COM-202: Signal Processing

Chapter 3.a: Introduction to Fourier Analysis

Overview

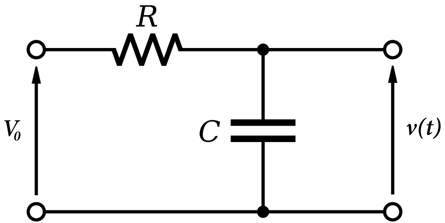
- Fourier analysis: concept and motivation
- the complex exponential
- the Fourier basis
- the DFT

What is a signal

quantitative description of a time-varying phenomenon

Continuous-time signals

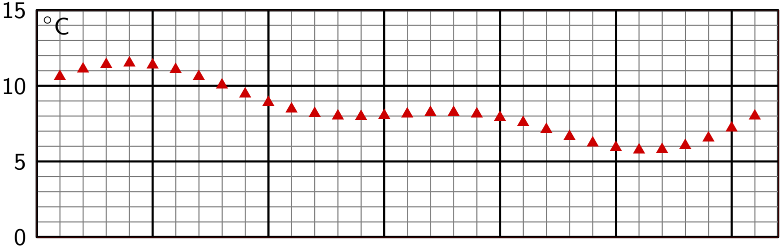
$$x(t) \qquad x : \mathbb{R} \to \mathbb{R}$$



$$v(t) = V_0(1 - e^{-\frac{t}{RC}})$$

Discrete-time signals

$$x[n] \qquad x : \mathbb{Z} \rightarrow \mathbb{C}$$



$$x[n] = \ldots \ 10.23, \ 10.73, \ 10.89, \ 11.17, \ 11.15, \ 10.86, \ \ldots$$

The unifying framework

all types of signals can be represented as vectors
in a suitable vector space

Signals as vectors in vector space

The "signal spaces" we will encounter

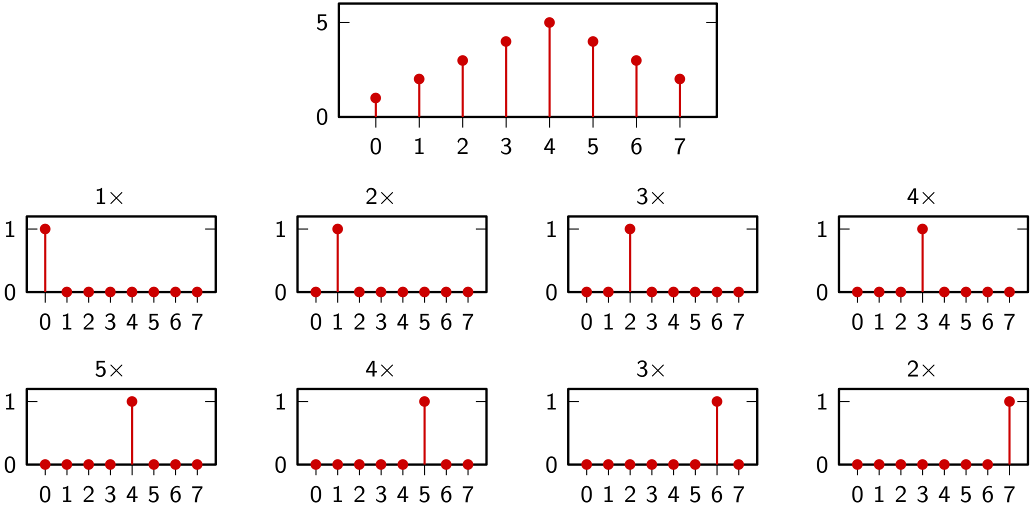
- $\mathbb{C}^N$: discrete-time finite-length signals of length N (and N-periodic signals)
- $\ell_2(\mathbb{Z})$: discrete-time infinite-length finite energy signals
- $L_2([a, b])$: continuous-time compact-support finite-energy signals
- $L_2(\mathbb{R})$: continuous-time finite-energy signals
- $f - BL$: continuous-time f-bandlimited signals

Fourier analysis

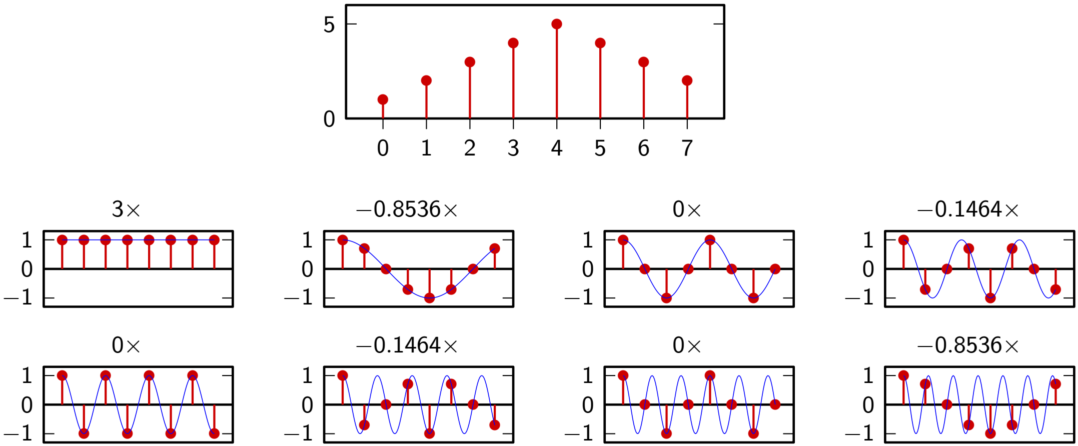
The fundamental idea

- signals are elements of a vector space
- vectors can be expressed as linear combinations of basis elements for any basis
- basis elements are the "atomic particles" of a signal
- the canonical basis is composed of instantaneous time elements
- the Fourier basis is composed of *oscillatory* elements

The time domain



The frequency domain



The Fourier transform

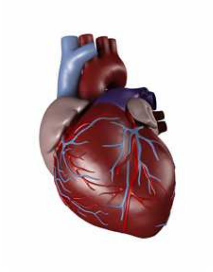
Goal: express *any* signal as a combination of periodic oscillations:

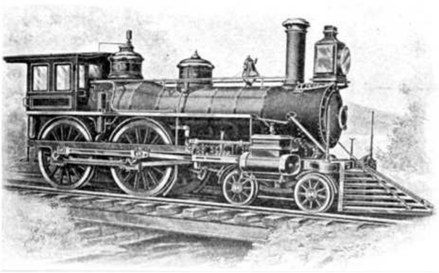
$$\mathbf{x} = \sum_k X_k \mathbf{w}_k$$

where $\{\mathbf{w}_k\}$ is an oscillatory basis for the signal space.

Fourier transform: an algorithm to compute the coefficients X_k
when $\{\mathbf{w}_k\}$ is a set of *harmonic* oscillations
(i.e., oscillations at multiples of a base frequency)

Oscillations are everywhere!

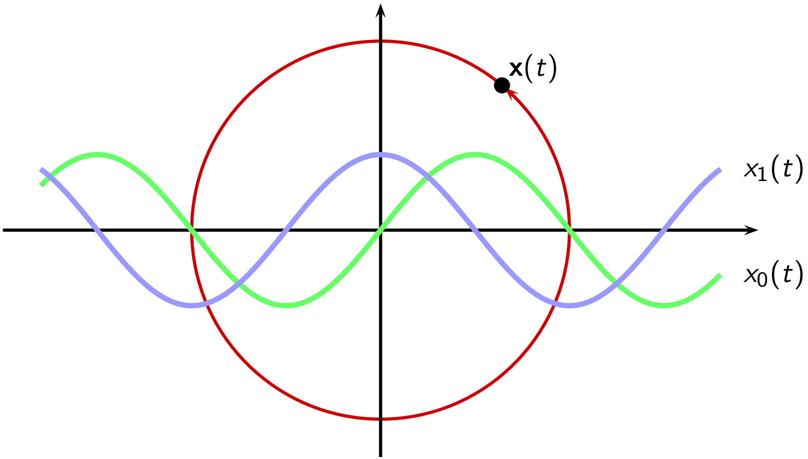




Oscillations are everywhere

- sustainable dynamic systems exhibit oscillatory behavior
- intuitively: things that don't move in circles can't last:
  - bombs
  - rockets
  - human beings...

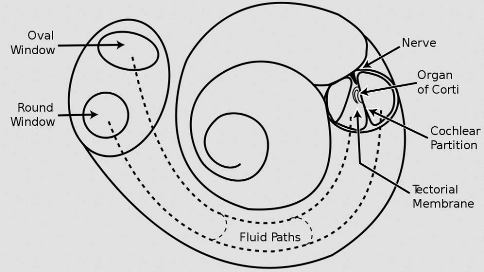
Oscillations are everywhere



You too can detect sinusoids!

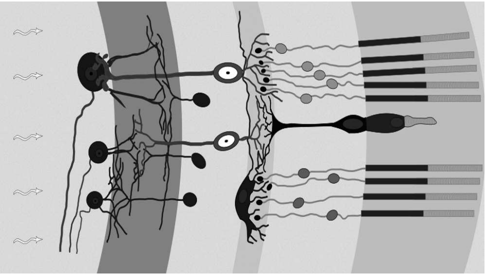
the human body has two receptors for sinusoidal signals:

cochlea (inner ear)



- air pressure oscillations
- frequencies from 20Hz to 20KHz
- wavelength from 17m to 1.7cm

rods and cones (retina)



- electromagnetic oscillations
- frequencies from 430THz to 790THz
- wavelength from 740nm to 380nm

The intuition

- humans analyze complex signals (audio, images) in terms of their sinusoidal components
- we can build instruments that “resonate” at one or multiple frequencies
  (tuning fork vs piano)
- the “frequency domain” seems to be as important as the time domain

Fundamental question

can we decompose any signal into sinusoidal elements?

yes, and Fourier showed us how to do it *exactly*!

analysis

- from time domain to frequency domain
- find the contribution of different frequencies
- discover “hidden” signal properties

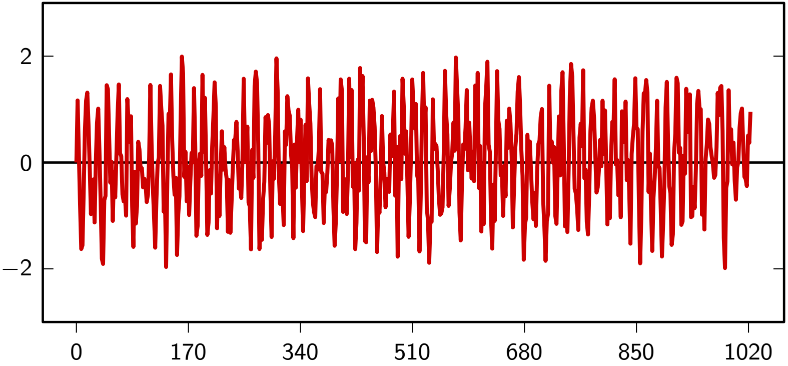
synthesis

- from frequency domain to time domain
- create signals with known frequency content
- fit signals to specific frequency regions

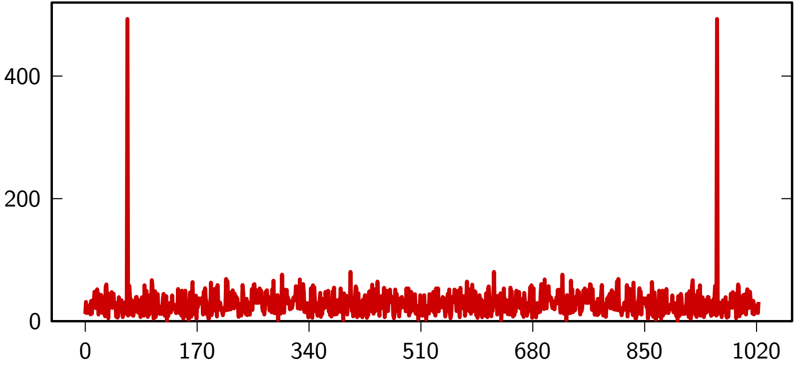
The mathematical setup

- let's start with finite-length signals (i.e. vectors in $\mathbb{C}^N$)
- Fourier analysis is a simple change of basis
- a change of basis is a change of perspective
- a change of perspective can reveal things (if the basis is good)

Mystery signal



Mystery signal in the Fourier basis



the complex exponential

Prerequisite Warning!

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

The complex exponential

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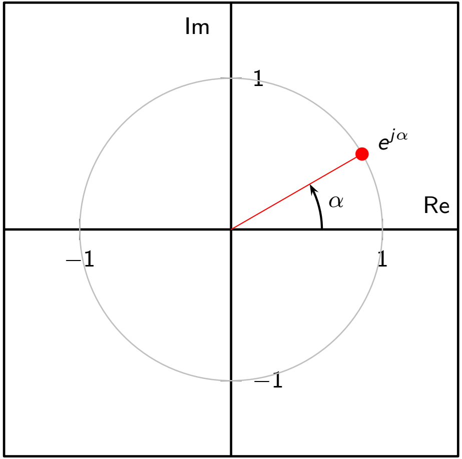
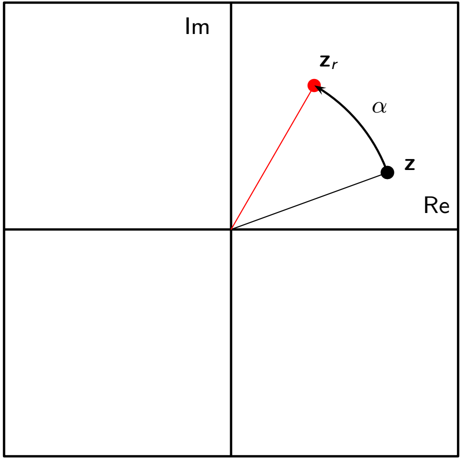


The complex exponential

$\mathbf{z}$: point on the complex plane; rotation: $\mathbf{z}_r = \mathbf{z}\, e^{j\alpha}$



Why complex exponentials?

- we can use complex numbers in digital systems, so why not?
- it makes sense: every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra

The advantages of complex exponentials

Example: change the phase of a cosine the “old-school” way

$$\cos(\omega n + \phi) = a\cos(\omega n) - b\sin(\omega n), \qquad a = \cos\phi,\ \ b = \sin\phi$$

- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

The advantages of complex exponentials

Example: change the phase of a pure cosine with complex exponentials

$$\cos(\omega n + \phi) = \mathrm{Re}\{e^{j(\omega n + \phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine “live” together
- phase shift is simple multiplication
- notation is simpler

The discrete-time oscillatory heartbeat

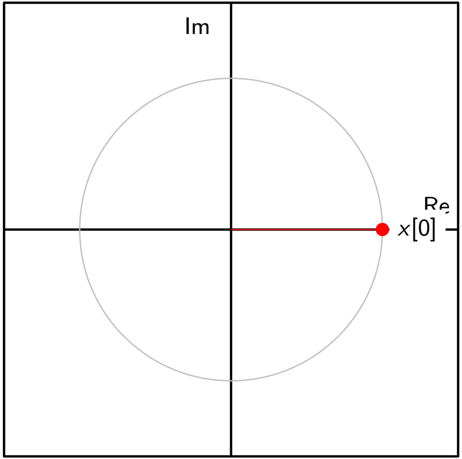
Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A

$$
\begin{aligned}
x[n] &= Ae^{j(\omega n+\phi)} \\
&= A[\cos(\omega n + \phi) + j\sin(\omega n + \phi)]
\end{aligned}
$$

The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



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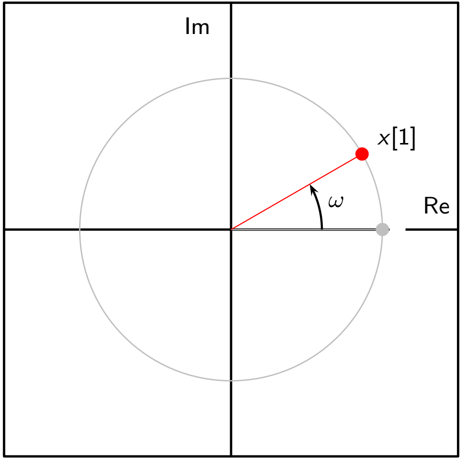

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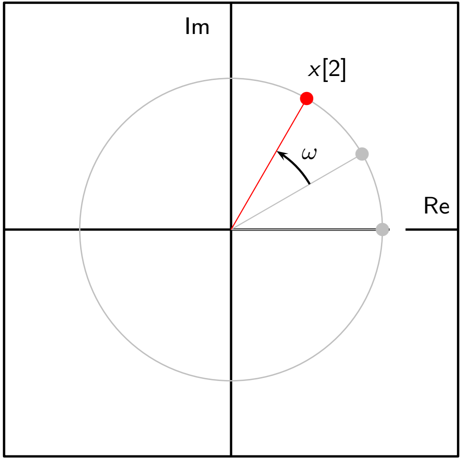

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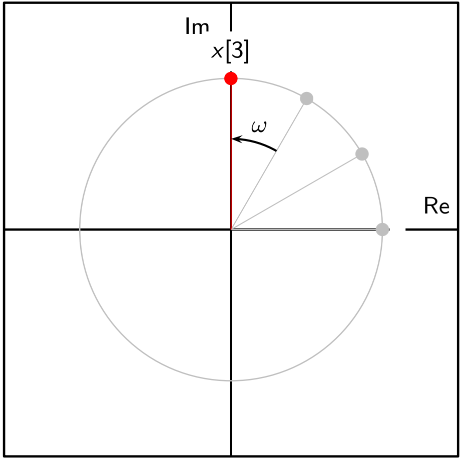

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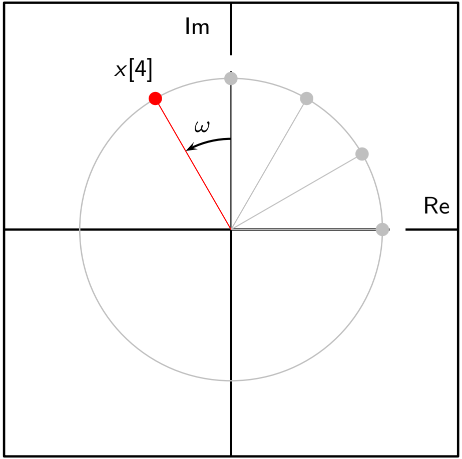

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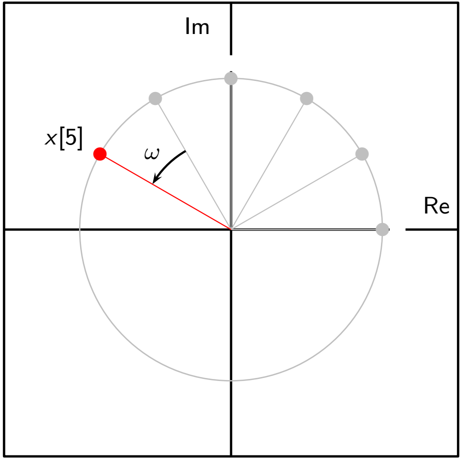

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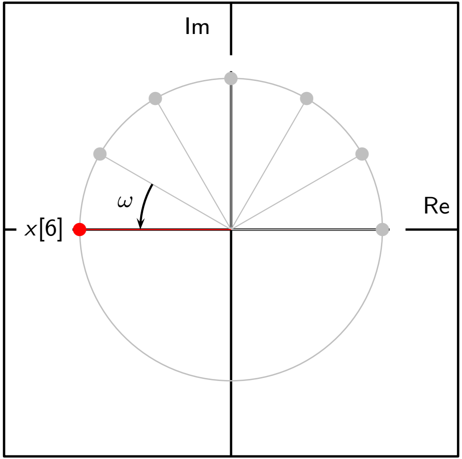

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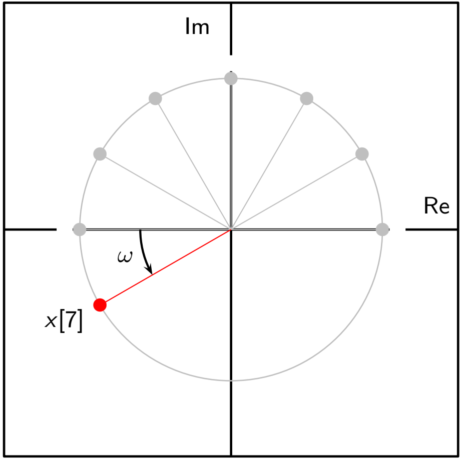

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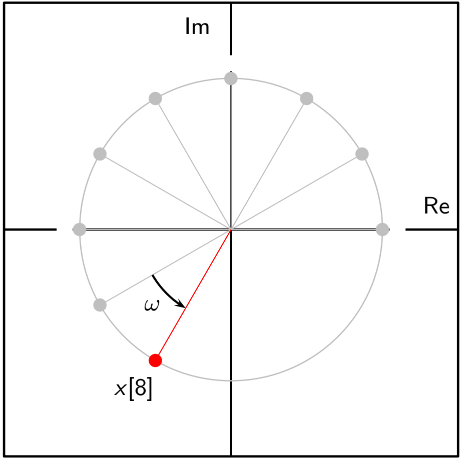

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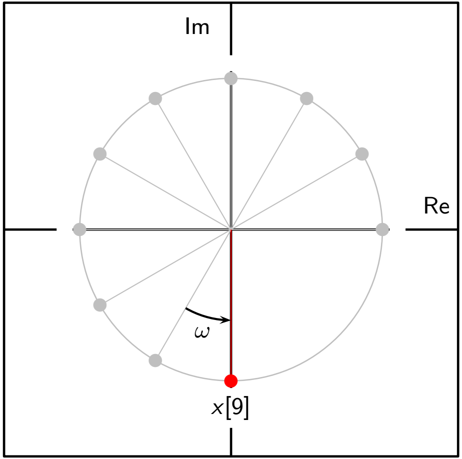

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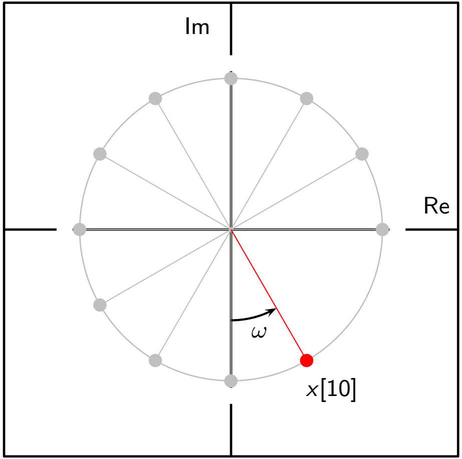

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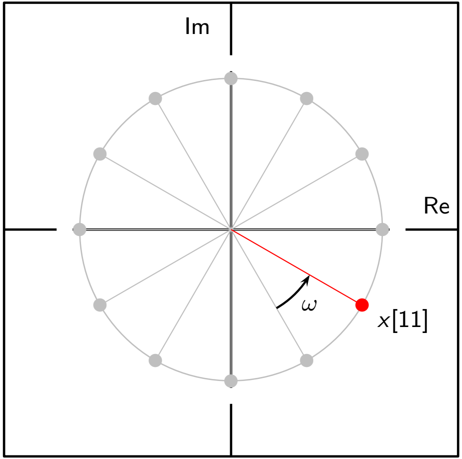

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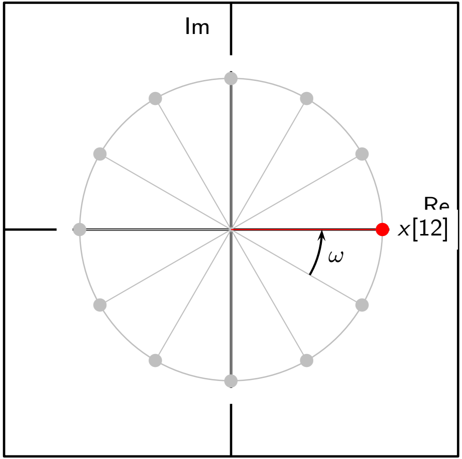

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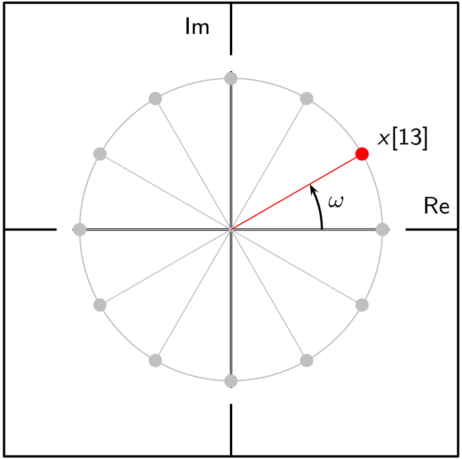

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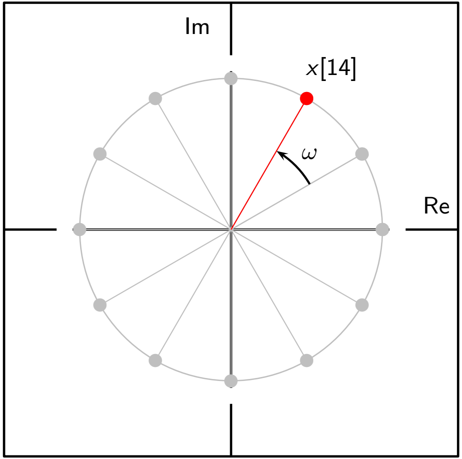
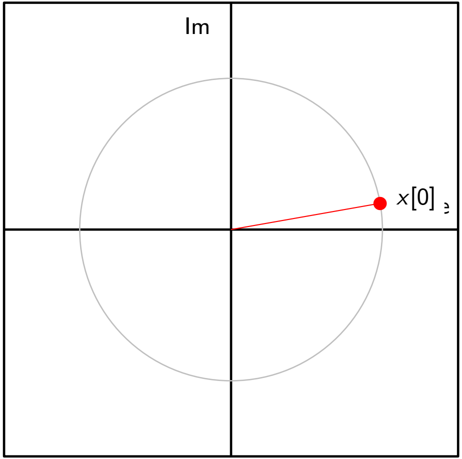


Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



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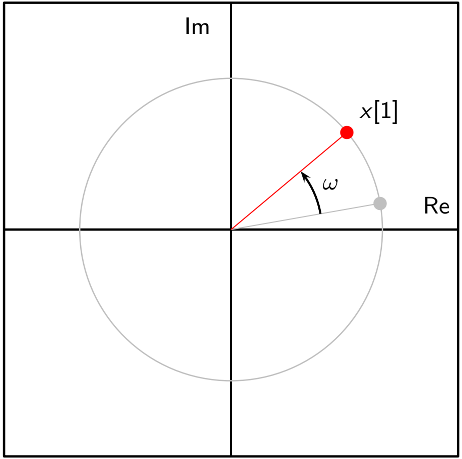

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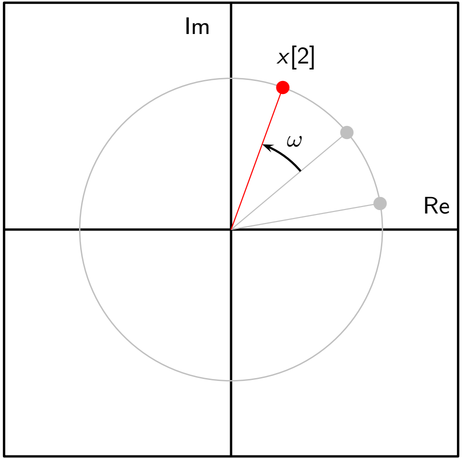

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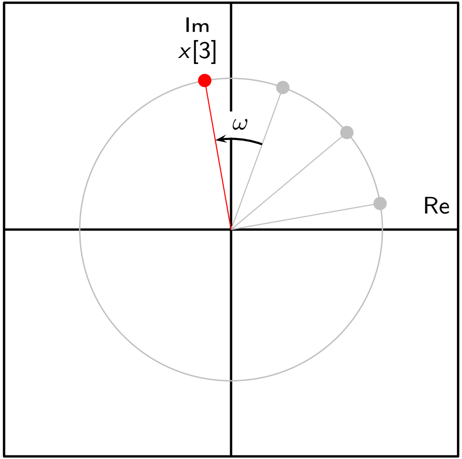

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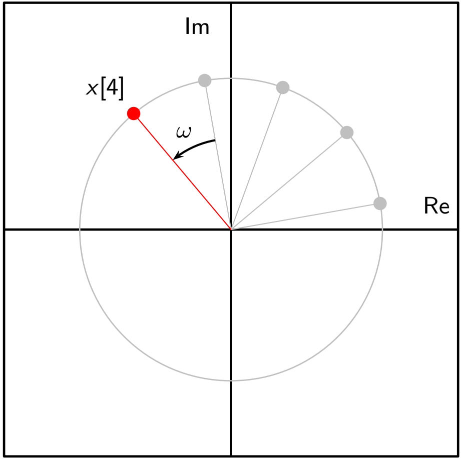

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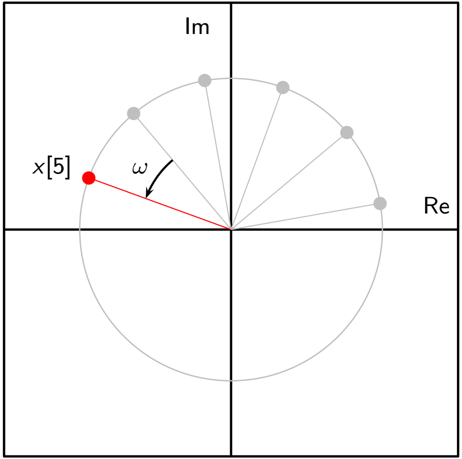

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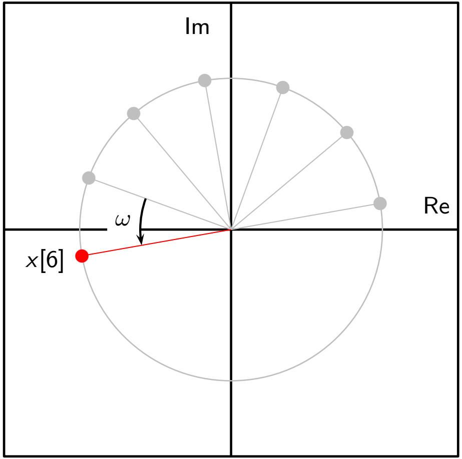

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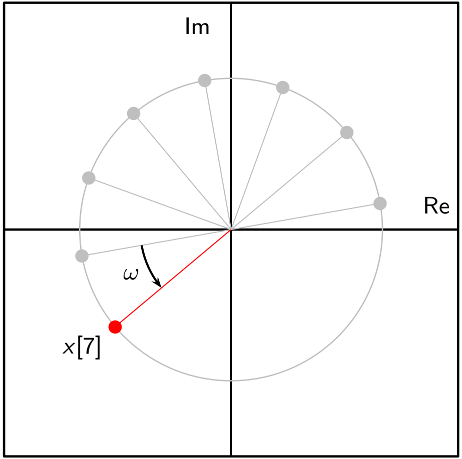

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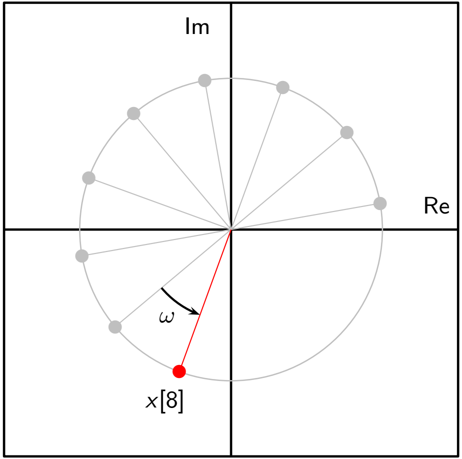

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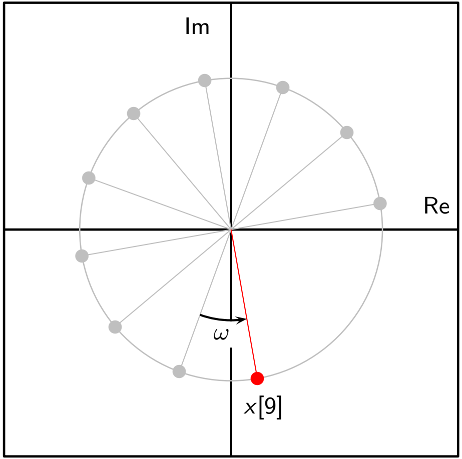

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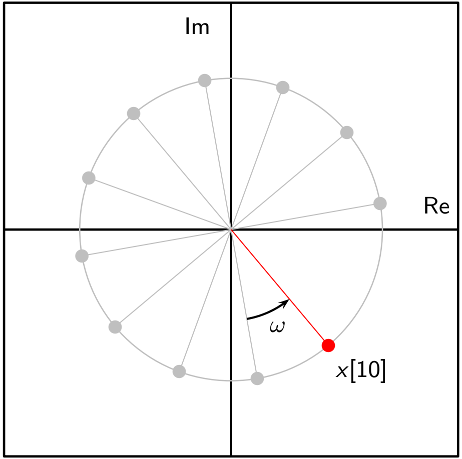

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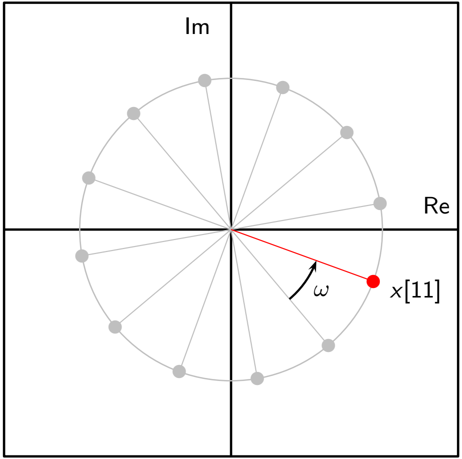

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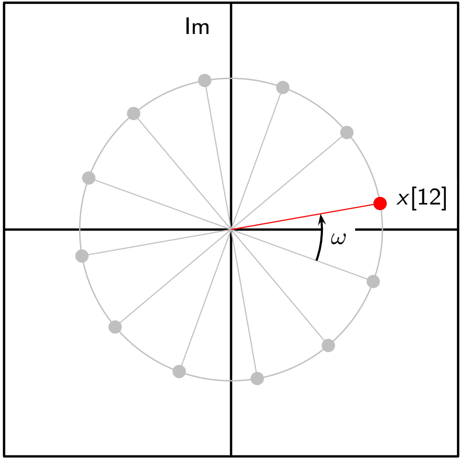

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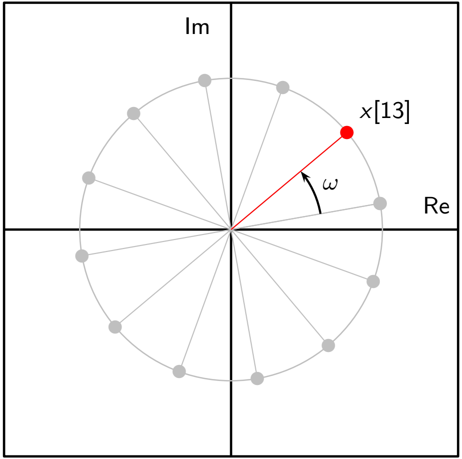

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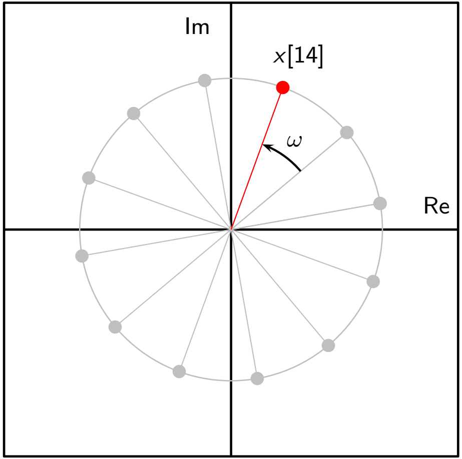

Careful: not every discrete-time sinusoid is periodic!

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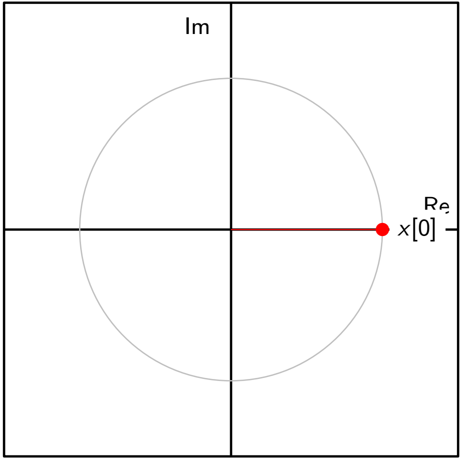

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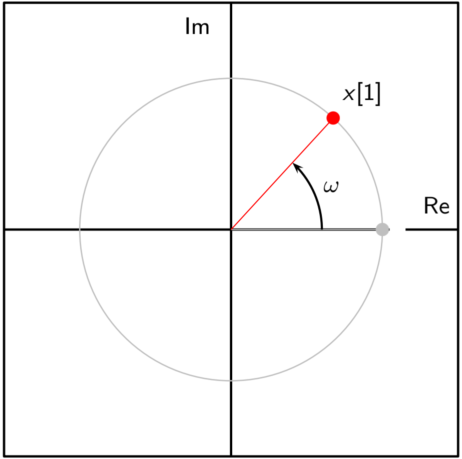

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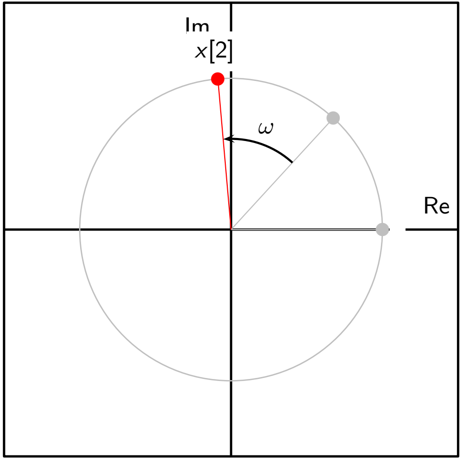

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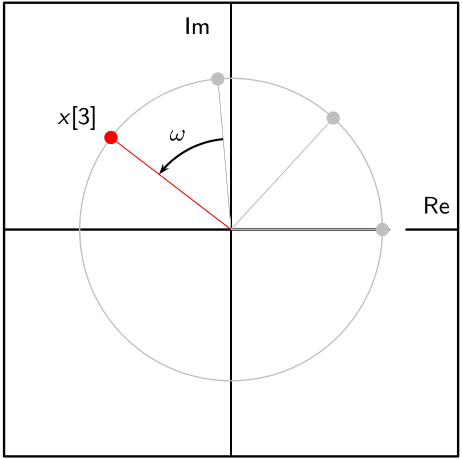

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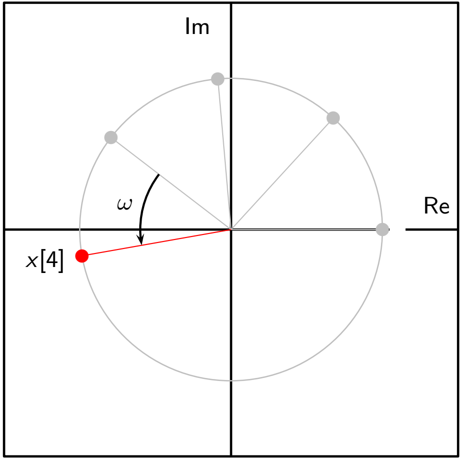

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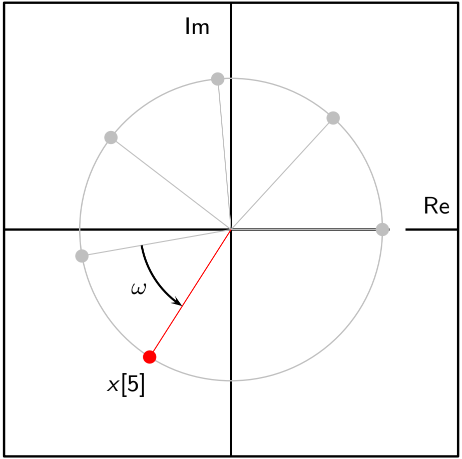

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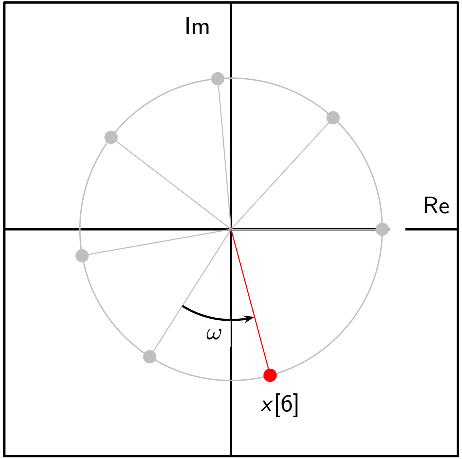

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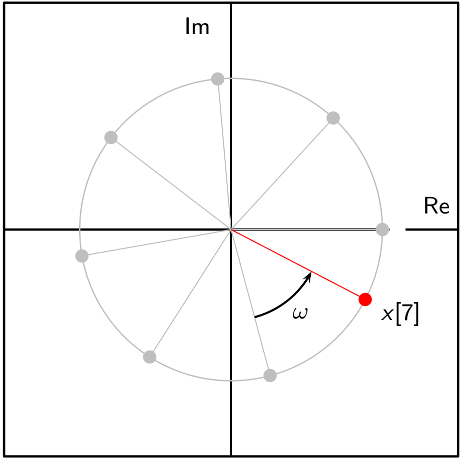

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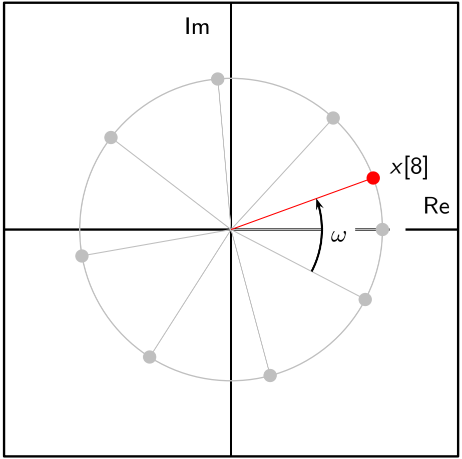

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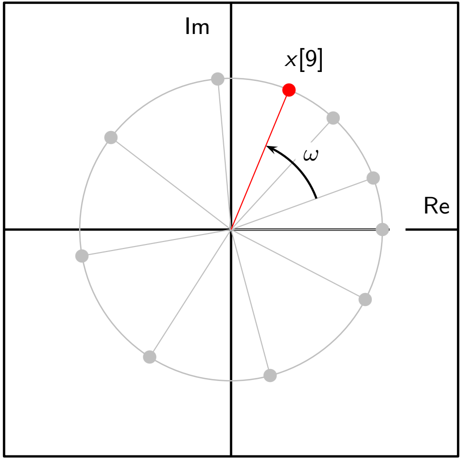

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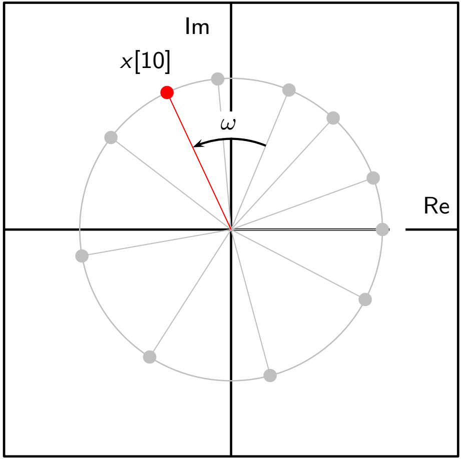

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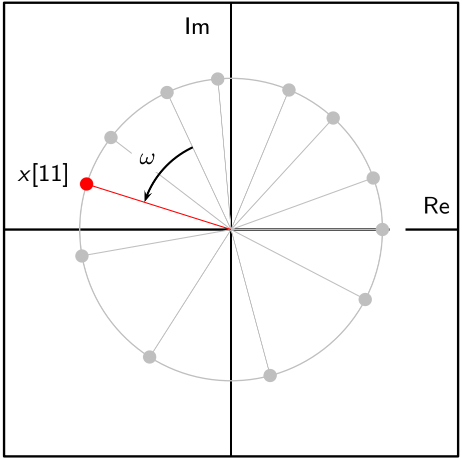

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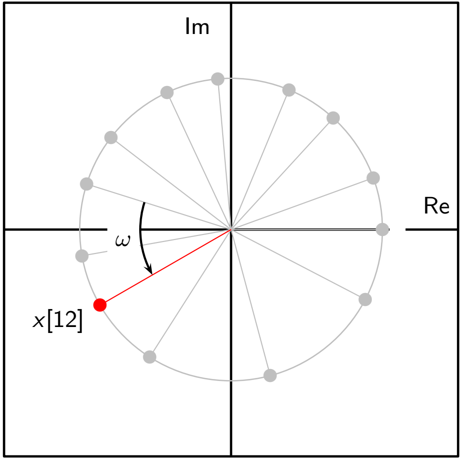

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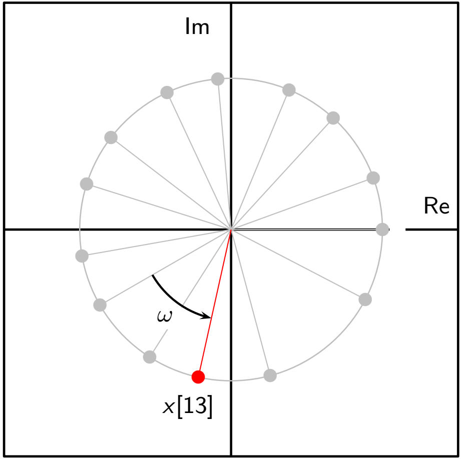

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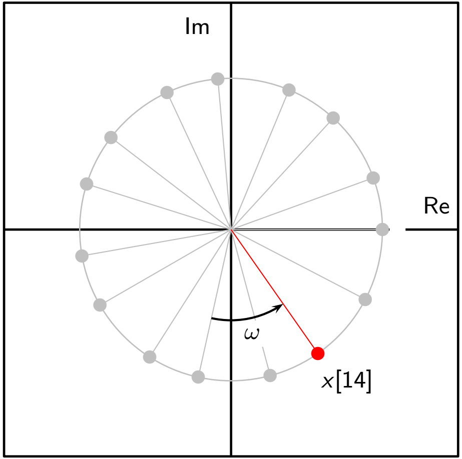

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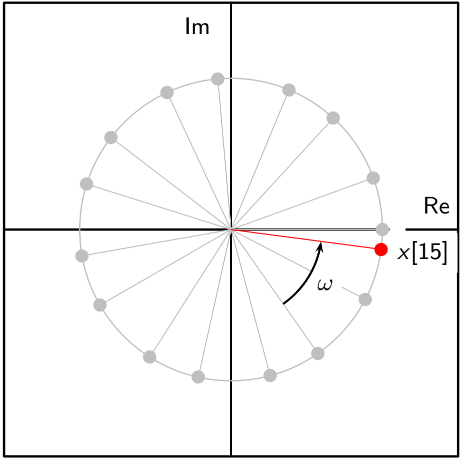

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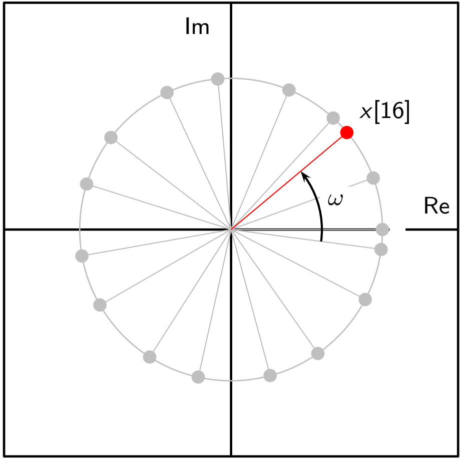

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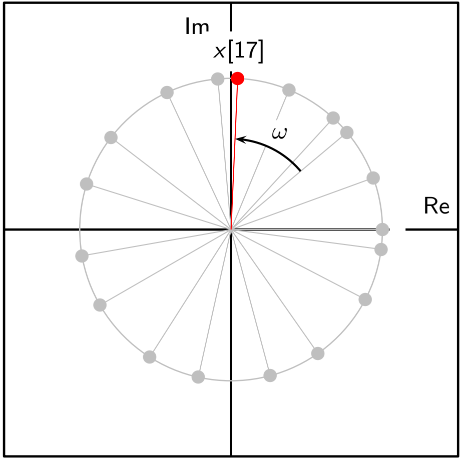

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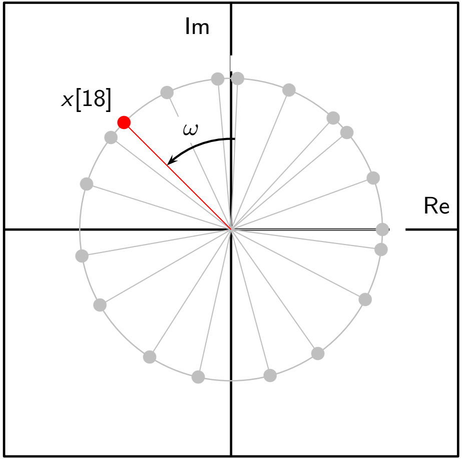


Frequencies yielding periodic discrete-time oscillations

$$e^{j\omega n} \text{ periodic in } n \iff \omega = \frac{M}{N} 2\pi, \quad M, N \in \mathbb{Z}$$

Frequencies yielding periodic discrete-time oscillations

$$x[n] = x[n+N], \qquad N \in \mathbb{N}^+$$

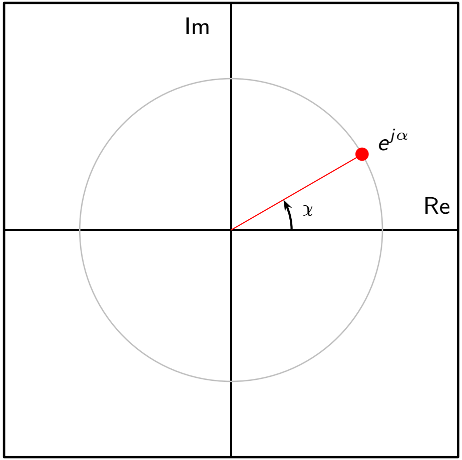
$$\begin{aligned}
e^{j(\omega n+\phi)} &= e^{j(\omega(n+N)+\phi)} \\
e^{j\omega n}e^{j\phi} &= e^{j\omega n}e^{j\omega N}e^{j\phi} \\
e^{j\omega N} &= 1 \\
\omega N &= M \cdot 2\pi, \qquad M \in \mathbb{Z} \\
\omega &= \frac{M}{N}2\pi
\end{aligned}$$

Phase periodicity of complex exponentials

$$e^{j\alpha} = e^{j(\alpha+2k\pi)} \quad \forall k \in \mathbb{Z}$$

(this is not about discrete-time, only about the mathematical object $e^{j\alpha}$)

Phase periodicity: one point, many names



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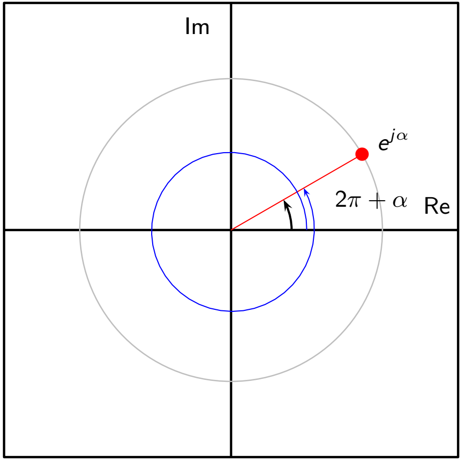

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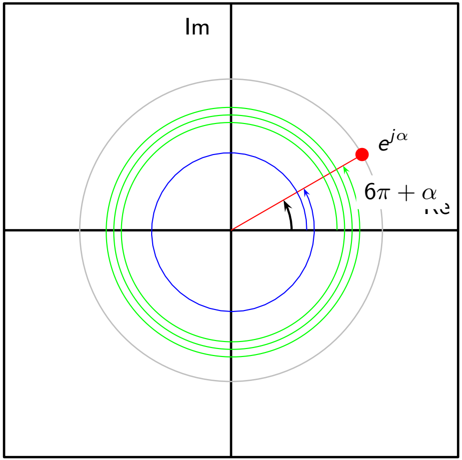

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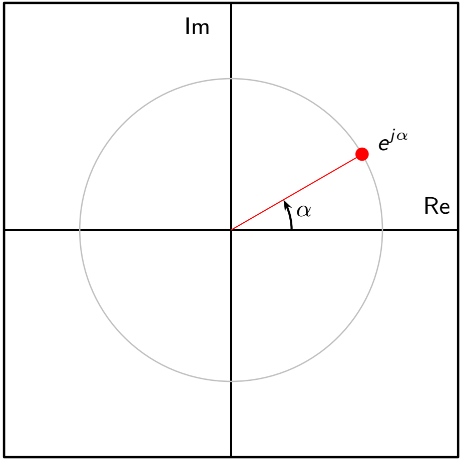

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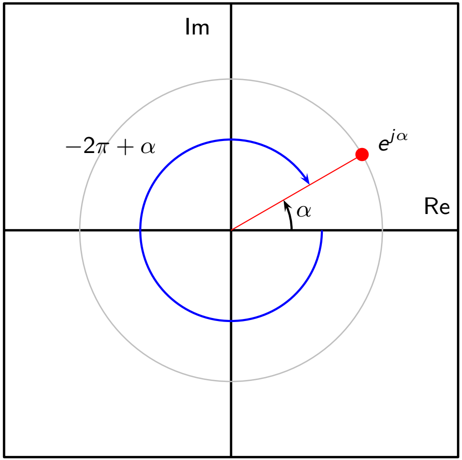
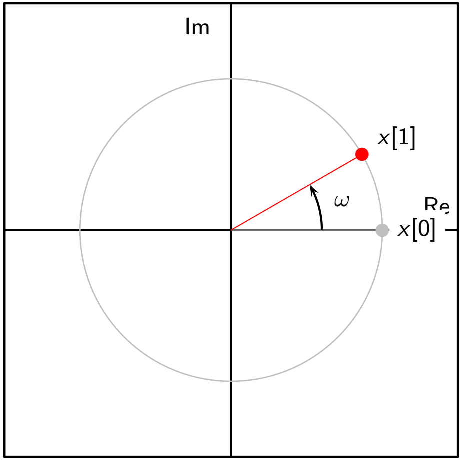


How “fast” can we oscillate in discrete time?

$$0 \leq \omega < 2\pi$$

Remember the complex exponential generating machine



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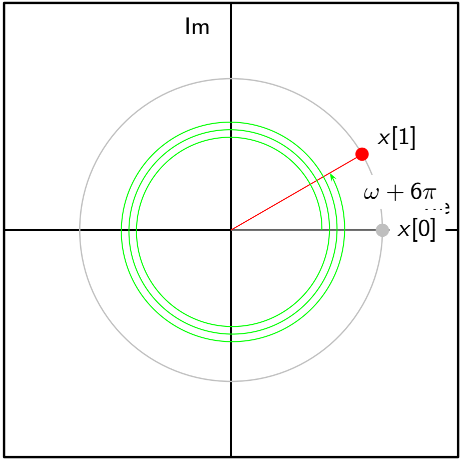
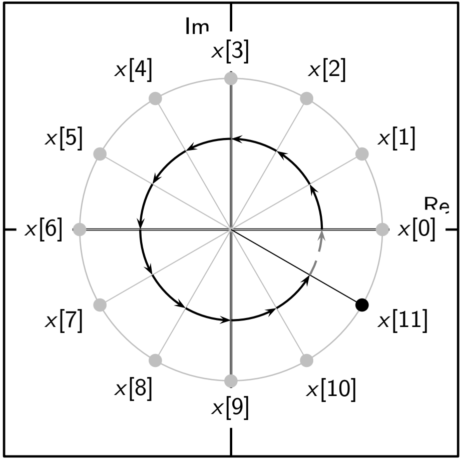


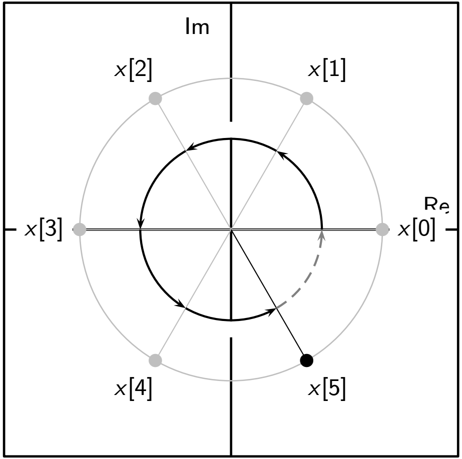
How “fast” can we go?

$$\omega = 2\pi/12$$



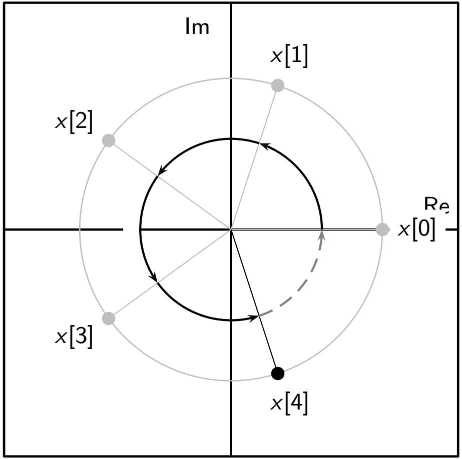
How “fast” can we go?

$$\omega = 2\pi/6$$



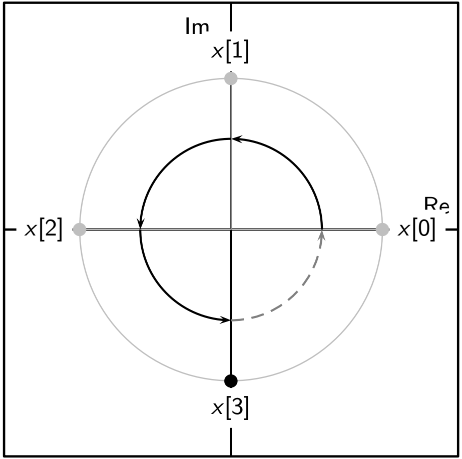
How “fast” can we go?

$$\omega = 2\pi/5$$



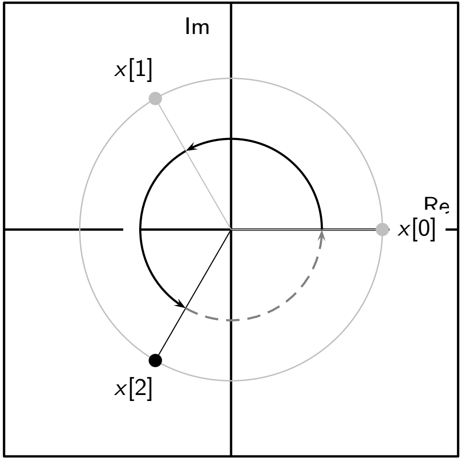
How “fast” can we go?

$$\omega = 2\pi/4$$



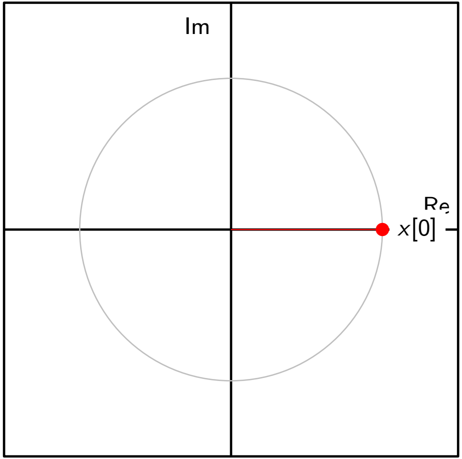
How "fast" can we go?

$$\omega = 2\pi/3$$



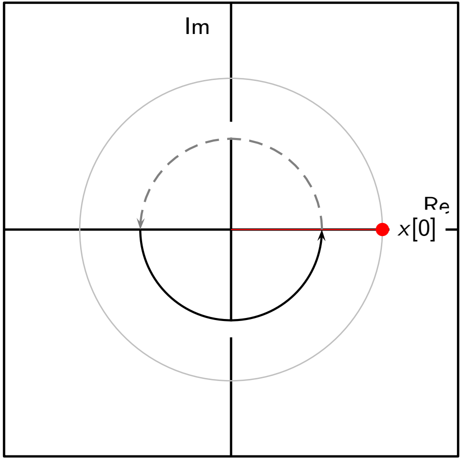
How “fast” can we go?

$$\omega = 2\pi/2 = \pi$$



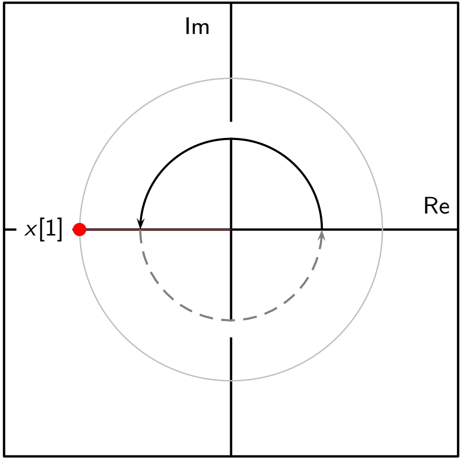
How "fast" can we go?

$$\omega = 2\pi/2 = \pi$$



How “fast” can we go?

$$\omega = 2\pi/2 = \pi$$



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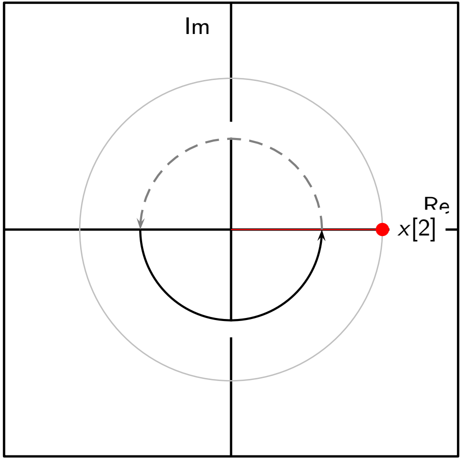

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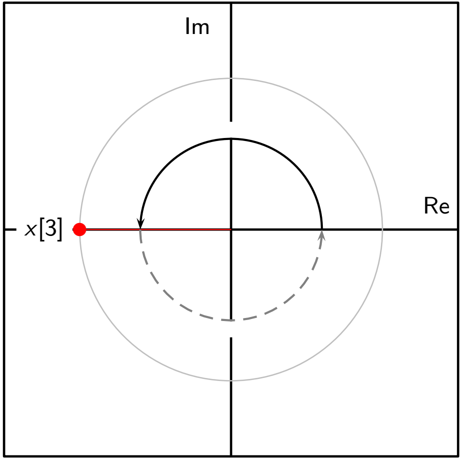

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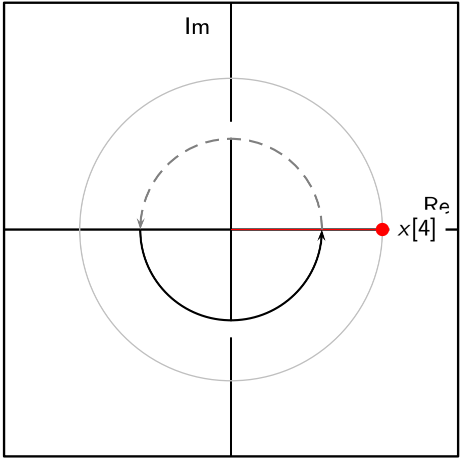

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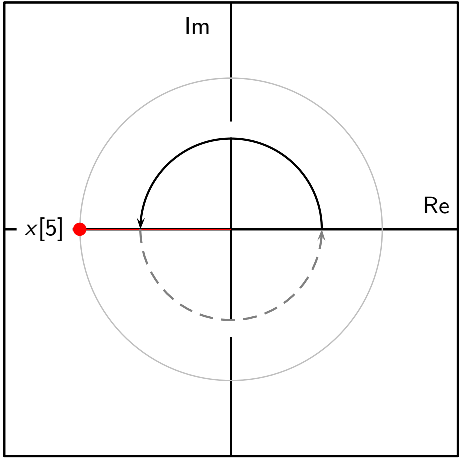
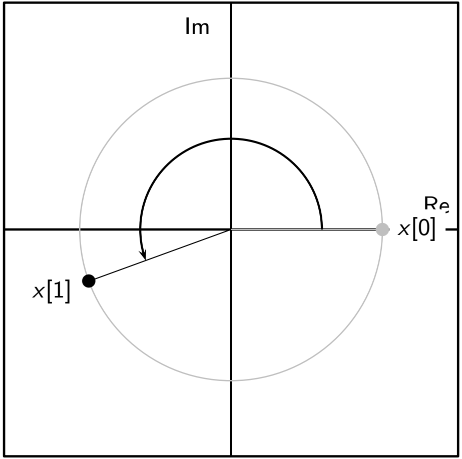
$$\omega = 2\pi/2 = \pi$$



What if we go "faster"?

$$\pi < \omega < 2\pi$$



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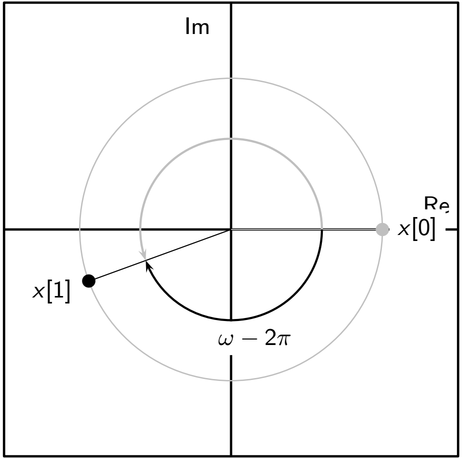
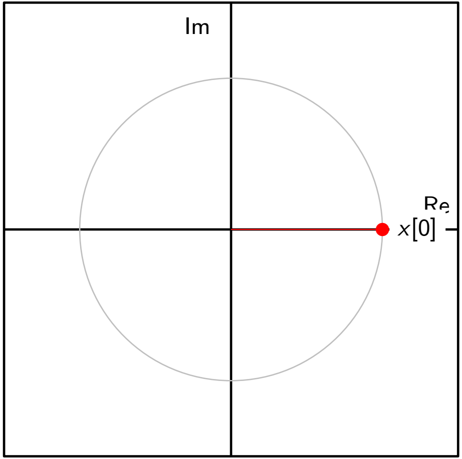


Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



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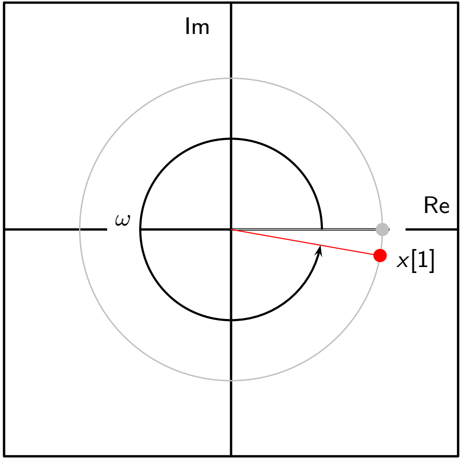

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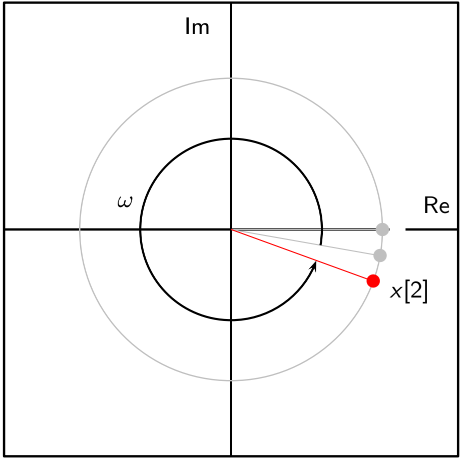

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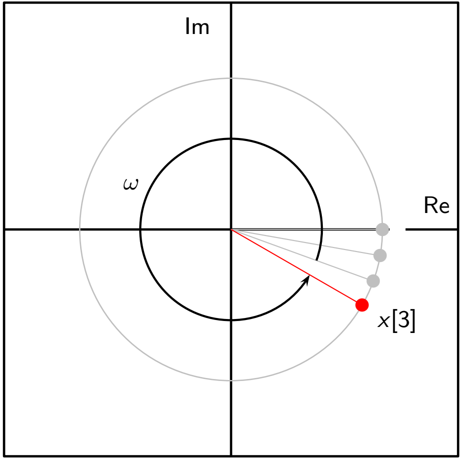

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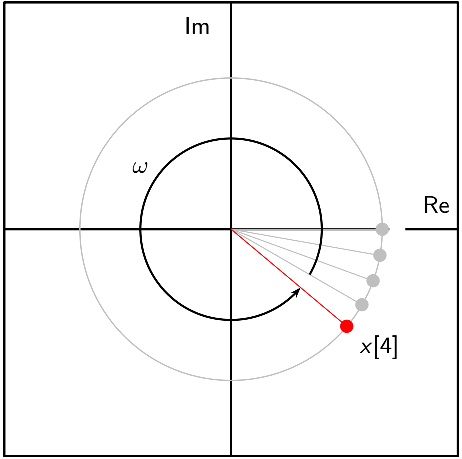

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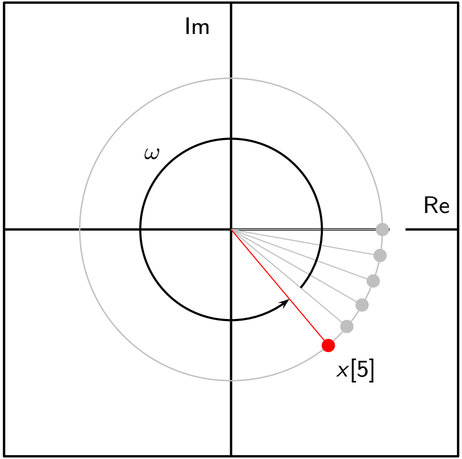

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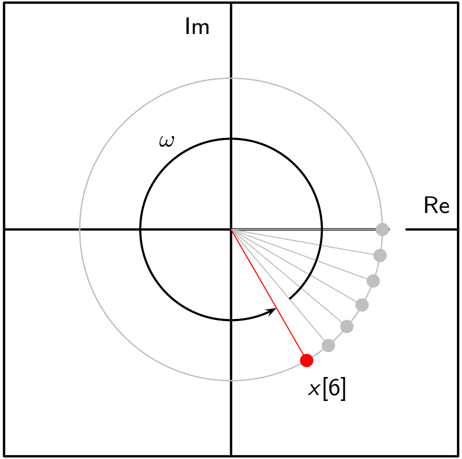

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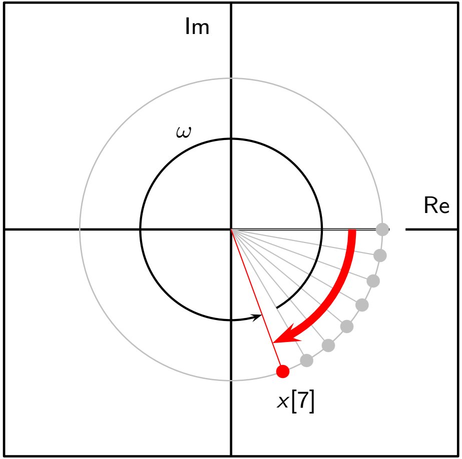


Summary

- $x[n] = e^{j(\omega n+\phi)}$ is the prototypical discrete-time oscillation
  - ω: angular frequency
  - ϕ: initial phase
- discrete-time oscillations are periodic ONLY if frequency is a rational multiple of 2π
- in discrete time, ω and $\omega + 2k\pi$ are indistinguishable frequencies

the Fourier basis

The Fourier Basis for $\mathbb{C}^N$

Claim: the set of N signals in $\mathbb{C}^N$

$$w_k[n] = e^{j\frac{2\pi}{N}nk}, \qquad n, k = 0, 1, \ldots, N-1$$

is an orthogonal basis in $\mathbb{C}^N$.

The Fourier Basis for $\mathbb{C}^N$

In vector notation:

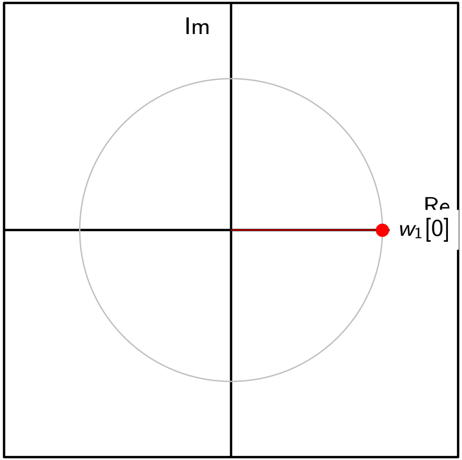
$$\{\mathbf{w}_k\}_{k=0,1,\ldots,N-1}$$

with

$$(w_k)_n = e^{j\frac{2\pi}{N}nk}$$

is an orthogonal basis in $\mathbb{C}^N$

Recall the complex exponential generating machine...



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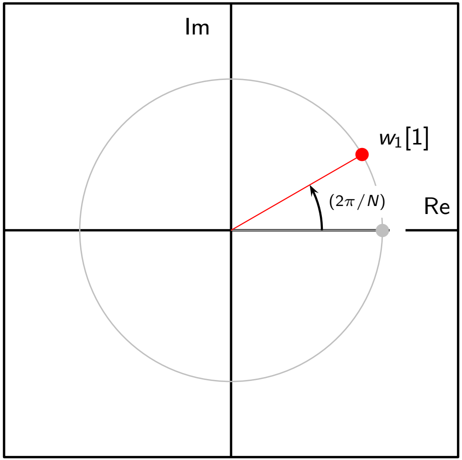

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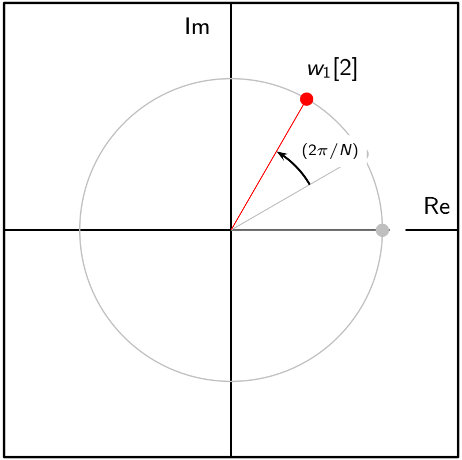

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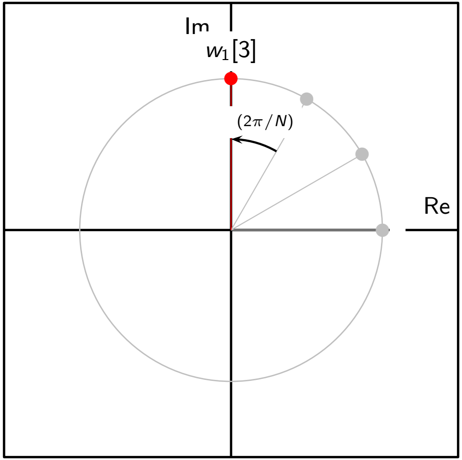

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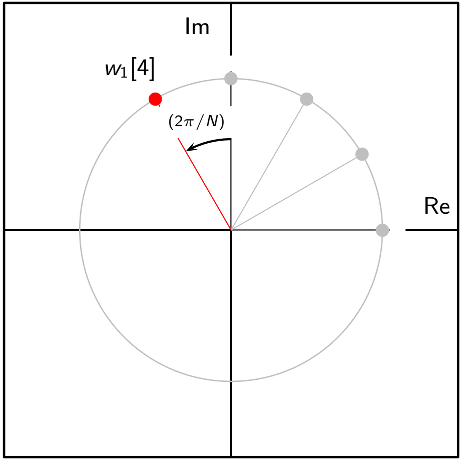

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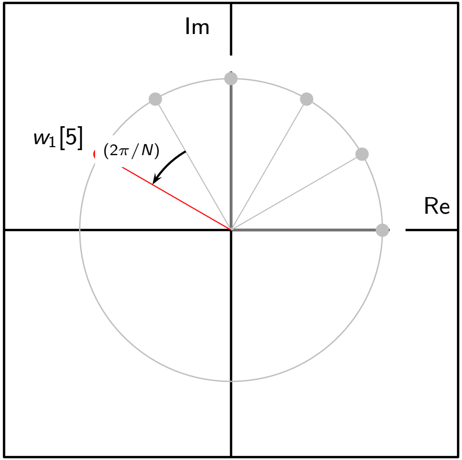

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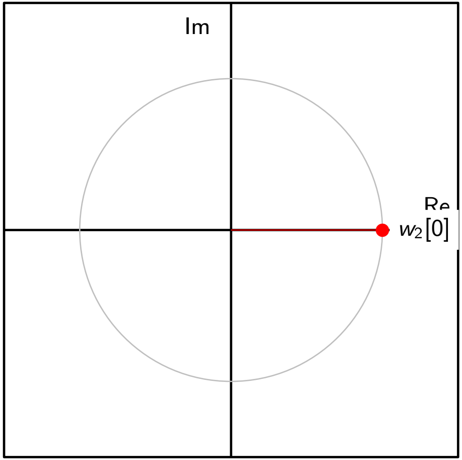

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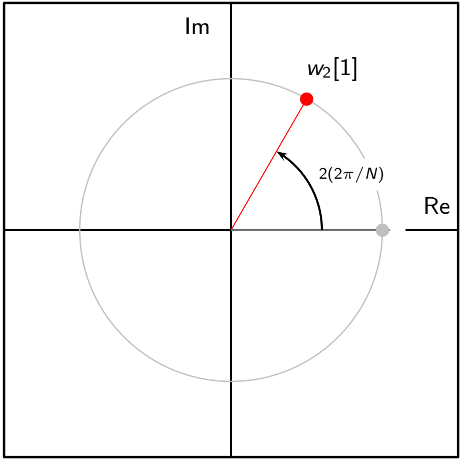

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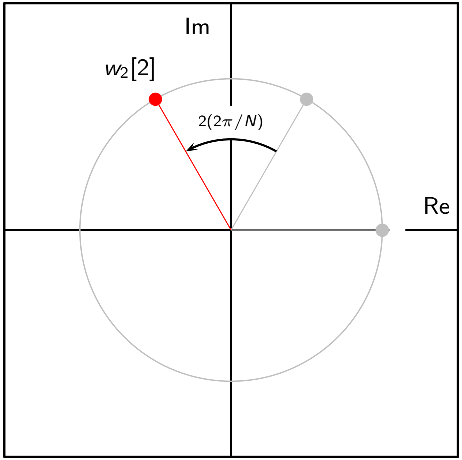

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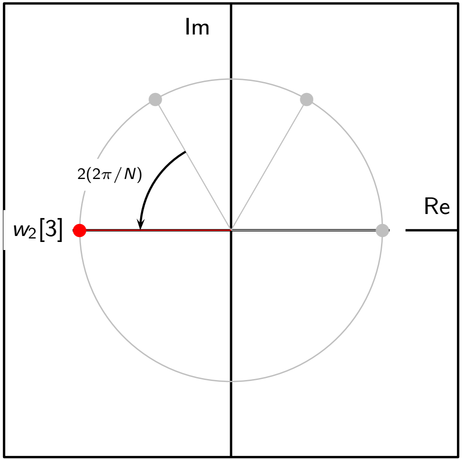

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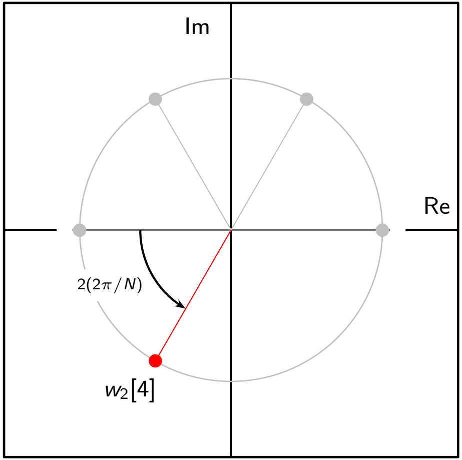

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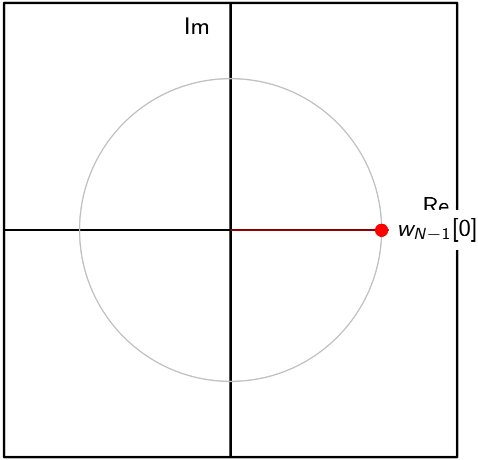

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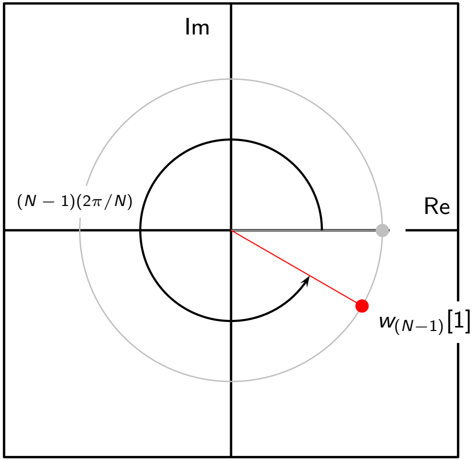

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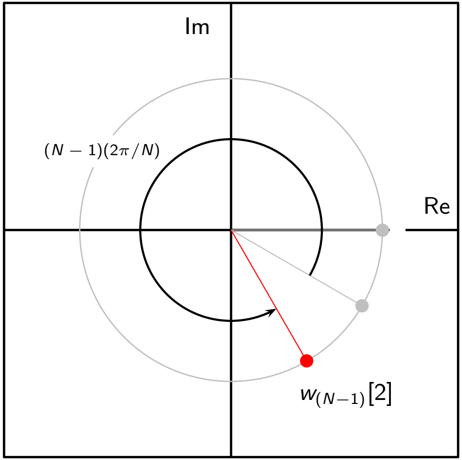

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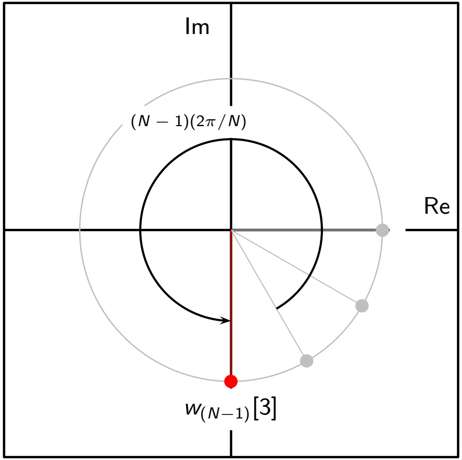

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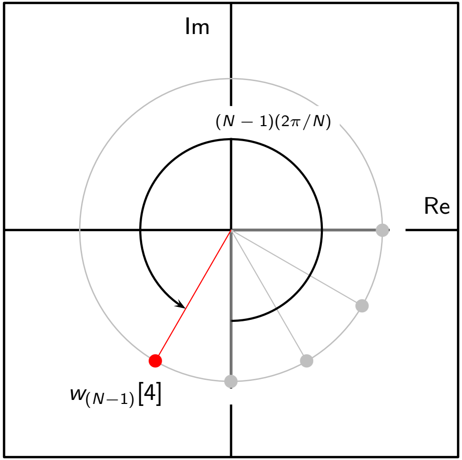
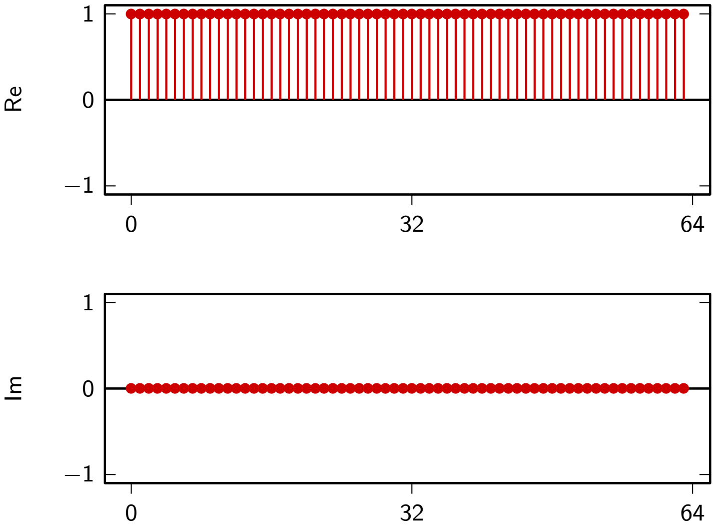
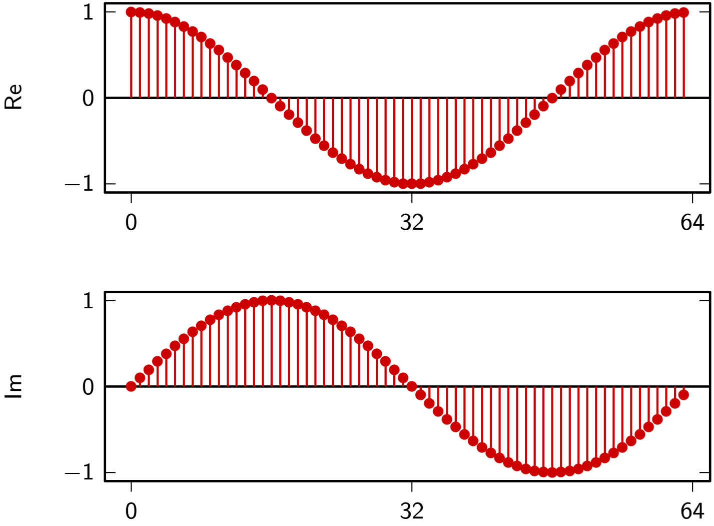


Basis vector $\mathbf{w}_0 \in \mathbb{C}^{64}$



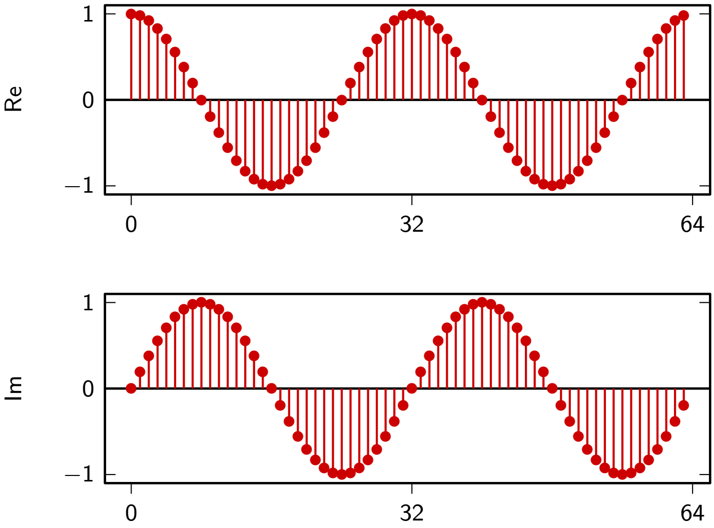
$\omega_0 = 0$

Basis vector $\mathbf{w}_1 \in \mathbb{C}^{64}$



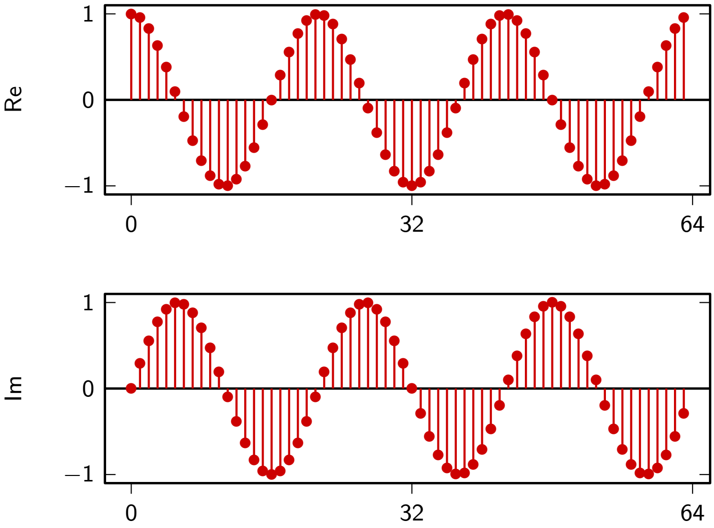
$$\omega_1 = \frac{2\pi}{64}$$

Basis vector $\mathbf{w}_2 \in \mathbb{C}^{64}$



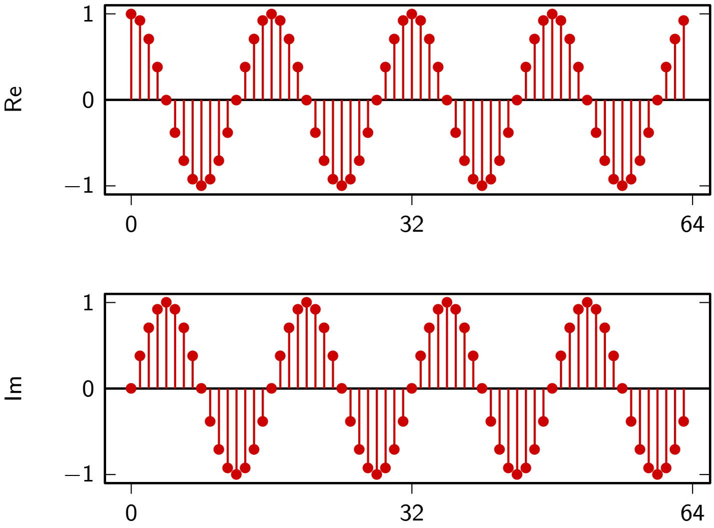
$$\omega_2 = 2\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_3 \in \mathbb{C}^{64}$



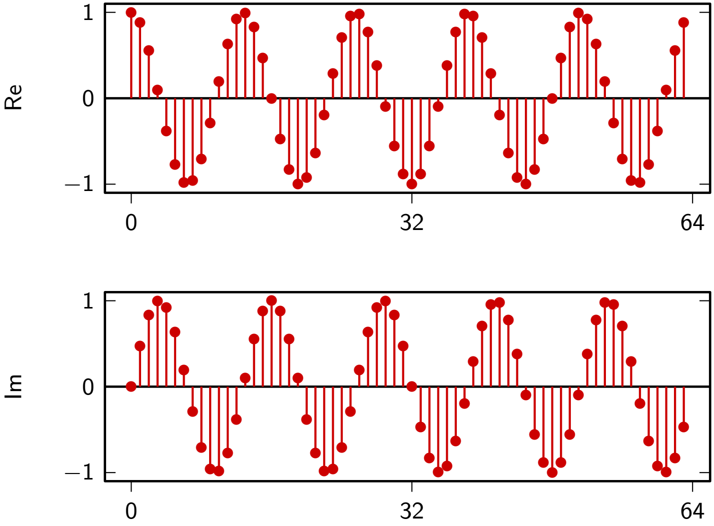
$$\omega_3 = 3\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_4 \in \mathbb{C}^{64}$



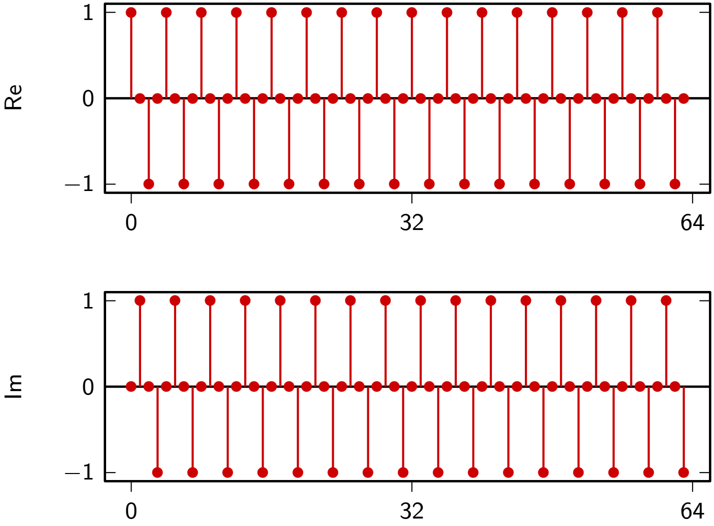
$$\omega_4 = 4\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_5 \in \mathbb{C}^{64}$



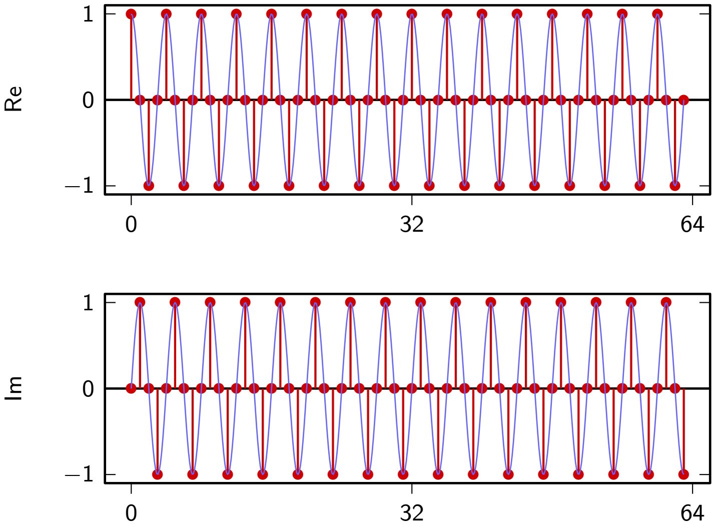
$$\omega_5 = 5\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{16} \in \mathbb{C}^{64}$



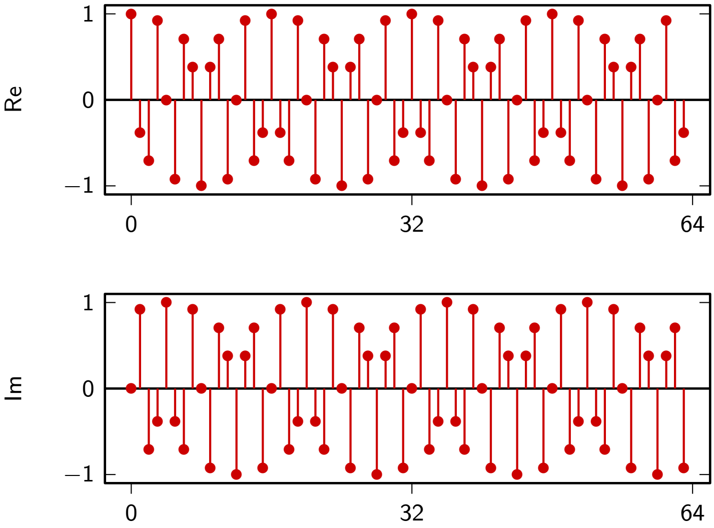
$$\omega_{16} = 16\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{16} \in \mathbb{C}^{64}$



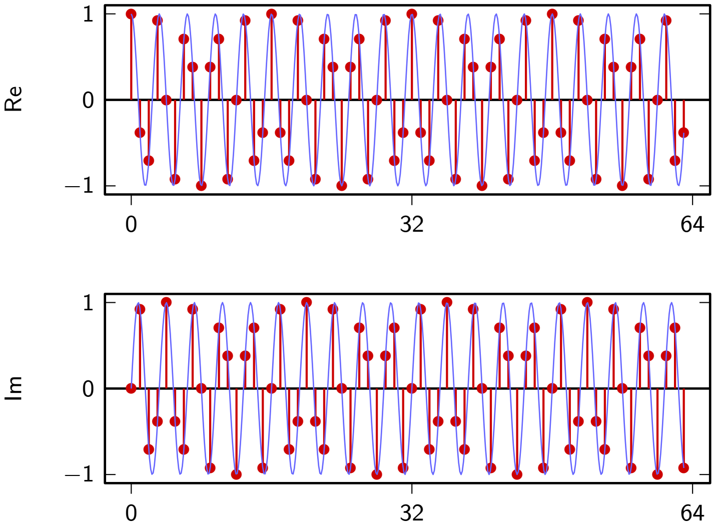
$$\omega_{16} = 16\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{20} \in \mathbb{C}^{64}$



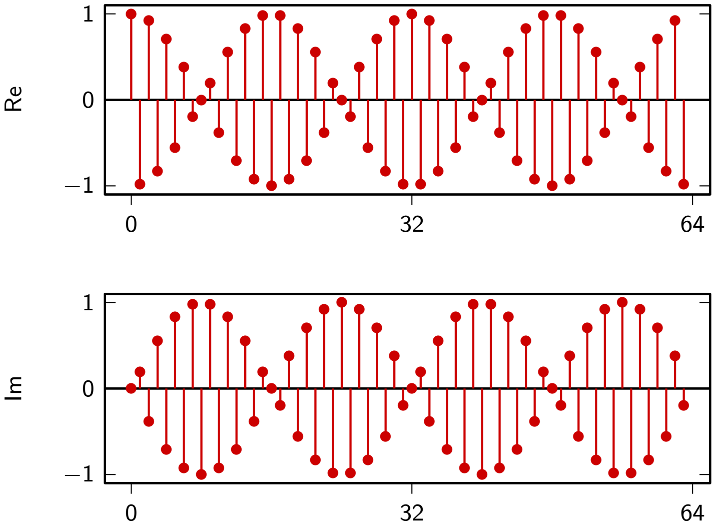
$$\omega_{20} = 20\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{20} \in \mathbb{C}^{64}$



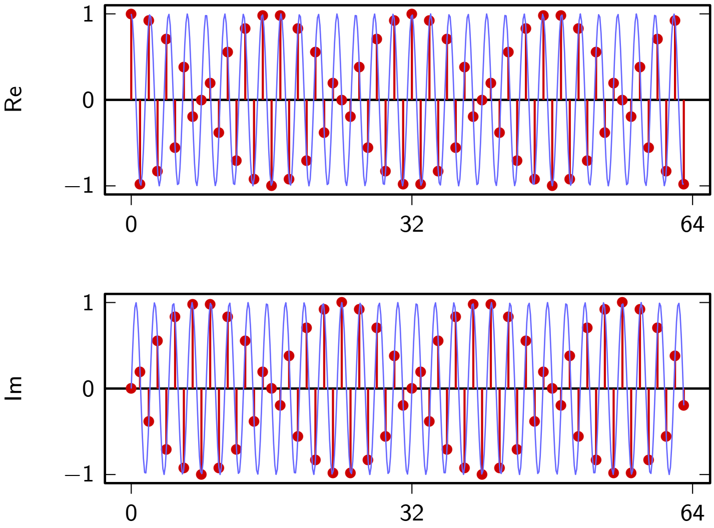
$$\omega_{20} = 20\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{30} \in \mathbb{C}^{64}$



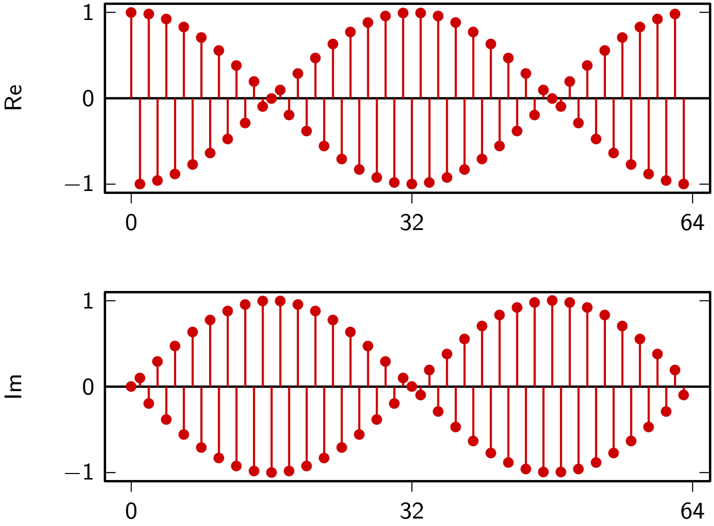
$$\omega_{30} = 30\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{30} \in \mathbb{C}^{64}$



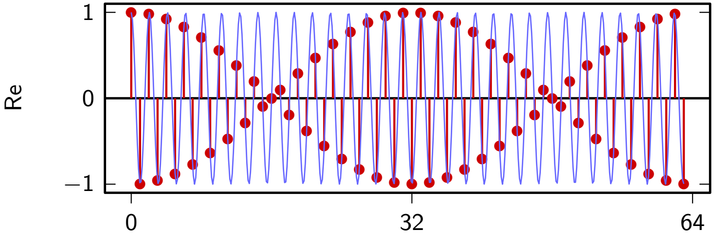
$$\omega_{30} = 30\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{31} \in \mathbb{C}^{64}$

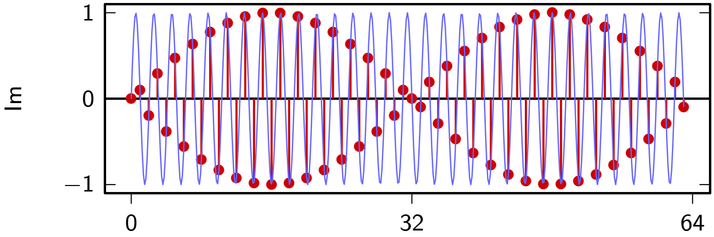


$$\omega_{31} = 31\,\frac{2\pi}{64}$$

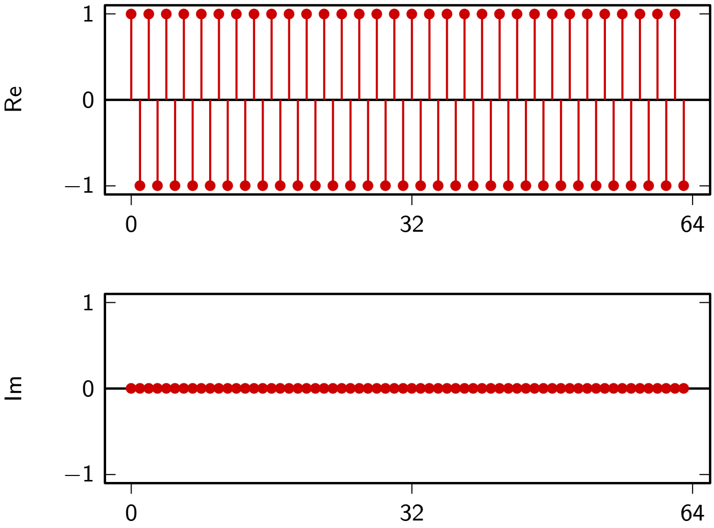
Basis vector $\mathbf{w}_{31} \in \mathbb{C}^{64}$



$$\omega_{31} = 31 \, \frac{2\pi}{64}$$

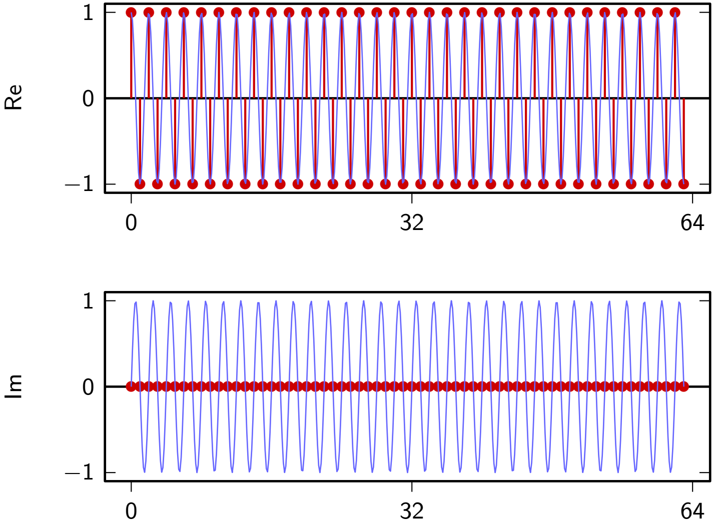


Basis vector $\mathbf{w}_{32} \in \mathbb{C}^{64}$



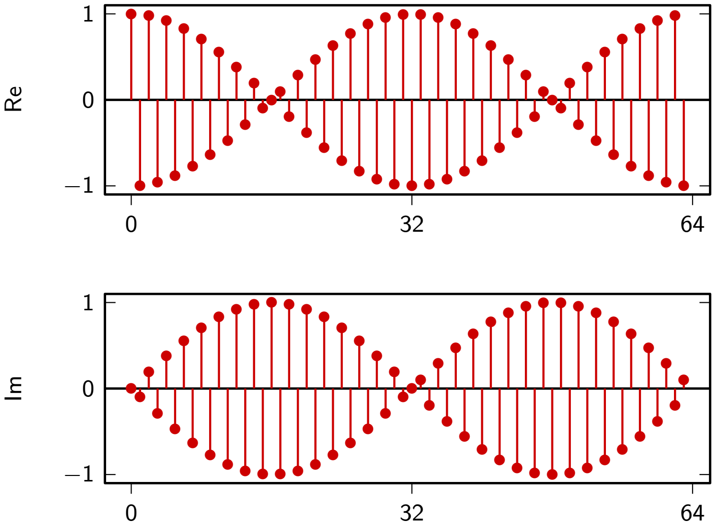
$$\omega_{32} = 32\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{32} \in \mathbb{C}^{64}$



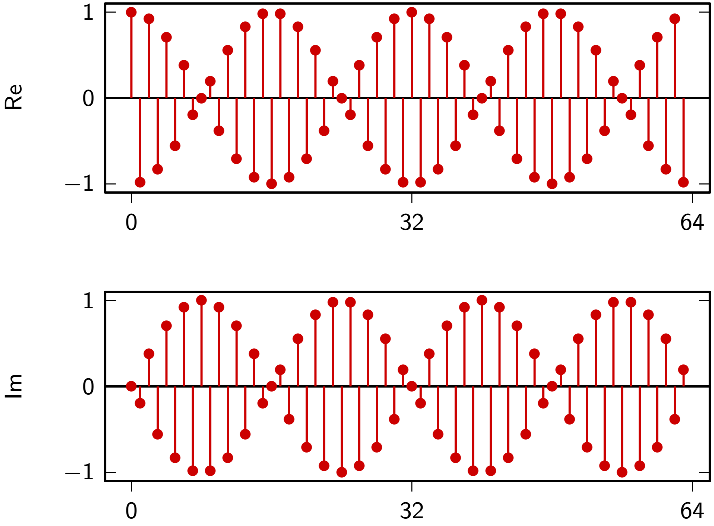
$$\omega_{32} = 32\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{33} \in \mathbb{C}^{64}$



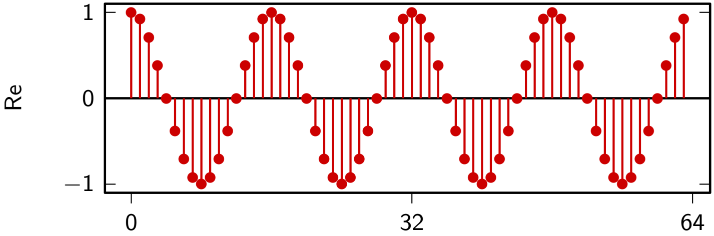
$$\omega_{33} = 33\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{34} \in \mathbb{C}^{64}$

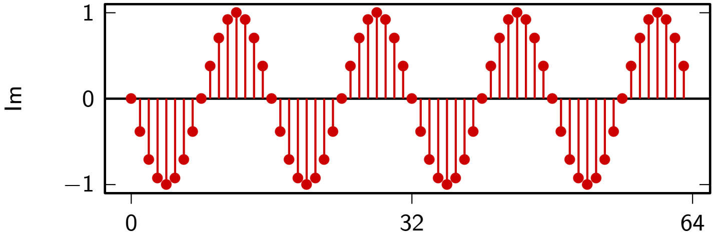


$$\omega_{34} = 34\,\frac{2\pi}{64}$$

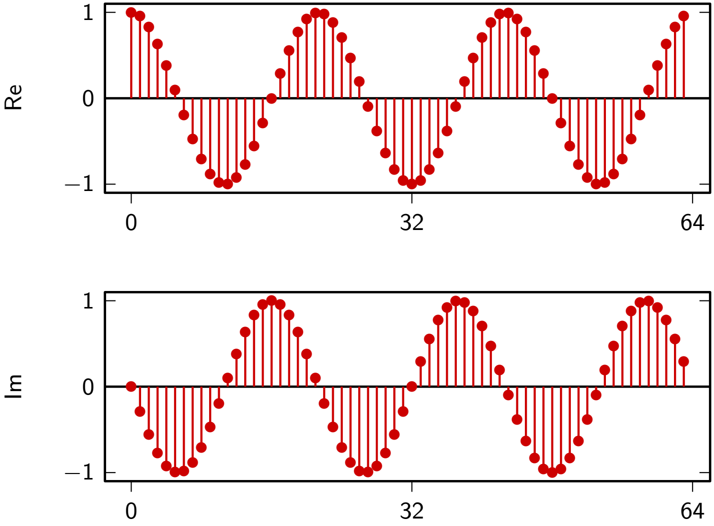
Basis vector $\mathbf{w}_{60} \in \mathbb{C}^{64}$



$$\omega_{60} = 60\,\frac{2\pi}{64}$$

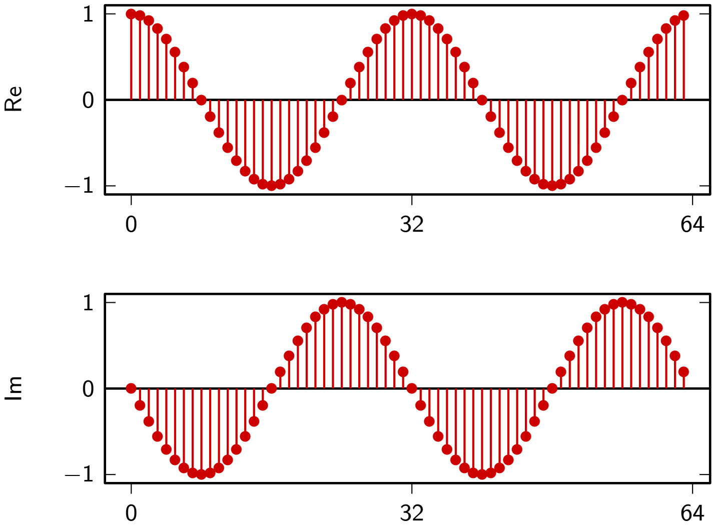


Basis vector $\mathbf{w}_{61} \in \mathbb{C}^{64}$



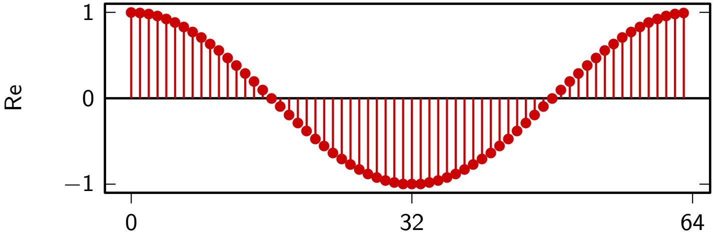
$$\omega_{61} = 61 \, \frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{62} \in \mathbb{C}^{64}$

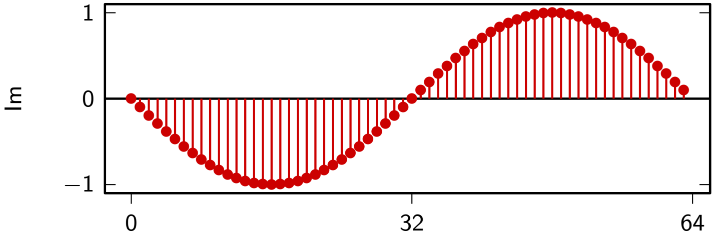


$$\omega_{62} = 62\,\frac{2\pi}{64}$$

Basis vector $\mathbf{w}_{63} \in \mathbb{C}^{64}$



$$\omega_{63} = 63 \, \frac{2\pi}{64}$$



Proof of orthogonality

$$
\begin{aligned}
\langle \mathbf{w}_k, \mathbf{w}_h \rangle &= \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}nk}\right)^* e^{j\frac{2\pi}{N}nh} \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(h-k)n} \\
&= \begin{cases} N & \text{for } h = k \\ \dfrac{1 - e^{j2\pi(h-k)}}{1 - e^{j\frac{2\pi}{N}(h-k)}} = 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

the Discrete Fourier Transform

The Fourier Basis for $\mathbb{C}^N$

The set of N orthogonal vectors

$$\{\mathbf{w}_k\}_{k=0,1,\dots,N-1}$$

where

$$\mathbf{w}_k = \begin{bmatrix} e^{j\frac{2\pi}{N}k\cdot 0} & e^{j\frac{2\pi}{N}k} & e^{j\frac{2\pi}{N}2k} & e^{j\frac{2\pi}{N}3k} & \dots & e^{j\frac{2\pi}{N}(N-1)k} \end{bmatrix}^T$$

is a basis for $\mathbb{C}^N$.

The Fourier Basis for $\mathbb{C}^N$

- N orthogonal vectors $\longrightarrow$ basis for $\mathbb{C}^N$
- vectors are not ortho*normal*; normalization factor would be $1/\sqrt{N}$
- for *practical* (i.e. algorithmic) reasons, we will keep the normalization factor explicit in the change of basis formulas

From the time domain to the frequency domain

- input data in terms of the orthonormal canonical basis:

$$\mathbf{x} = \sum_{n=0}^{N-1} x_n \mathbf{e}_n$$

- same data in terms of the orthogonal Fourier basis

$$\mathbf{X} = \sum_{k=0}^{N-1} X_k \mathbf{w}_k$$

- because of orthognonality, the N new coordinates can be easily computed as

$$X_k = \langle \mathbf{w}_k, \mathbf{x} \rangle$$

Basis expansion using the Fourier basis

Analysis formula:

$$X_k = \langle \mathbf{w}_k, \mathbf{x} \rangle$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}_k$$

Basis expansion in algorithmic form (using data arrays)

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *"time" domain*

Change of basis in matrix form

- Fourier basis is orthognonal so we can build a change-of-basis matrix
- the N new coordinates are

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

where

$$\mathbf{W}[k, n] = \langle \mathbf{w}_k, \mathbf{e}_n \rangle = e^{-j\frac{2\pi}{N}nk}$$

The Fourier matrix

The Fourier matrix is obtained by stacking the conjugate-transposes of the basis vectors:

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_0^H \\ \mathbf{w}_1^H \\ \mathbf{w}_2^H \\ \vdots \\ \mathbf{w}_{N-1}^H \end{bmatrix}$$

The Fourier matrix is unitary up to a constant:

$$\mathbf{W}^H\mathbf{W} = \mathbf{W}\mathbf{W}^H = N\mathbf{I}$$

Change of basis in matrix form

Analysis formula:

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$$

For those keeping track of details...

Our definition for the inner product in $\mathbb{C}^N$ conjugates the first term:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_n x_n^* y_n$$

This preserves the structure of the Fourier formulas:

$$X_k = \langle \mathbf{w}_k, \mathbf{x} \rangle \qquad \text{single coefficient}$$

$$\mathbf{X} = \mathbf{W}\mathbf{x} \qquad \text{full set of coefficients}$$

Structure of the Fourier matrix

Define $W_N = e^{-j\frac{2\pi}{N}}$
(or simply W when N is evident from the context)

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W^1 & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{2(N-1)} \\ & & & \dots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)^2} \end{bmatrix}$$

DFT Matrix

Because of the aliasing property of the complex exponential

$$W_N^m = W_N^{(m \mod N)}$$

Example:

$$\begin{aligned} W_8^{11} &= e^{-j\frac{2\pi}{8}11} \\ &= e^{-j\frac{2\pi}{8}8} e^{-j\frac{2\pi}{8}3} \\ &= W_8^3 \end{aligned}$$

Small DFT matrices: $N = 2, 3$

$$W_2 = e^{-j\frac{2\pi}{2}} = -1$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$W_3 = e^{-j\frac{2\pi}{3}} = -(1 + j\sqrt{3})/2$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W & W^2 \\ 1 & W^2 & W^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W & W^2 \\ 1 & W^2 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -(1 + j\sqrt{3})/2 & -(1 - j\sqrt{3})/2 \\ 1 & -(1 - j\sqrt{3})/2 & -(1 + j\sqrt{3})/2 \end{bmatrix}$$

Small DFT matrices: $N = 4$

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Small DFT matrices: $N = 5$

$$\mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W^6 & W^8 \\ 1 & W^3 & W^6 & W^9 & W^{12} \\ 1 & W^4 & W^8 & W^{12} & W^{16} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W & W^3 \\ 1 & W^3 & W & W^4 & W^2 \\ 1 & W^4 & W^3 & W^2 & W \end{bmatrix}$$

Small DFT matrices: $N = 6$

$$\mathbf{W}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \\ 1 & W^2 & W^4 & W^6 & W^8 & W^{10} \\ 1 & W^3 & W^6 & W^9 & W^{12} & W^{15} \\ 1 & W^4 & W^8 & W^{12} & W^{16} & W^{20} \\ 1 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \\ 1 & W^2 & W^4 & 1 & W^2 & W^4 \\ 1 & W^3 & 1 & W^3 & 1 & W^3 \\ 1 & W^4 & W^2 & 1 & W^4 & W^2 \\ 1 & W^5 & W^4 & W^3 & W^2 & W \end{bmatrix}$$