

and, by solving for n positive, negative, and equal to zero, the only possible value for the offset is $c = -1/2$. The signal $\mathbf{s} = \mathbf{u}_{-1/2}$ is in fact called the *signum* sequence, formally defined as

$$s[n] = \begin{cases} +1/2 & n \geq 0 \\ -1/2 & n < 0 \end{cases} \tag{4.36}$$

and with generalized DTFT

$$S(\omega) = \frac{1}{1 - e^{-j\omega}}.$$

Since $u[n] = s[n] + 1/2$, the DTFT of the unit step is finally

$$U(\omega) = \frac{1}{1 - e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega). \tag{4.37}$$