COM-202: Signal Processing

Chapter 3.b: The Discrete Fourier Transform

Overview

- the Fourier basis for $\mathbb{C}^N$ (recap)
- the DFT: properties and examples
- interpreting a DFT plot

The DFT algorithm

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *time domain*

The DFT is an orthogonal change of basis

Analysis formula:

$$X_k = \langle \mathbf{w}_k, \mathbf{x} \rangle$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}_k$$

Change of basis in matrix form

Because of orthogonality, the change of basis can be expressed as a matrix-vector multiplication.

Analysis formula:

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$$

The DFT matrix

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W^1 & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{2(N-1)} \\ & & & \dots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)^2} \end{bmatrix} \qquad W = e^{-j\frac{2\pi}{N}}$$

The DFT is obviously linear

$$\mathsf{DFT}\left\{\alpha\,\mathbf{x}+\beta\,\mathbf{y}\right\}=\alpha\,\mathsf{DFT}\left\{\mathbf{x}\right\}+\beta\,\mathsf{DFT}\left\{\mathbf{y}\right\}$$

Time and frequency reversal

$$x[-n \mod N] \overset{\text{DFT}}{\longleftrightarrow} X[-k \mod N]$$

or

$$\text{DFT}\left\{\mathcal{R}\mathbf{x}\right\} = \mathcal{R}\mathbf{X}$$

Time and frequency reversal (conjugation)

$$x^*[n] \overset{\mathrm{DFT}}{\longleftrightarrow} X^*[-k \mod N]$$

or

$$\mathrm{DFT}\{\mathbf{x}^*\} = \mathcal{R}\mathbf{X}^*$$

Circular shift (time domain)

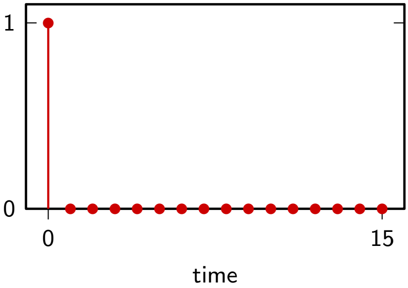
$$x[(n - n_0) \mod N] \overset{\text{DFT}}{\longleftrightarrow} e^{-j\frac{2\pi}{N}n_0 k} X[k]$$

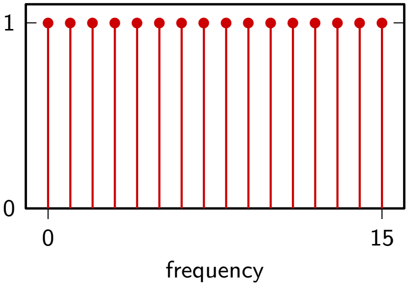
Circular shift (frequency domain)

$$e^{j\frac{2\pi}{N}nk_0}x[n] \overset{\text{DFT}}{\longleftrightarrow} X[(k-k_0) \mod N]$$

DFT of $\delta[n] \in \mathbb{C}^N$

$$\sum_{n=0}^{N-1} \delta[n]\, e^{-j\frac{2\pi}{N}nk} = 1 \quad \forall k$$





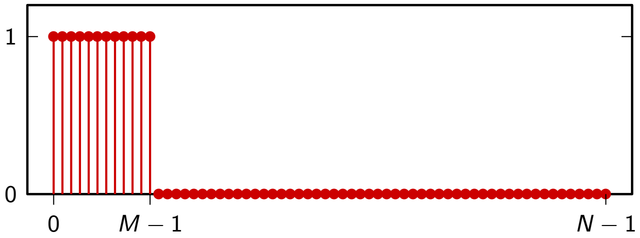
DFT of $\mathbf{e}_m = \delta[n-m] \in \mathbb{C}^N$

$$\sum_{n=0}^{N-1} \delta[n-m]\, e^{-j\frac{2\pi}{N}nk} = e^{-j\frac{2\pi}{N}mk}$$

$$\text{DFT}\left\{\mathbf{e}_m\right\} = \mathbf{w}_m^*$$

DFT of length-M step in $\mathbb{C}^N$

$$x[n] = \sum_{m=0}^{M-1} \delta[n-m], \quad n = 0, 1, \ldots, N-1$$



DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \sum_{m=0}^{M-1} \text{DFT}\left\{\delta[n-m]\right\} = \sum_{m=0}^{M-1} e^{-j\frac{2\pi}{N}mk}$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM} - e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right]}$$

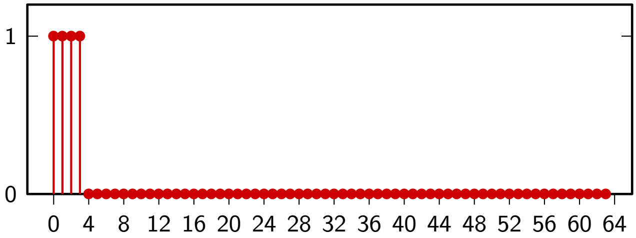
$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

DFT of length-M step in $\mathbb{C}^N$

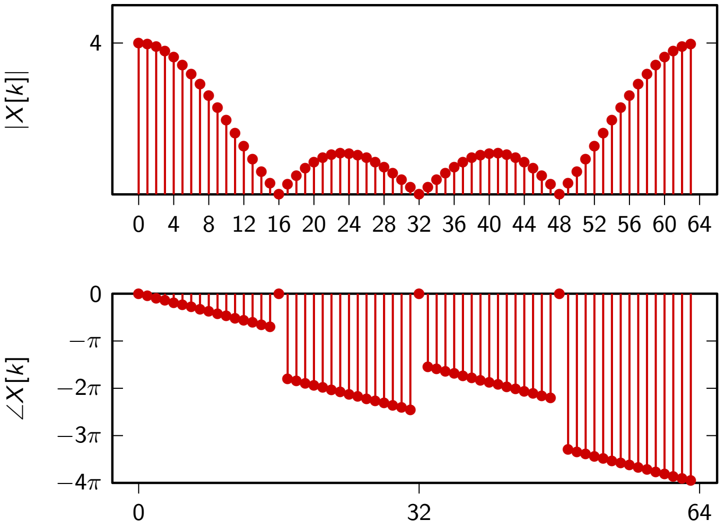
$$X[k] = \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

- $X[0] = M$, from the definition of the sum
- $X[k] = 0$ if Mk/N nonzero integer $(0 < k < N)$
- $\angle X[k]$ linear in k (except at sign changes for the real part)

Length-4 step in $\mathbb{C}^N$



DFT of length-4 step in $\mathbb{C}^{64}$

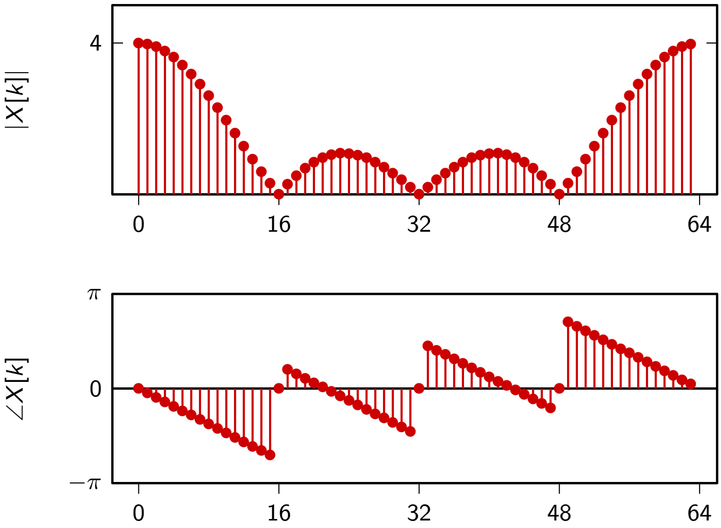


Wrapping the phase

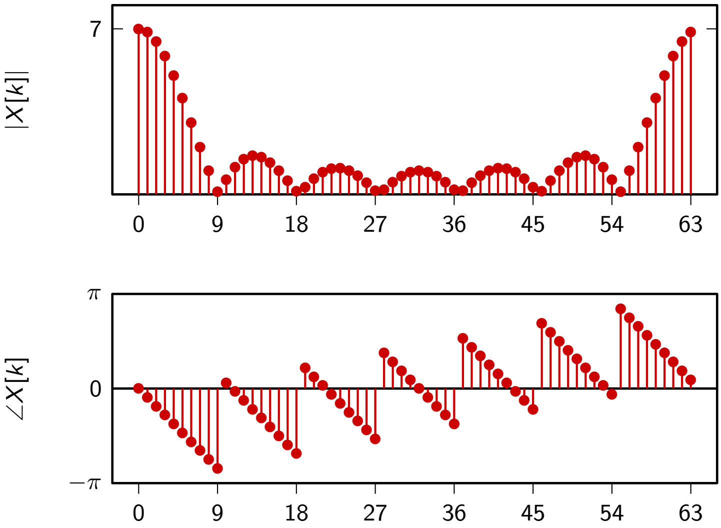
Often the phase is displayed "wrapped" over the $[-\pi, \pi]$ interval.

- most numerical packages return wrapped phase
- phase can be unwrapped by adding multiples of 2π

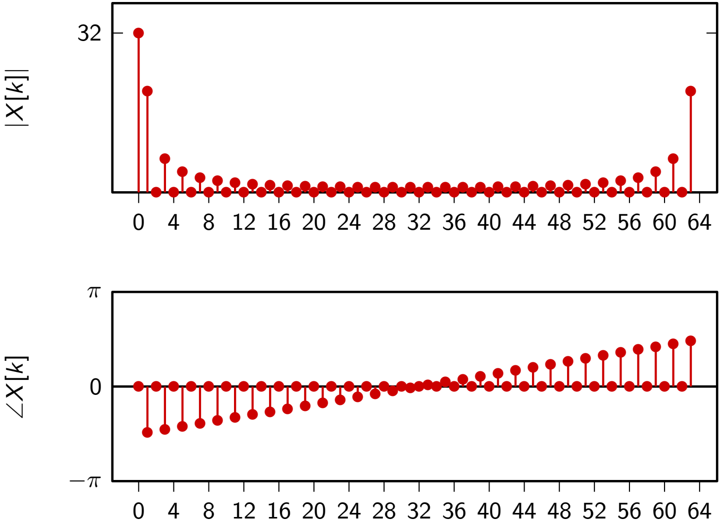
DFT of length-4 step in $\mathbb{C}^{64}$ (phase wrapped)



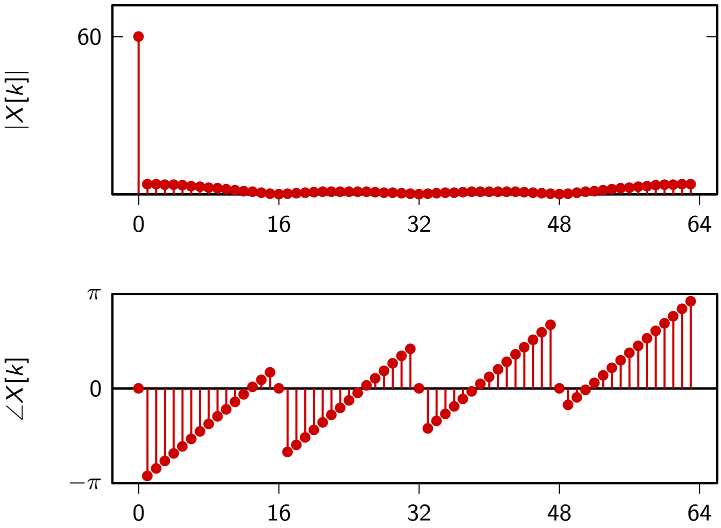
DFT of length-7 step in $\mathbb{C}^{64}$ (phase wrapped)



DFT of length-32 step in $\mathbb{C}^{64}$ (phase wrapped)

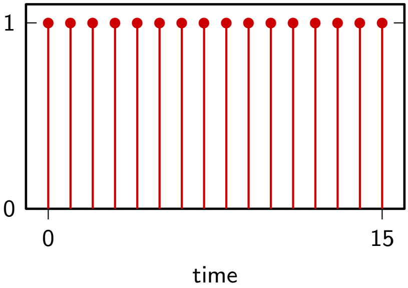


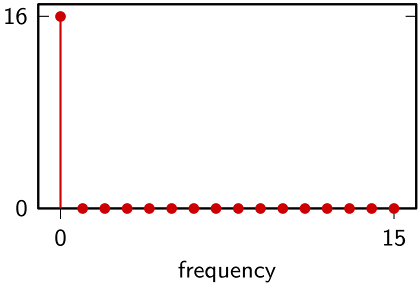
DFT of length-60 step in $\mathbb{C}^{64}$ (phase wrapped)



DFT of $x[n] = 1, \quad x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = N\delta[k]$$





First example of time-frequency duality

$$\mathrm{DFT}\{\delta[n]\} = 1$$

$$\mathrm{DFT}\{1\} = N\delta[k]$$

signals maximally concentrated in time are maximally spread out in frequency, and vice-versa

DFT of harmonic oscillations

harmonic oscillations in $\mathbb{C}^N$ complete a full number of periods over N steps

$$x[n] = e^{j\omega_m n}, \qquad \omega_m = \frac{2\pi}{N} m$$

$$\begin{aligned} X[k] &= \langle \mathbf{w}_k, \mathbf{x} \rangle \\ &= \langle \mathbf{w}_k, \mathbf{w}_m \rangle \\ &= N\delta[m-k] \end{aligned}$$

$$\mathrm{DFT}\left\{\mathbf{w}_m\right\} = N\mathbf{e}_m$$

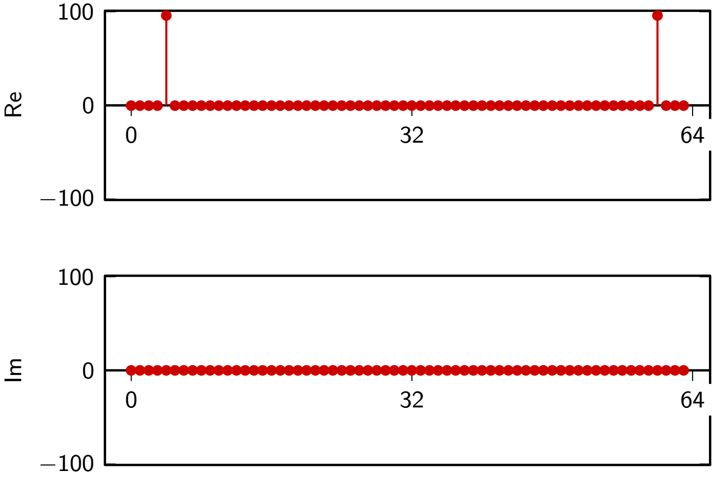
DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n\right) \\
&= 3\cos\left(\frac{2\pi}{64}\, 4\, n\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right] \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right] \qquad \leftarrow e^{-j\frac{2\pi}{64}4n} = e^{j\left(-\frac{2\pi}{64}4n + 2\pi n\right)} \\
&= \frac{3}{2}\left(w_4[n] + w_{60}[n]\right)
\end{aligned}
$$

DFT of $x[n] = 3\cos(2\pi/16\, n), \quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
X[k] &= \langle \mathbf{w}_k, \mathbf{x} \rangle \\
&= \langle \mathbf{w}_k, \frac{3}{2}(\mathbf{w}_4 + \mathbf{w}_{60}) \rangle \\
&= \frac{3}{2}\langle \mathbf{w}_k, \mathbf{w}_4 \rangle + \frac{3}{2}\langle \mathbf{w}_k, \mathbf{w}_{60} \rangle \\
&= \begin{cases} (3/2)N = 96 & \text{for } k = 4,\ 60 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$



DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3), \quad x[n] \in \mathbb{C}^{64}$

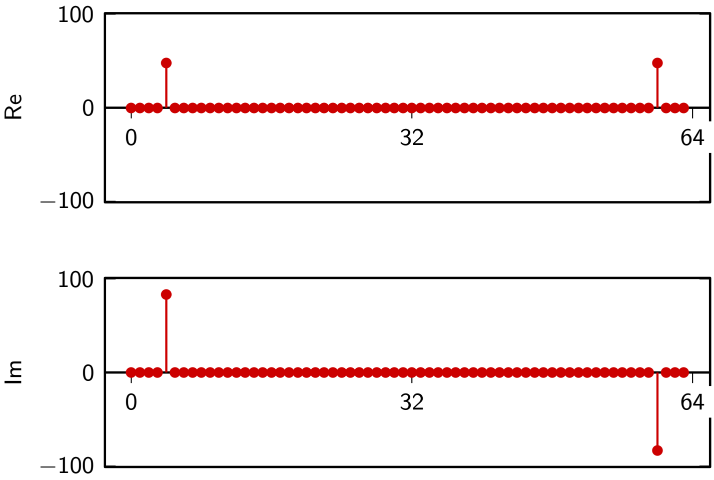
$$
\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n + \frac{\pi}{3}\right) \\
&= 3\cos\left(\frac{2\pi}{64}\, 4\, n + \frac{\pi}{3}\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi}{64}4n} e^{-j\frac{\pi}{3}}\right] \\
&= \frac{3}{2}\left(e^{j\frac{\pi}{3}}\, w_4[n] + e^{-j\frac{\pi}{3}}\, w_{60}[n]\right)
\end{aligned}
$$

DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3), \quad x[n] \in \mathbb{C}^{64}$

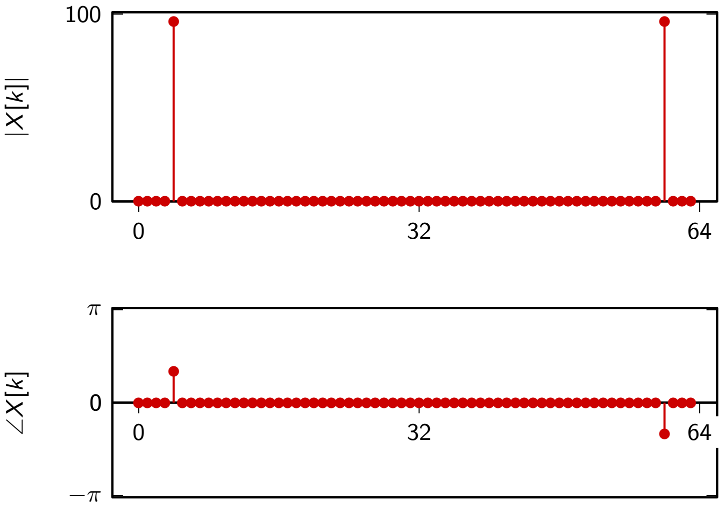
$$X[k] = \langle \mathbf{w}_k, \mathbf{x} \rangle$$

$$= \begin{cases} 96 e^{j\frac{\pi}{3}} & \text{for } k = 4 \\ 96 e^{-j\frac{\pi}{3}} & \text{for } k = 60 \\ 0 & \text{otherwise} \end{cases}$$

DFT of $x[n] = 3\cos(2\pi/16\,n + \pi/3)$, $\quad x[n] \in \mathbb{C}^{64}$



DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3), \quad x[n] \in \mathbb{C}^{64}$



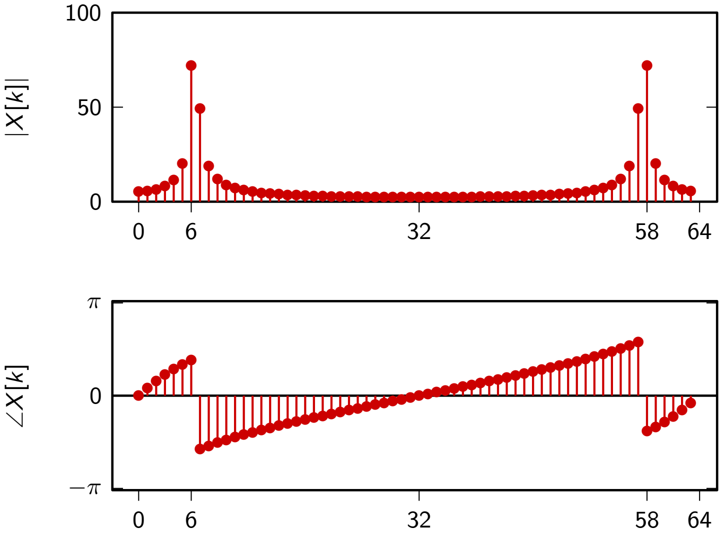
DFT of $x[n] = 3\cos(2\pi/10\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$\frac{2\pi}{64}6 < \frac{2\pi}{10} < \frac{2\pi}{64}7$$

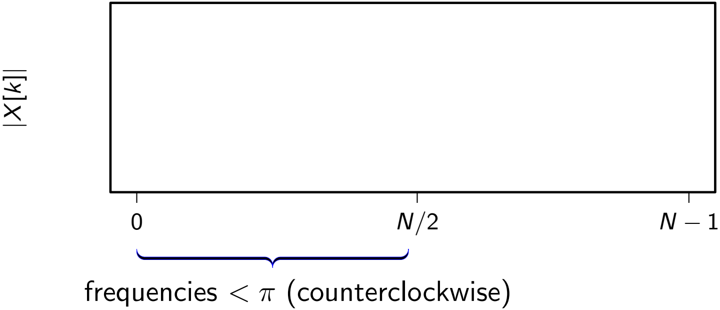
The DFT is a numerical algorithm!

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

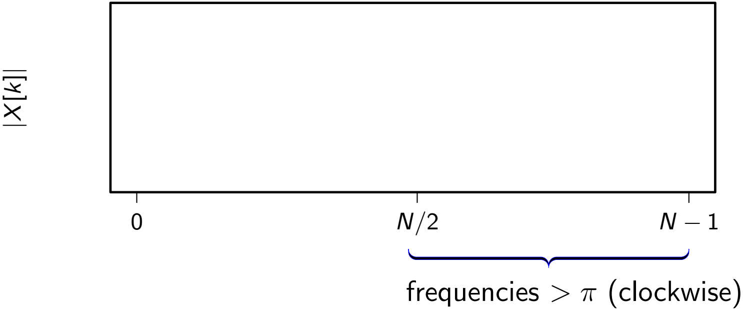
DFT of $x[n] = 3\cos(2\pi/10\, n)$, $x[n] \in \mathbb{C}^{64}$



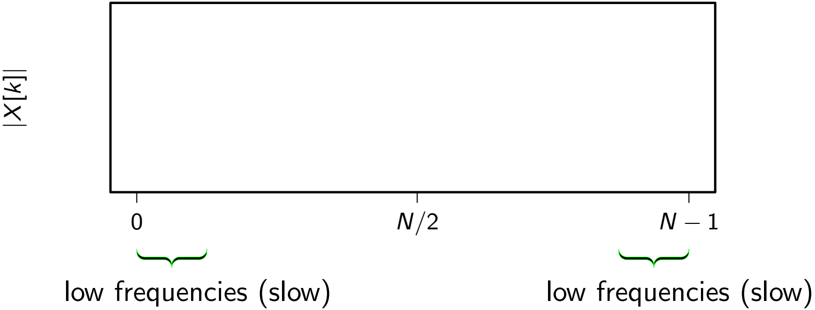
Interpreting a DFT plot



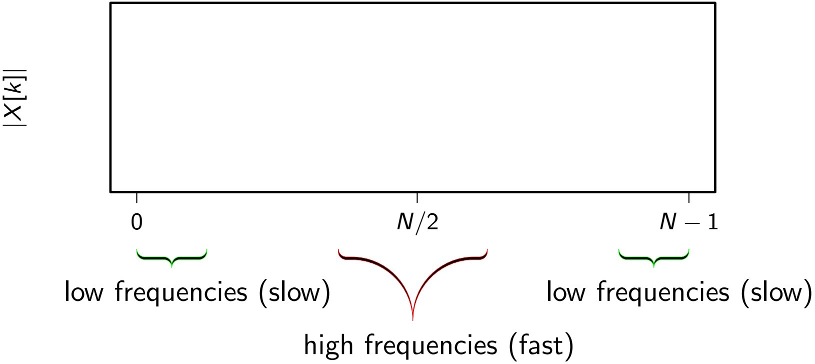
Interpreting a DFT plot



Interpreting a DFT plot

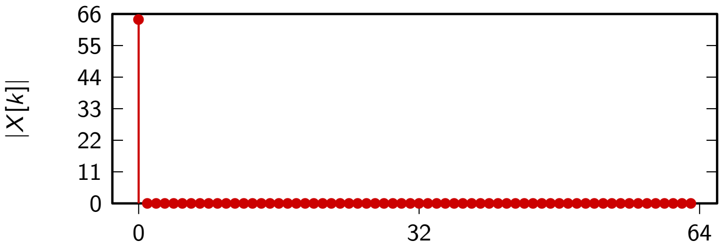


Interpreting a DFT plot



Interpreting a DFT plot

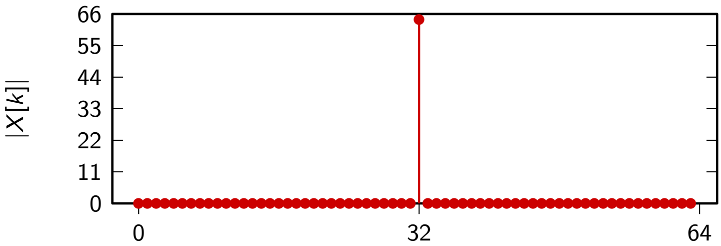
$x[n] = 1$ (slowest signal)



only lowest frequency

Interpreting a DFT plot

$$x[n] = \cos \pi n = (-1)^n \text{ (fastest signal)}$$



only highest frequency

Energy distribution

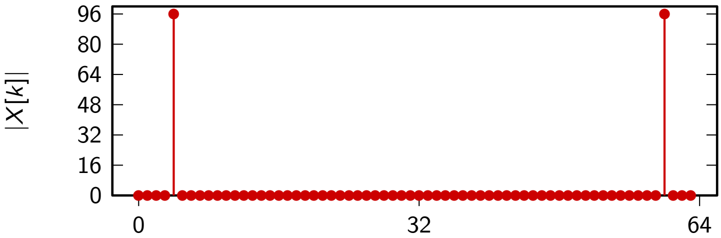
Recall Parseval's Theorem: $\|\mathbf{x}\|^2 = \sum |\alpha_k|^2$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

square magnitude of k-th DFT coefficient
proportional to signal's energy at frequency $\omega = (2\pi/N)k$

Interpreting a DFT plot

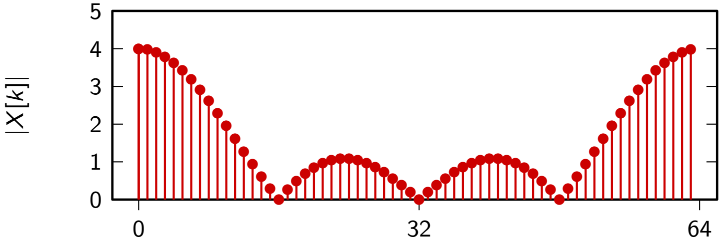
$$x[n] = 3\cos(2\pi/16\, n) \text{ (sinusoid)}$$



energy concentrated on single frequency
(counterclockwise and clockwise combine to give real signal)

Interpreting a DFT plot

$$x[n] = u[n] - u[n-4] \text{ (step)}$$

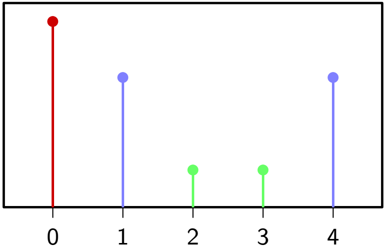


energy mostly in low frequencies

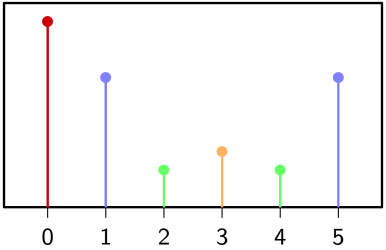
DFT of real signals

For real signals the DFT is "symmetric" in magnitude:

$$|X[k]| = |X[N-k]| \text{ for } k = 1, 2, \ldots, \lfloor N/2 \rfloor$$



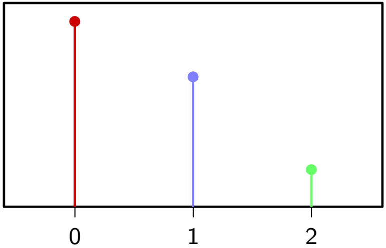
$N = 5$, odd length



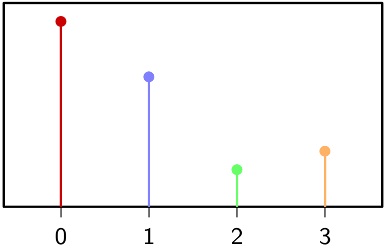
$N = 6$, even length

DFT of real signals

For real signals, magnitude plots need only $\lfloor N/2 \rfloor + 1$ points



$N = 5$, odd length



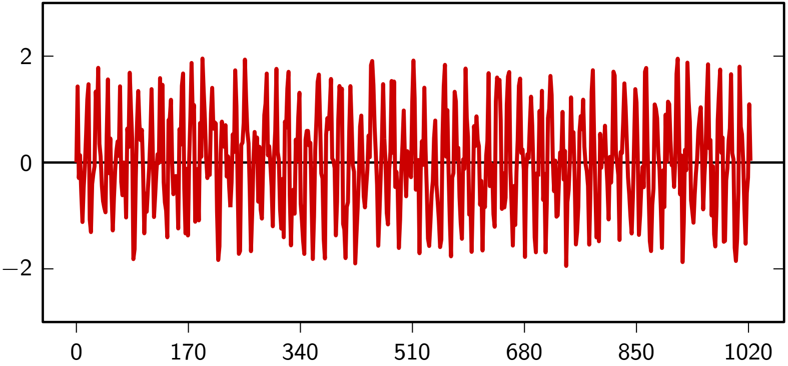
$N = 6$, even length

the DFT as an analysis tool

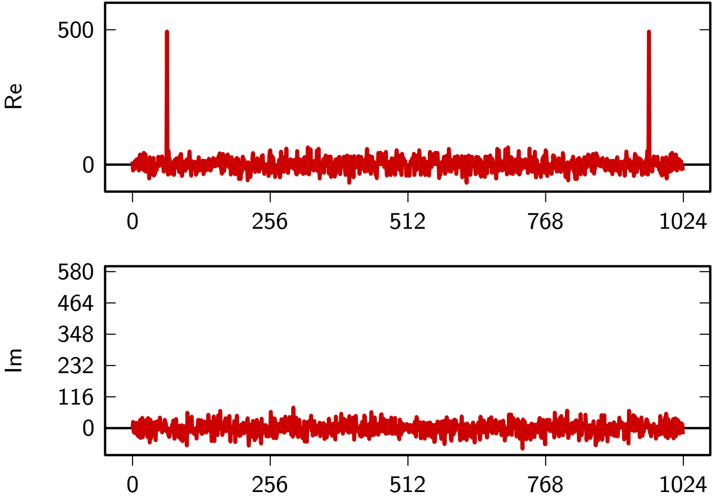
Overview

- DFT analysis examples
- Labeling the DFT axes

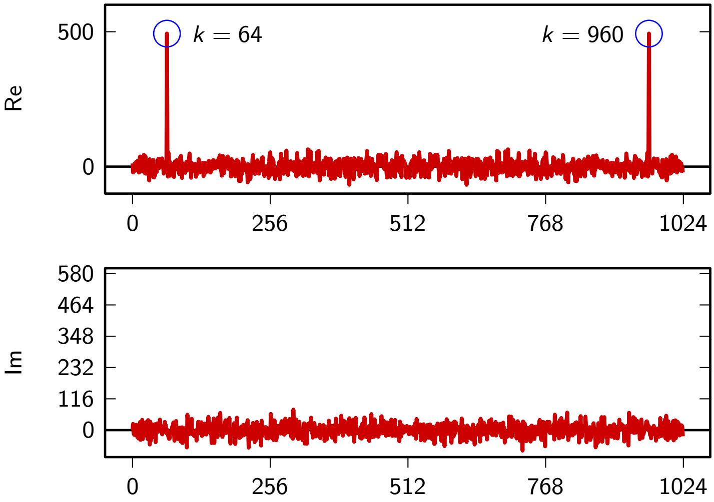
Mystery signal revisited



Mystery signal revisited



Mystery signal revisited



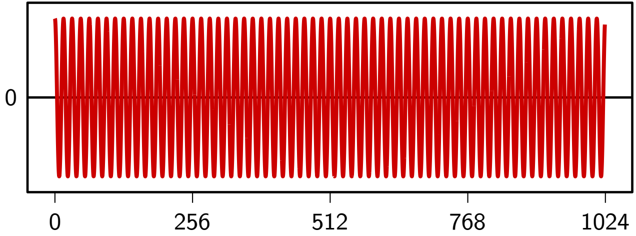
Mystery signal revisited

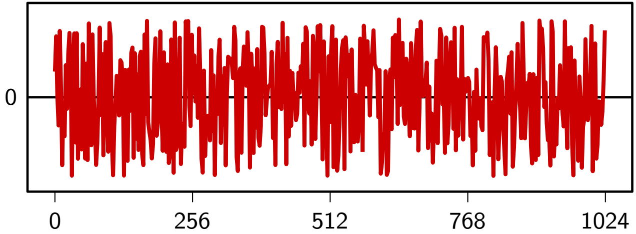
$$x[n] = \cos(\omega n + \phi) + \eta[n]$$

with

$$\begin{aligned} \phi &= 0 \\ \omega &= \frac{2\pi}{1024} 64 \end{aligned}$$

Mystery signal unveiled

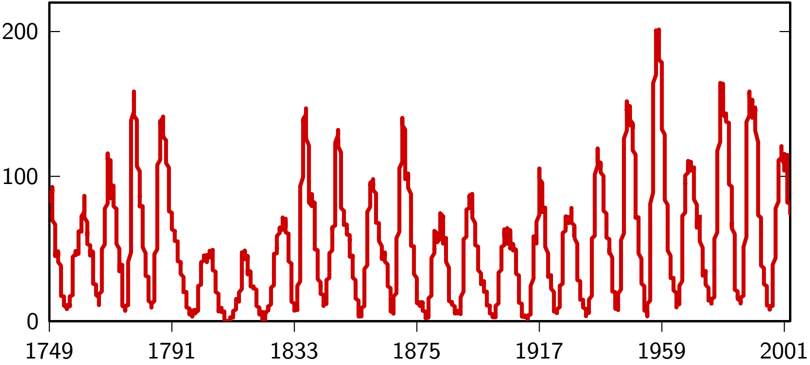




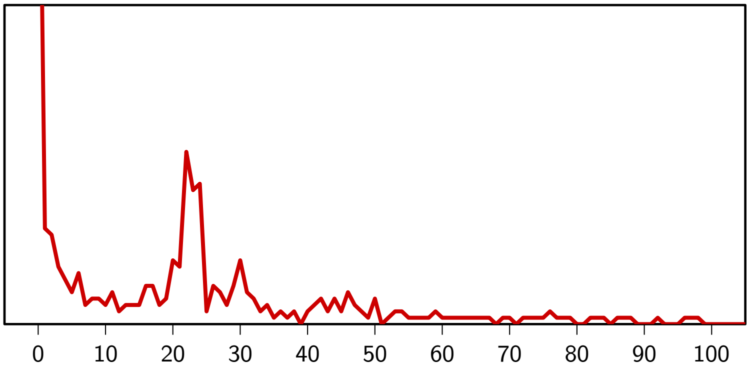
Solar spots

- sunspot number: $s = 10 \times \#$ of clusters $+ \#$ of spots
- data set from 1749 to 2003, 2904 months

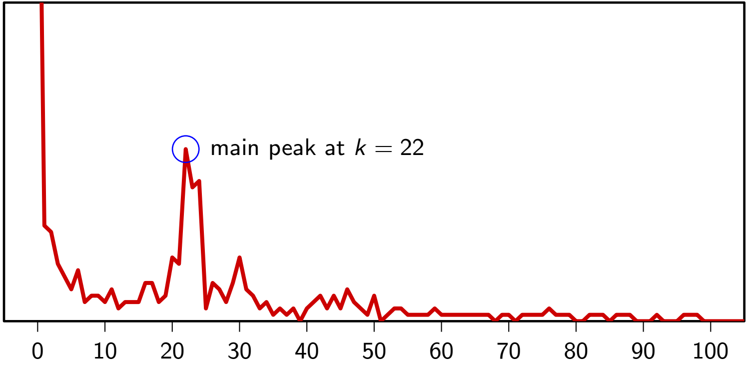
Solar spots



Solar spots



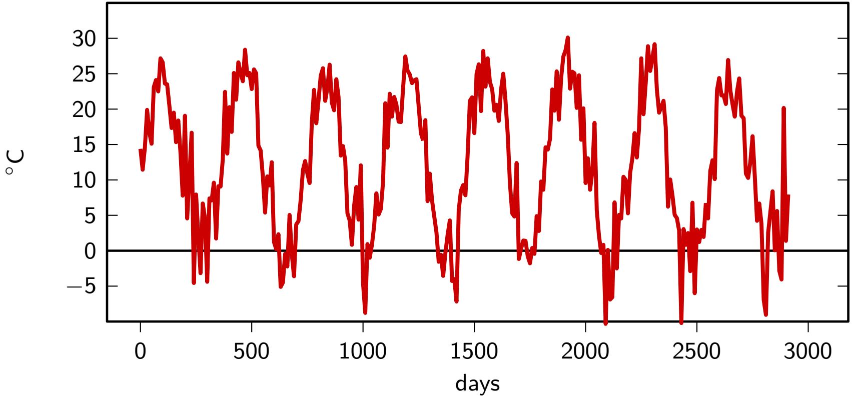
Solar spots



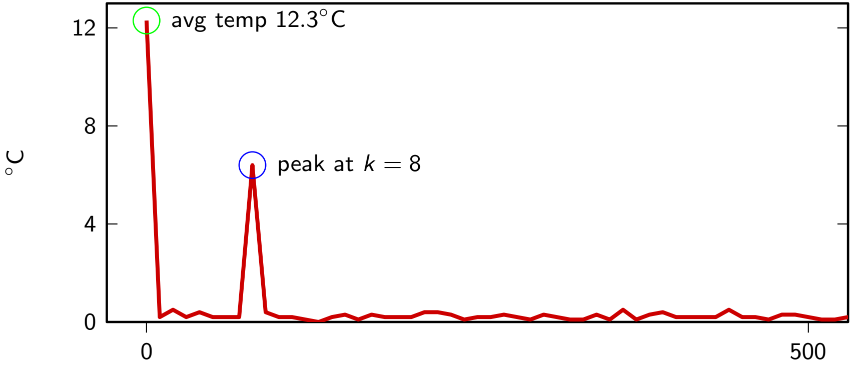
Solar spots

- DFT main peak for $k = 22$
- 22 cycles over 2904 months
- period: $\dfrac{2904}{22} \approx 11$ years

Daily temperature (2920 days)



Daily temperature: DFT



first few hundred DFT coefficients
(in magnitude and normalized by the length of the temperature vector)

Daily temperature

- average value (0-th DFT coefficient): 12.3°C
- DFT main peak for $k = 8$, value 6.4°C
- 8 cycles over 2920 days
- period: $\frac{2920}{8} = 365$ days
- temperature excursion: $12.3^{\circ}\text{C} \pm 12.8^{\circ}\text{C}$

Daily temperature

In case you're wondering why $\pm 12.8°$:

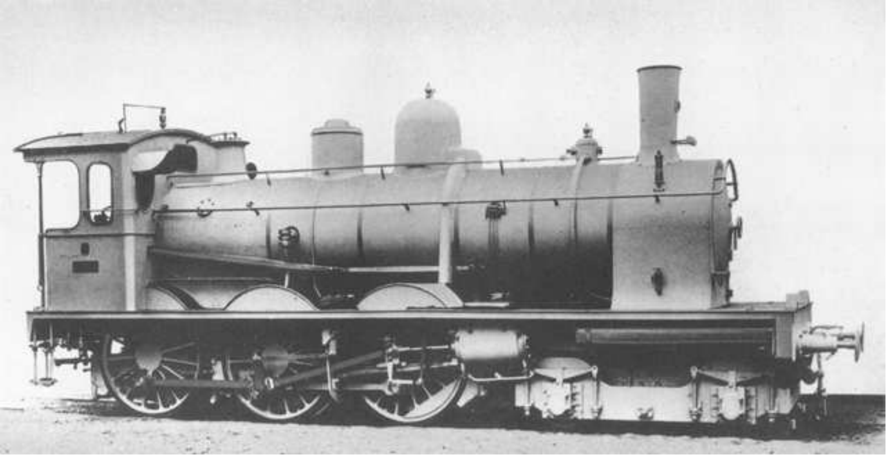
$$\mathrm{DFT}\left\{A\cos\left(\frac{2\pi}{N}Mn\right)\right\}[k] = \begin{cases} \frac{A}{2}N & \text{for } k = M, N-M \\ 0 & \text{otherwise} \end{cases}$$

Labeling the frequency axis

If we know the "clock" of the system T_s (or its frequency F_s)

- fastest (positive) frequency is $\omega = \pi$
- sinusoid at $\omega = \pi$ needs two samples to do a full revolution
- time between samples: $T_s = 1/F_s$ seconds
- real-world period for fastest sinusoid: $2T_s$ seconds
- real-world frequency for fastest sinusoid: $F_s/2$ Hz

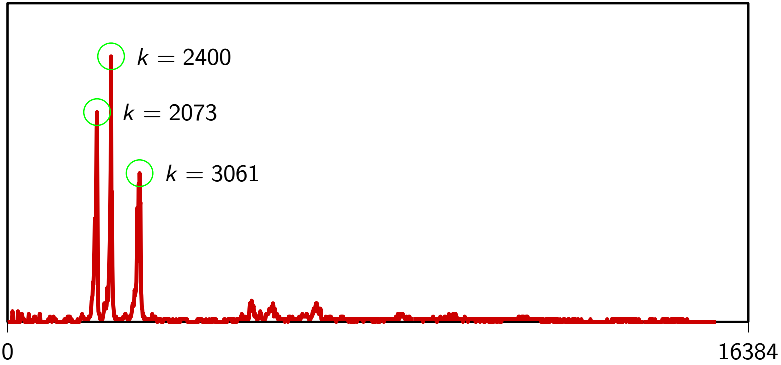
Example: train whistle



Play

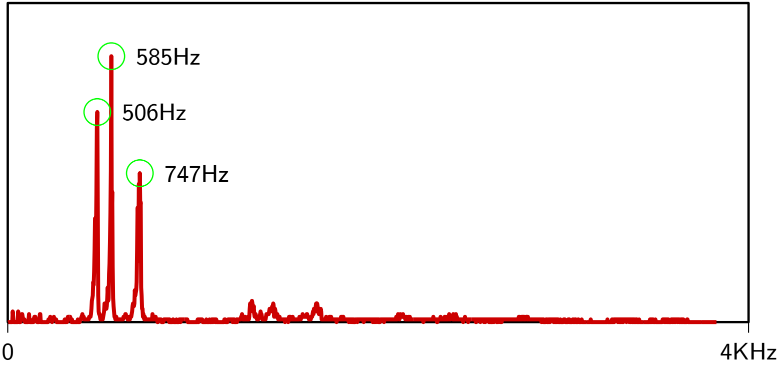
Example: train whistle

32768 samples (the "clock" of the system $F_s = 8000\text{Hz}$)



Example: train whistle

the "clock" of the system $F_s = 8000\text{Hz}$



Example: train whistle

if we look up the frequencies:

B minor chord