Exercise 1. DTFT

Consider the signal

$$x[n] = \begin{cases} 0 & n = 0 \\ 1/n^3 & n \neq 0. \end{cases}$$

Show that its DTFT $X(\omega)$ exists and that the real part of $X(\omega)$ is equal to zero. [*Note: you do not need to compute the DTFT explicitly.*]

Solution:

To show that the DTFT exists, it suffices to show that the signal has finite energy, that is, $\sum_n |x[n]|^2 < \infty$. We have

$$S = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 2\sum_{n=1}^{\infty} \frac{1}{n^6} = 2\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^3$$

Since $(1/x)^3 \leq (1/x)$ for all $x \geq 1$ we can write

$$S = 2\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^3 \leq 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} < \infty.$$

Since $x[-n] = -x[n]$, the signal is antisymmetric and therefore its DTFT is purely imaginary; to show the result formally:

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^3}(e^{-j\omega n} - e^{j\omega n}) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^3}(-j\sin(\omega n) - j\sin(\omega n)) \\ &= j\left(-2\sum_{n=1}^{\infty} \frac{\sin(\omega n)}{n^3}\right) \end{aligned}$$

which is purely imaginary.

Exercise 2. DTFT

Consider the function $f(\omega) = |\cos(\omega/2)|$:

(a) show that $f(\omega)$ is a valid DTFT (namely, that it is 2π-periodic and square integrable)

(b) find the sequence $x[n]$ whose DTFT $X(\omega)$ is equal to $f(\omega)$.

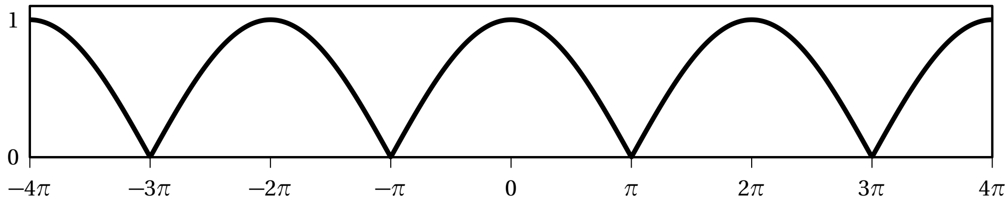
Solution:

(a) *the function* $f(\omega)$ *is* 2π*-periodic since*

$$f(\omega+2k\pi)=|\cos(\omega/2+k\pi)|=|(-1)^k\cos(\omega/2)|=|\cos(\omega/2)|=f(\omega).$$

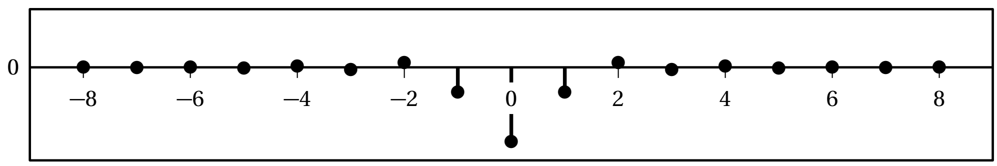
Additionally, $f(\omega)\in L_2([-\pi,\pi])$ *since*

$$\int_{-\pi}^{\pi}|\cos(\omega/2)|^2 d\omega=\frac{1}{2}[\omega+\sin\omega]_{-\pi}^{\pi}=\pi$$



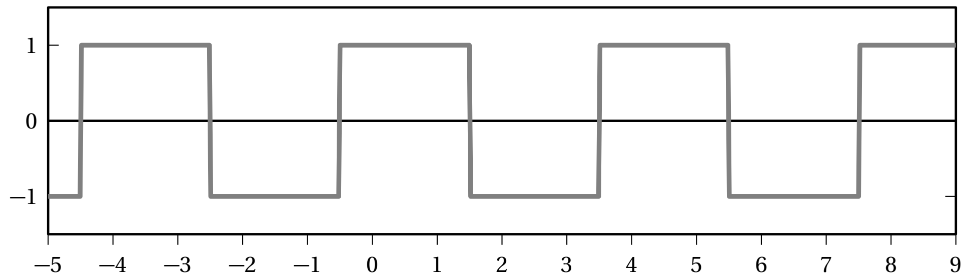
(b) *using the inverse formula for the DTFT:*

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos(\omega/2)|e^{j\omega n}d\omega\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega/2)e^{j\omega n}d\omega\\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}(e^{j\omega/2}+e^{-j\omega/2})e^{j\omega n}d\omega\\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}e^{j\omega(n+1/2)}+e^{j\omega(n-1/2)}d\omega\\
&= \frac{-j}{4\pi}\left(\frac{1}{n+1/2}e^{j\omega(n+1/2)}\Big|_{-\pi}^{\pi}+\frac{1}{n-1/2}e^{j\omega(n-1/2)}\Big|_{-\pi}^{\pi}\right)\\
&= \frac{-j}{4\pi}\left(\frac{2j(-1)^n}{n+1/2}-\frac{2j(-1)^n}{n-1/2}\right)\\
&= \frac{2}{\pi}\,\frac{(-1)^{n+1}}{4n^2-1}
\end{aligned}$$



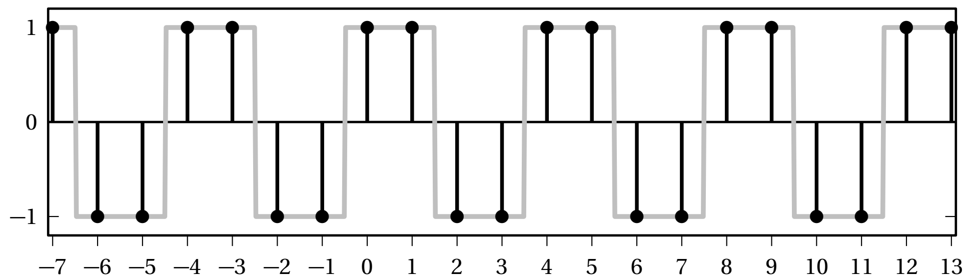
Exercise 3. DFT/DFS

The continuous-time square wave $x_c(t)$, a portion of which is shown in the figure below, is converted into the discrete-time sequence $x[n] = x_c(nT_s)$ with $T_s = 1$ and $n \in \mathbb{Z}$. Compute the appropriate frequency-domain representation for $x[n]$.



Solution:

The discrete-time samples look like this:



so that $x[n] = \tilde{x}[n]$ a periodic sequence with period 4. In this case the natural frequency-domain representation is the DFS, namely the four coefficients

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{3} \tilde{x}[n] e^{-j\frac{2\pi}{4}kn} \qquad k = 0,1,2,3 \\
&= 1 + e^{-j\frac{\pi}{2}k} - e^{-j\pi k} - e^{-j\frac{3\pi}{2}k} \\
&= 1 + e^{-j\frac{\pi}{2}k} - (-1)^k - e^{j\frac{\pi}{2}k} \\
&= 1 - (-1)^k - 2j\sin((\pi/2)k) \\
&= \begin{cases} 0 & k = 0,2 \\ 2-2j & k = 1 \\ 2+2j & k = 3 \end{cases} .
\end{aligned}$$

Exercise 4. DTFT

Sketch the *approximate* magnitude of the Fourier transform of the sequence

$$x[n] = \begin{cases} \left(\frac{99}{100}\right)^n + \left(j\frac{99}{100}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Solution:

We can write

$$x[n] = \alpha^n u[n] + (j\alpha)^n u[n]$$

with $\alpha = 0.99$. By linearity, therefore,

$$X(\omega) = \frac{1}{1-\alpha e^{-j\omega}} + \frac{1}{1-j\alpha e^{-j\omega}}.$$

To sketch the approximate magnitude of the DTFT first recall that $j = e^{j\frac{\pi}{2}}$, so that

$$X(\omega) = \frac{1}{1-\alpha e^{-j\omega}} + \frac{1}{1-\alpha e^{j\frac{\pi}{2}} e^{-j\omega}} = A(\omega) + A(\omega - \pi/2)$$

with

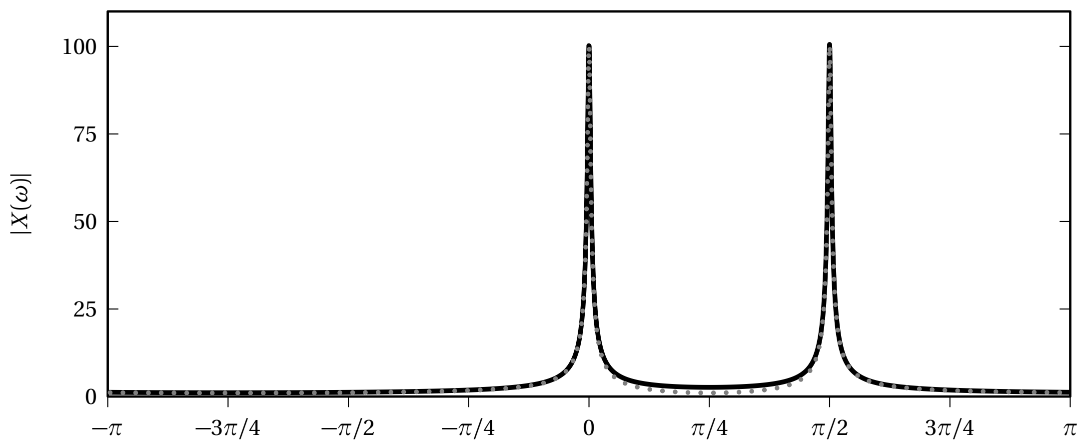
$$A(\omega) = \frac{1}{1-\alpha e^{-j\omega}}.$$

With $\alpha = 0.99$, $A(0) = 100$, $A(\pi/2) = 1/\sqrt{1+\alpha^2} \approx 0.7$ and $A(\pi) = 1/1.99 \approx 0.5$. Since the decay is very fast we can write

$$|X(\omega)| \approx |A(\omega)| + |A(\omega - \pi/2)|$$

that is, the DTFT magnitude will be approximately the superposition of two copies of $|A(\omega)|$, one centered in zero and the other in $\pi/2$. This is plotted with a black line in the figure below.

Note that the approximation is actually very good: in the figure the exact magnitude is plotted with gray dots and the curves are almost identical.



Exercise 5. DFT

Compute the DFT of the $\mathbb{C}^4$ vector $\mathbf{x} = [1\ \ 1\ -1\ -1]^T$

Solution:

the DFT matrix for $\mathbb{C}^4$ *is*

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

By computing the matrix-vector product $\mathbf{X} = \mathbf{W}\mathbf{x}$ *it is easy to obtain* $\mathbf{X} = [0\ \ (2-2j)\ \ 0\ \ (2+2j)]^T$

Exercise 6. DTFT

Consider the finite-support sequence

$$x[n] = \begin{cases} 1/6 & \text{for } 0 \le n < 6 \\ 0 & \text{otherwise} \end{cases}$$

Next, consider the family of complex-valued finite-support sequences

$$x_k[n] = x[n]\, e^{-j\omega_k n}$$

where $\omega_k = (2\pi/6)k$.

(a) sketch $|X(\omega)|$, the magnitude of the DTFT of $x[n]$; be as precise as possible

(b) sketch $|X_k(\omega)|$, the magnitude of the DTFT of $x_k[n]$, for $k = 1$ and $k = 4$

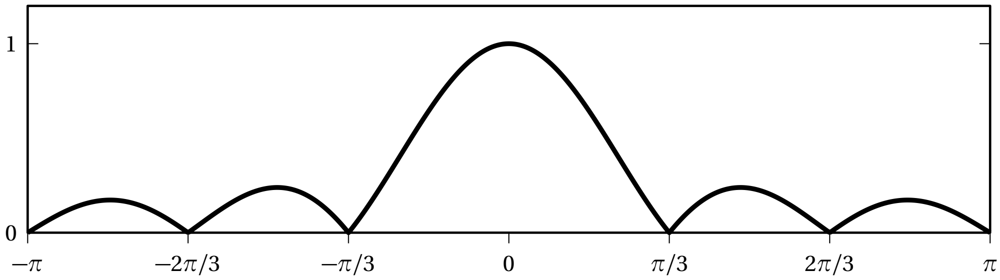
(c) prove that $\displaystyle\sum_{k=0}^{5} X_k(\omega) = 1$

Solution:

(a) *the sequence corresponds to the impulse response of a moving average filter of length six; the magnitude response is*

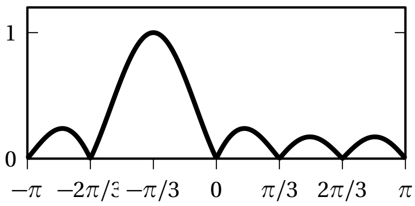
$$|X(\omega)| = \left|\frac{1}{6}\,\frac{\sin(3\omega)}{\sin(\omega/2)}\right|$$

so it will be equal to zero for $\omega = \pm\pi/3, \pm 2\pi/3, \pm\pi$ *and equal to 1 (by continuity) for* $\omega = 0$*:*

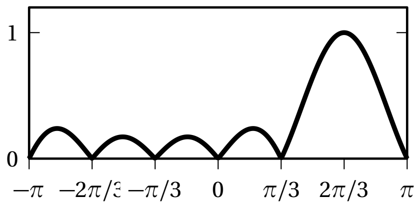


(b) *multiplication by* $e^{-j\omega_k n}$ *in time corresponds to a left shift by* $\omega_k = k(\pi/3)$ *in frequency. Because of the* 2π*-periodicity of the spectrum, the shift appears as a circular shift over the* $[-\pi, \pi]$ *range:*

$|X_1(\omega)|$



$|X_4(\omega)|$



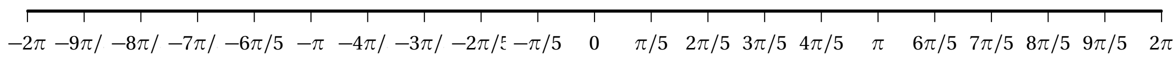
(c)

$$\begin{aligned}
\sum_{k=0}^{5} X_k(\omega) &= \sum_{k=0}^{5} \mathit{DTFT}\{x_k[n]\} \\
&= \mathit{DTFT}\left\{\sum_{k=0}^{5} x_k[n]\right\} \qquad \text{(by linearity)} \\
&= \mathit{DTFT}\left\{\frac{1}{6}\sum_{k=0}^{5} e^{-j\frac{2\pi}{6}nk}\right\} \\
&= \mathit{DTFT}\left\{\frac{1}{6}\mathit{DFT}\{1\}\right\} \qquad \text{(DFT in } \mathbb{C}^6\text{)} \\
&= \mathit{DTFT}\{\delta[n]\} = 1
\end{aligned}$$

Exercise 7. DTFT

Plot the DTFT of the signal $x[n] = \text{sinc}(5n/2)$.

Hint: you can either work mostly in the time domain using simple trigonometry (but careful with the value of $x[0]$) or you can work mostly in the frequency domain by considering $x[n]$ as a continuous-time sinc sampled with period $T_s = 5/2$; in this case the grid below can be of help.



Solution:

The standard derivation of the rect-sinc DTFT pair starts from a rect in frequency. Since the cutoff of the rect must be less than π, the inverse DTFT returns a signal of the form $\text{sinc}(\alpha n)$ with $\alpha < 1$ and therefore we cannot use the rect-sinc pair formula if, like in this case, $\alpha > 1$.

Working in the time domain: *by exploting the 2π-periodicity of the sine, we have for $n \neq 0$:*

$$\text{sinc}(5n/2) = \frac{\sin 5\pi n/2}{5\pi n/2} = \frac{\sin(2\pi n + \pi n/2)}{5\pi n/2} = (1/5)\frac{\sin \pi n/2}{\pi n/2} = (1/5)\,\text{sinc}(n/2)$$

For $n = 0$, $x[n] = 1$ while the above expression is equal to $1/5$. Therefore we can write

$$x[n] = (1/5)\,\text{sinc}(n/2) + (4/5)\delta[n]$$

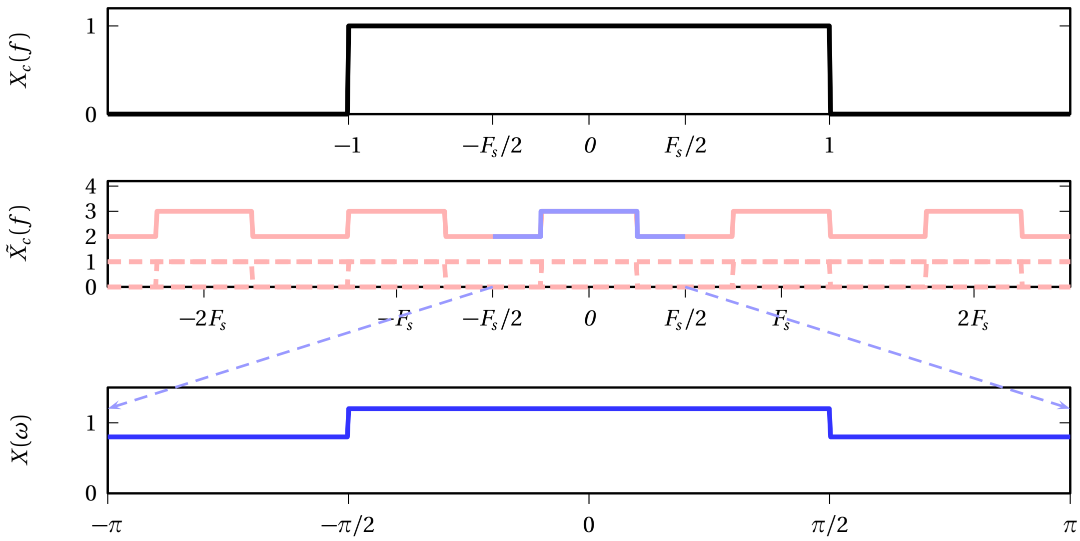
We can now use the standard sinc-rect transform pair formula to obtain

$$X(\omega) = 4/5 + (2/5)\,\text{rect}(\omega/\pi)$$

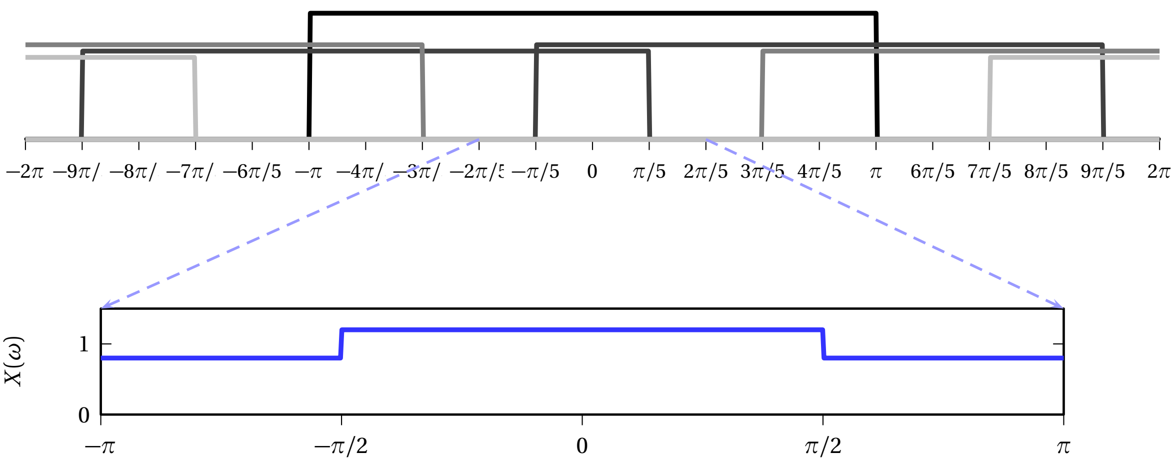
Working in the frequency domain: *The signal $x(t) = \text{sinc}(t)$ is 1-BL (its CTFT is $X(f) = \text{rect}(f)$) so the maximum sampling period to avoid aliasing i T_1. Since $T_s = 5/2 > 1$ there will be aliasing and the DTFT of the sampled sequence will be*

$$X(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega}{2\pi} F_s + k F_s\right)$$

with $F_s = 1/T_s = 2/5$. We can determine the shape of the DTFT graphically:



The overlaps are illustrated in detail by this figure, using an artificial different height for each spectral copy:



so that

$$X(\omega) = \begin{cases} 6/5 & \text{for } |\omega| < \pi/2 \\ 4/5 & \text{otherwise} \end{cases} \qquad \text{extended by } 2\pi\text{-periodicity}$$

Exercise 8. DFT

Describe the subspace of signals in $\mathbb{C}^4$ whose DFT coefficients are all purely imaginary (or zero). (For instance, we can describe the subspace of signals in $\mathbb{C}^3$ whose DFT coefficients are all purely imaginary as the set of vectors of the form $\begin{bmatrix} ja & b & -b^* \end{bmatrix}^T$ where $a \in \mathbb{R}$ and $b \in \mathbb{C}$.)

Solution:

Given the Fourier matrix $\mathbf{W}$ *for* $\mathbb{C}^4$ *we need to determine the set of vectors for which*

$$\Re\{\mathbf{Wx}\} = \Re\{\mathbf{W}\}\Re\{\mathbf{x}\} - \Im\{\mathbf{W}\}\Im\{\mathbf{x}\} = 0$$

Set $\mathbf{x} = \begin{bmatrix} a + j\alpha & b + j\beta & c + j\gamma & d + j\delta \end{bmatrix}^T$*; since*

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

we need to find the values for which

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -j & 0 & j \\ 0 & 0 & 0 & 0 \\ 0 & j & 0 & -j \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = 0$$

This results in the following system of equations

$$\begin{cases} a + b + c + d = 0 \\ a - b + c - d = 0 \\ a - c + \beta - \delta = 0 \\ a - c - \beta + \delta = 0 \end{cases}$$

from which we have $a = c = 0$, $b = -d$ *and* $\beta = \delta$. *The set of vector is therefore of the form* $\begin{bmatrix} je & g & jf & -g^* \end{bmatrix}^T$ *with* $e, f \in \mathbb{R}$ *and* $g \in \mathbb{C}$.

Exercise 9. DFT

Given a vector $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix} \in \mathbb{C}^N$ and its DFT $\mathbf{X} = \begin{bmatrix} X_0 & X_1 & \dots & X_{N-1} \end{bmatrix}$ show that

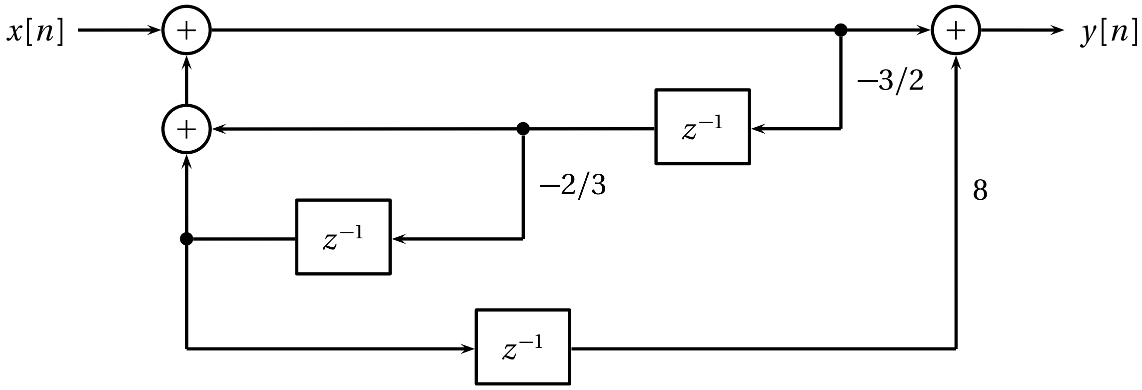
$$\mathbf{x}^* = \text{IDFT}\left\{\begin{bmatrix} X_0^* & X_{N-1}^* & X_{N-2}^* & \dots & X_1^* \end{bmatrix}\right\}$$

Solution:

$$
\begin{aligned}
IDFT\left\{\left[X_0^* \quad X_{N-1}^* \quad X_{N-2}^* \quad \dots \quad X_1^*\right]\right\}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X_k^* \, e^{j\frac{2\pi}{N}n(N-k)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X_k^* \, e^{-j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \left(X_k \, e^{j\frac{2\pi}{N}nk}\right)^* \\
&= \left(\frac{1}{N}\sum_{k=0}^{N-1} X_k \, e^{j\frac{2\pi}{N}nk}\right)^* \\
&= x_n^*
\end{aligned}
$$

Exercise 10. System analysis

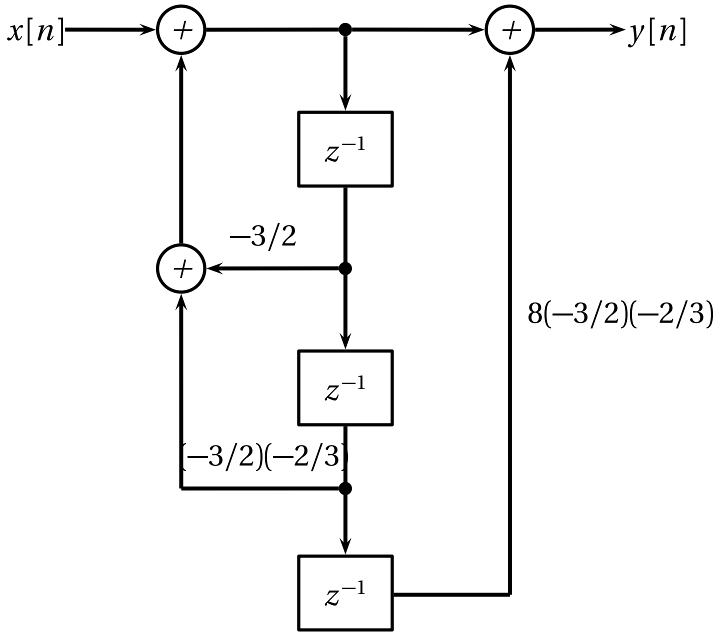
Consider the causal system described by the following block diagram:



Compute its transfer function $H(z)$ and determine if the system is stable.

Solution:

The easiest way to solve the problem is to re-arrange the block diagram in standard form:



From this, the transfer function is derived immediately as

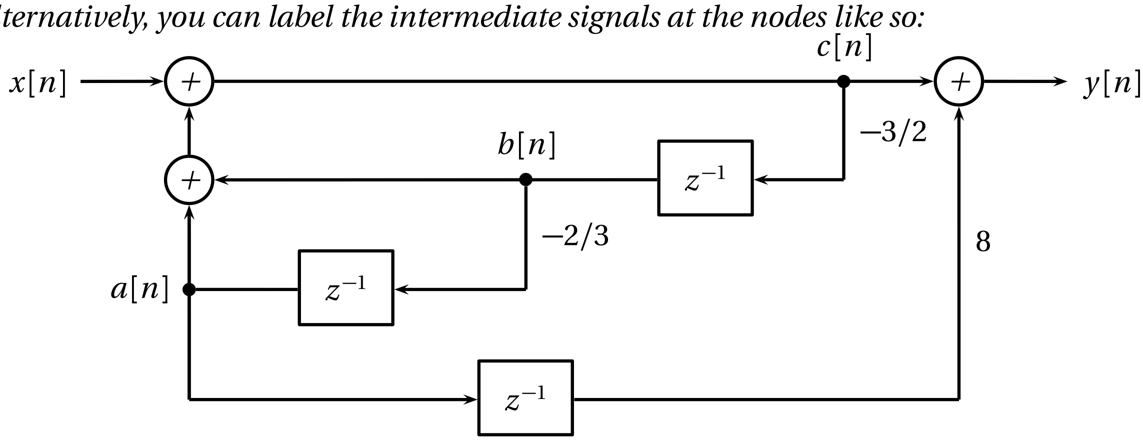
$$H(z) = \frac{1 + 8z^{-3}}{1 + (3/2)z^{-1} - z^{-2}}$$

By finding the roots of numerator and denominator we can write

$$H(z) = \frac{(1+2z^{-1})(1-2z^{-1}+4z^{-2})}{(1+2z^{-1})(1-(1/2)z^{-1})} = \frac{1-2z^{-1}+4z^{-2}}{1-(1/2)z^{-1}}$$

so that the system is stable since it only has one pole and the pole is inside the unit circle.

Alternatively, you can label the intermediate signals at the nodes like so:



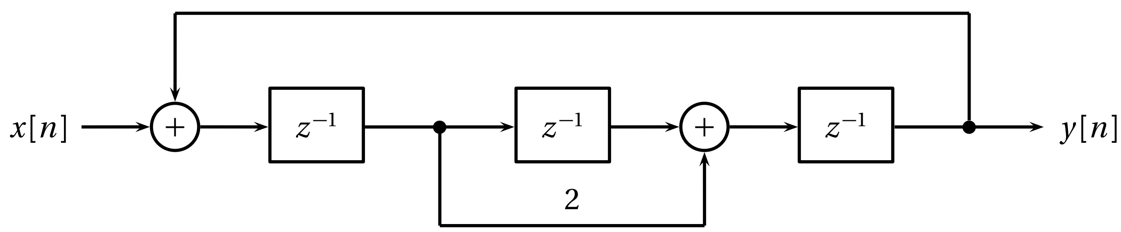
From this, using the z-transform, we can write

$$A(z) = -(2/3)z^{-1}B(z)$$
$$B(z) = -(3/2)z^{-1}C(z)$$
$$C(z) = X(z) + A(z) + B(z)$$
$$Y(z) = C(z) + 8z^{-1}A(z)$$

and, by substituting back, we obtain $H(z) = Y(z)/X(z)$*.*

Exercise 11. System analysis

Consider the causal system implemented by the following block diagram:



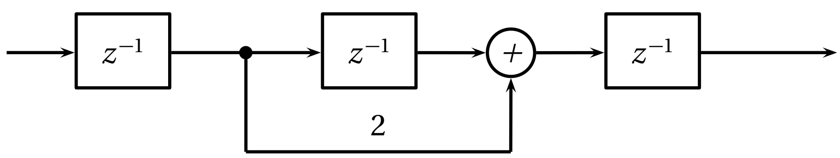
(a) compute the system's transfer function $H(z)$

(b) plot the system's poles and zeros on the complex plane

(c) determine if the system stable

Solution:

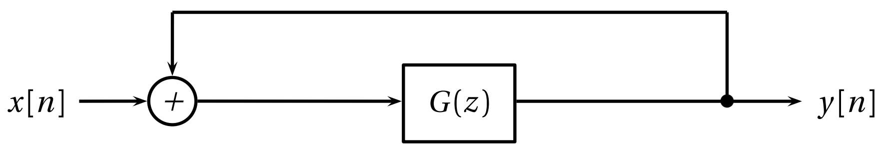
First consider the FIR subsystem inside the feedback loop and call its transfer function $G(z)$



From simple inspection it is easy to see that

$$G(z) = 2z^{-2} + z^{-3}$$

Now we can redraw the system as a simple feedback loop:



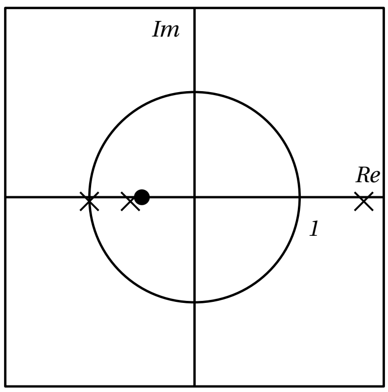
so that

$$H(z) = \frac{G(z)}{1 - G(z)} = \frac{2z^{-2} + z^{-3}}{1 - 2z^{-2} - z^{-3}}.$$

Using simple factorization:

$$H(z) = 2z^{-2} \frac{1 + (1/2)z^{-1}}{1 - z^{-2} - z^{-2}(1 + z^{-1})} = 2z^{-2} \frac{1 + (1/2)z^{-1}}{(1 + z^{-1})(1 - z^{-1} - z^{-2})}$$

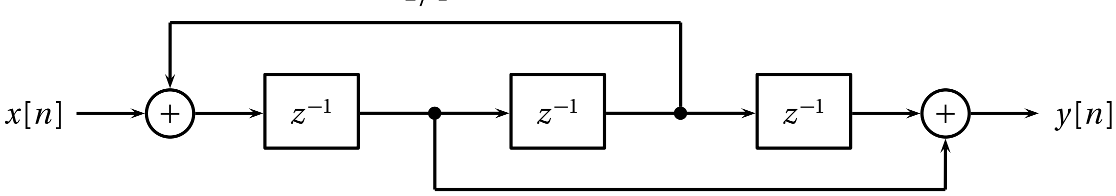
so that the transfer function has a zero in $z = -1/2$ *and poles in* -1 *and* $(1 \pm \sqrt{5})/2$*:*



Since one of the poles is outsite the unit circle, the system is not stable.

Exercise 12. System analysis

Consider the causal system implemented by the following block diagram:



(a) Compute the system's transfer function $H(z)$

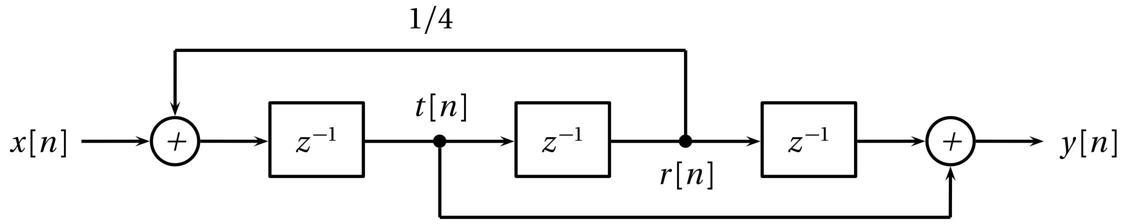
(b) Plot the system's poles and zeros on the complex plane

(c) Sketch the magnitude of the system's frequency response $|H(\omega)|$

(d) Draw another block diagram that implements the same transfer function $H(z)$ as a cascade of a second-order direct form II structure and a simple delay.

Solution:

First set two auxiliary variables $t[n]$ and $r[n]$ like so:



(a) From simple inspection:

$$y[n] = t[n] + r[n-1]$$
$$r[n] = t[n-1]$$
$$t[n] = x[n-1] + (1/4)r[n-1]$$

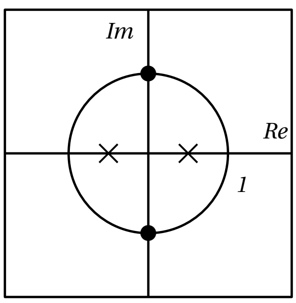
which, in the z-domain, becomes

$$Y(z) = T(z) + z^{-1}R(z)$$
$$R(z) = z^{-1}T(z)$$
$$T(z) = z^{-1}X(z) + (1/4)z^{-1}R(z)$$

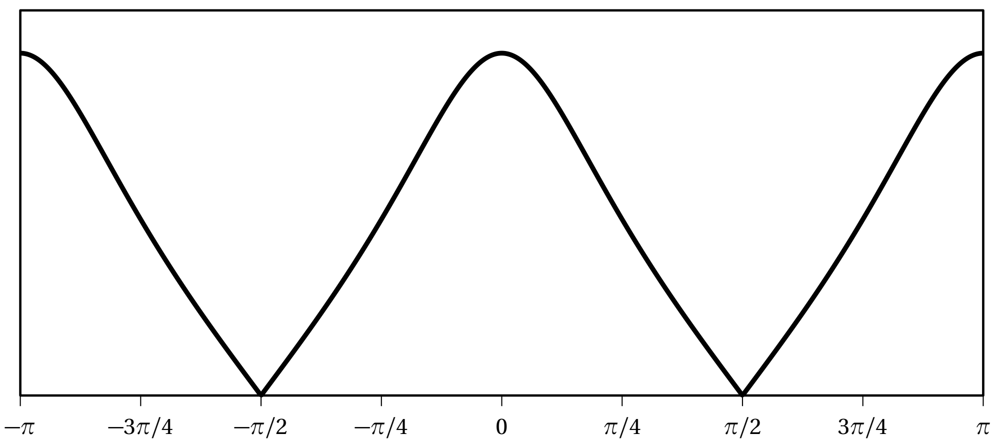
Solving for $X(z)$ and $Y(z)$ yields

$$H(z) = \frac{X(z)}{Y(z)} = z^{-1}\frac{1+z^{-2}}{1-(1/4)z^{-2}}$$

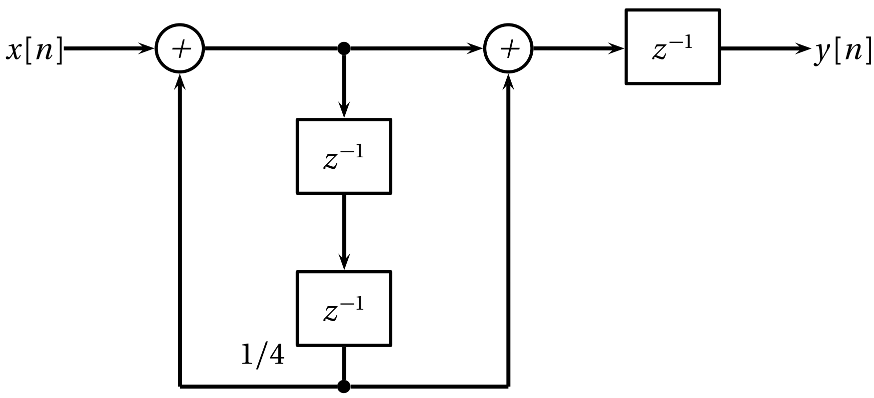
(b) the roots of the numerator are $\pm j$ and those of the denominator are $\pm 1/2$ therefore



(c) The filter is an approximate stopband

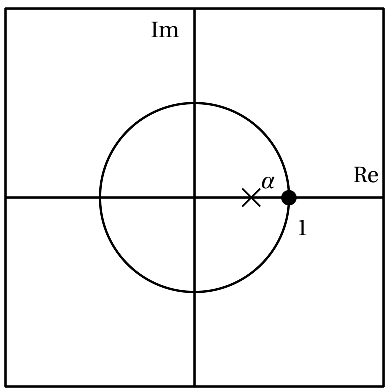


(d)



Exercise 13. Pole-zero plot

Compute the impulse response of the causal filter with the following pole-zero plot:



Solution:

The system has a pole in $z = \alpha$ and a zero in $z = 1$. We can write the transfer function of the system as

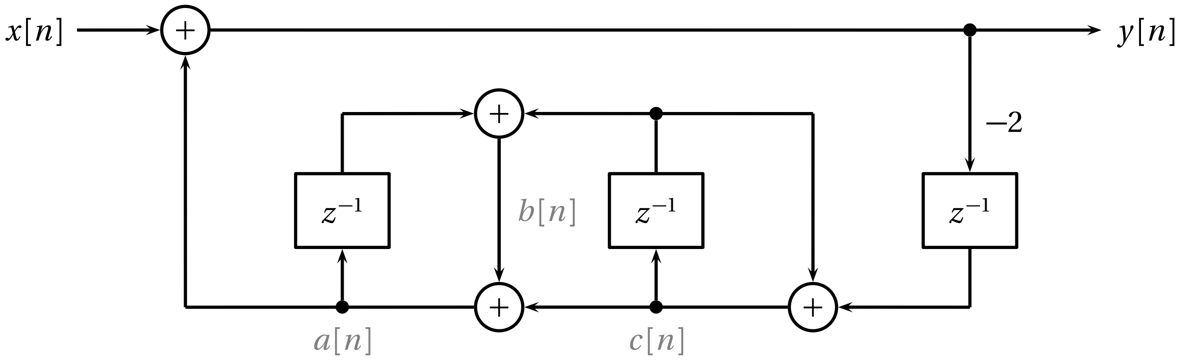
$$H(z) = \frac{1 - z^{-1}}{1 - \alpha z^{-1}} = \frac{1}{1 - \alpha z^{-1}} - z^{-1} \frac{1}{1 - \alpha z^{-1}}$$

A first order section with a pole in $z = \alpha$ has a transfer function $G(z) = 1/(1 - \alpha z^{-1})$ and impulse response $g[n] = \alpha^n u[n]$. Therefore the impulse response of the above system is

$$h[n] = g[n] - g[n-1] = \alpha^n u[n] - \alpha^{n-1} u[n-1] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \alpha^{n-1}(\alpha - 1) & n > 0 \end{cases}$$

Exercise 14. System analysis

Consider the causal system described by the following block diagram:



Compute its transfer function $H(z) = Y(z)/X(z)$.

Solution:

Consider the intermediate signals $a[n], b[n], c[n]$ as in the above figure. In the z-domain we have

$$\begin{aligned} Y(z) &= X(z) + A(z) \\ A(z) &= B(z) + C(z) \\ B(z) &= z^{-1} A(z) + z^{-1} C(z) \\ C(z) &= z^{-1} C(z) - 2z^{-1} Y(z) \end{aligned}$$

Using the third equation with the second

$$A(z) = z^{-1} A(z) + z^{-1} C(z) + C(z) \quad \Rightarrow \quad A(z) = \frac{1 + z^{-1}}{1 - z^{-1}} C(z)$$

while the fourth equation gives

$$C(z) = \frac{-2z^{-1}}{1-z^{-1}} Y(z)$$

Replacing these results in the first equation:

$$Y(z) = X(z) - 2z^{-1} \frac{1+z^{-1}}{(1-z^{-1})^2} Y(z)$$

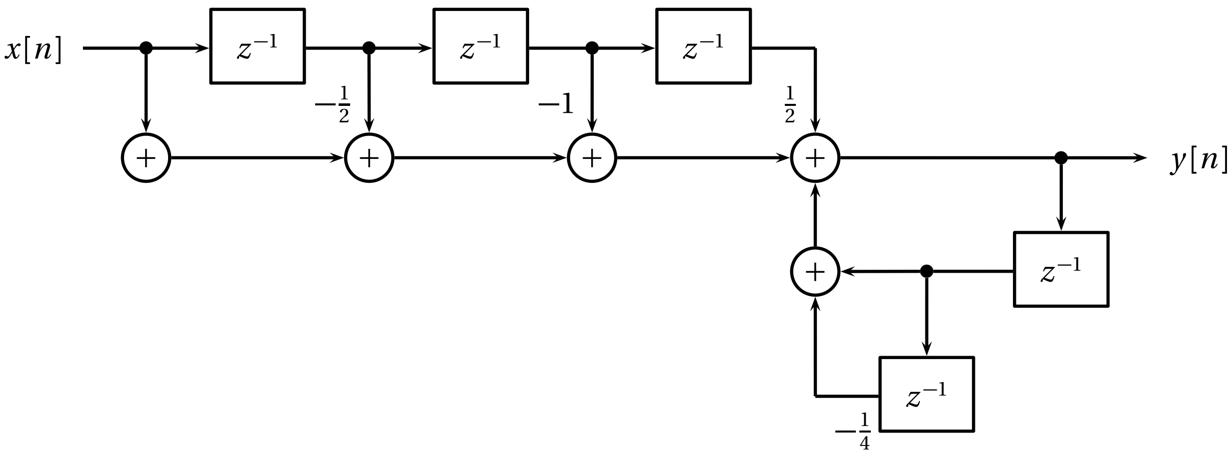
$$\left[1 + 2z^{-1} \frac{1+z^{-1}}{(1-z^{-1})^2}\right] Y(z) = \left[\frac{1 - 2z^{-1} + z^{-2} + 2z^{-1} + 2z^{-2}}{(1-z^{-1})^2}\right] Y(z) = X(z)$$

so that finally

$$H(z) = \frac{(1-z^{-1})^2}{1+3z^{-2}}$$

Exercise 15. System analysis

Consider the causal system described by the following block diagram:



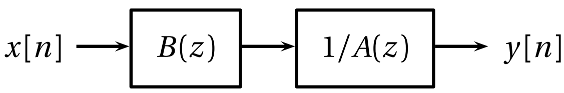
(a) Compute its transfer function $H(z) = Y(z)/X(z)$.

(b) Is the system stable?

(c) Draw a block diagram that implements the same transfer function using just two delay blocks.

Solution:

The system can be seen as the cascade of an FIR and an IIR filters



where

$$B(z) = 1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{1}{2}z^{-3}$$

and

$$A(z) = 1 - z^{-1} + \frac{1}{4}z^{-2}.$$

Since we will need to determine the stability of the system later, we can already factorize $A(z)$ *by simple inspection as*

$$A(z) = (1 - \frac{1}{2}z^{-1})^2.$$

We can also try to see if the root of $A(z)$ *is also a root of* $B(z)$*: indeed* $B(1/2) = 0$*. We can now factor* $B(z)$ *either by performing polynomial division or by noticing that both* $+1$ *and* -1 *are also roots; we have*

$$B(z) = (1 - \frac{1}{2}z^{-1})(1 - z^{-2}).$$

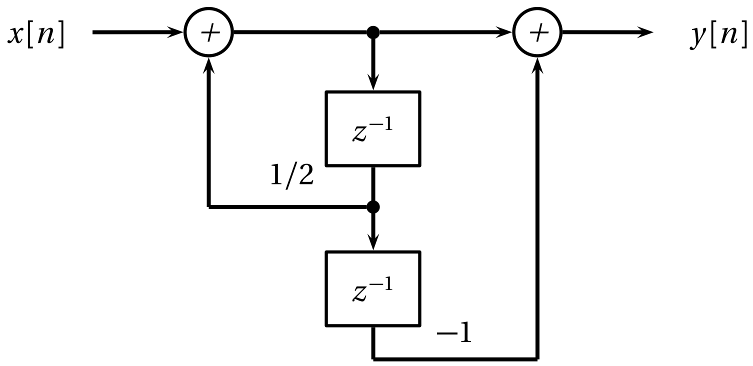
With this:

(a) The global transfer function is

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 - z^{-2}}{1 - \frac{1}{2}z^{-1}}.$$

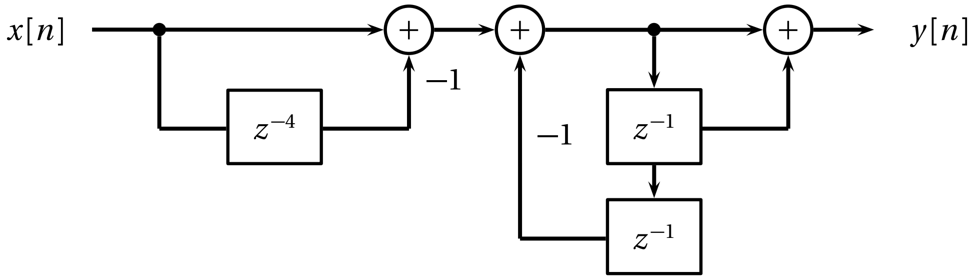
(b) The pole of the system is in $z = 1/2$ *so the system is stable*

(c) The system is an incomplete second order section, so we can use the standard Direct Form II like so:



Exercise 16. System analysis

Consider the causal system implemented by the following block diagram:



(a) (12p) find the impulse response of the system

(b) (4p) is the system stable?

(c) (4p) what is the value of the system's frequency response at $\omega = 0$?

Solution:

The block diagram implements the cascade of two filters: the first is a simple feedforward section with a delay of four samples, while the second is an incomplete second order section. Directly from the diagarm, we can determine their transfer functions as

$$H_1(z) = 1 - z^{-4}$$
$$H_2(z) = \frac{1+z^{-1}}{1+z^{-2}}.$$

The overall transfer function is

$$H(z) = (1-z^{-4})\frac{1+z^{-1}}{1+z^{-2}} = (1-z^{-2})(1+z^{-1}) = 1 + z^{-1} - z^{-2} - z^{-3}.$$

(a) The resulting filter is therefore FIR with impulse response

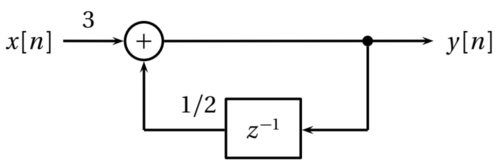
$$h[n] = \begin{cases} 1 & n = 0,1 \\ -1 & n = 2,3 \\ 0 & \text{otherwise.} \end{cases}$$

(b) The system stable because it's FIR.

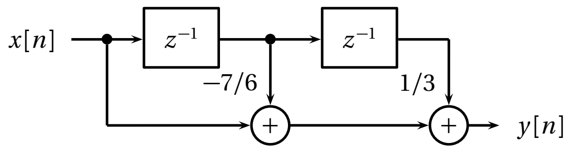
(c) The impulse response is even and antisymmetric, so the frequency response will be zero in zero.

Exercise 17. System analysis

Consider the following block diagrams, implementing the causal LTI systems $H_1(z)$ and $H_2(z)$:

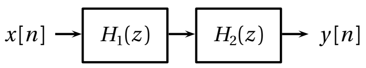


$H_1(z)$

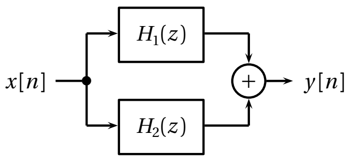


$H_2(z)$

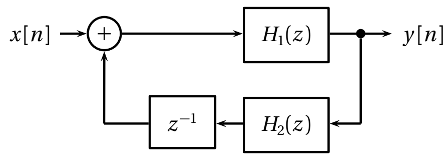
The two systems are then connected together in three different configurations to create three new causal LTI systems:



$H_A(z)$



$H_B(z)$



$H_C(z)$

Determine the stability of $H_A(z)$ [5p], $H_B(z)$, and $H_C(z)$. Please justify your answers.

Solution:

From the block diagrams we can immediately see that:

- $H_1(z)$ *is a first-order feedback loop (i.e. a leaky integrator) with transfer function*

$$H_1(z) = \frac{3}{1-(1/2)z^{-1}};$$

  the single pole of the filter is in $z = 1/2$ *so* $H_1(z)$ *is stable.*

- $H_2(z)$ *is a pure feedforward structure yielding the transfer function* $H_2(z) = 1-(7/6)z^{-1} + (1/3)z^{-2}$ *which, being an FIR, is also stable. Since this will be useful later, by finding the zeros of the transfer function, we can factor* $H_2(z)$ *as*

$$H_2(z) = (1-(2/3)z^{-1})(1-(1/2)z^{-1}).$$

With this:

(a) $H_A(z) = H_1(z)H_2(z)$ *which is stable because it is a cascade of stable subsystems*

(b) $H_B(z) = H_1(z) + H_2(z)$ *which is stable because both subsystems are stable and they operate independently*

(c) $H_C(z)$ *uses the two subsystems in a feedback loop; the overall transfer function can be computed from the input-output relation in the* z*-domain:*

$$Y(z) = H_1(z)[X(z) + z^{-1}H_2(z)Y(z)]$$

which yields

$$H_C(z) = \frac{H_1(z)}{1 - z^{-1}H_1(z)H_2(z)} = \frac{1}{1/H_1(z) - z^{-1}H_2(z)}$$

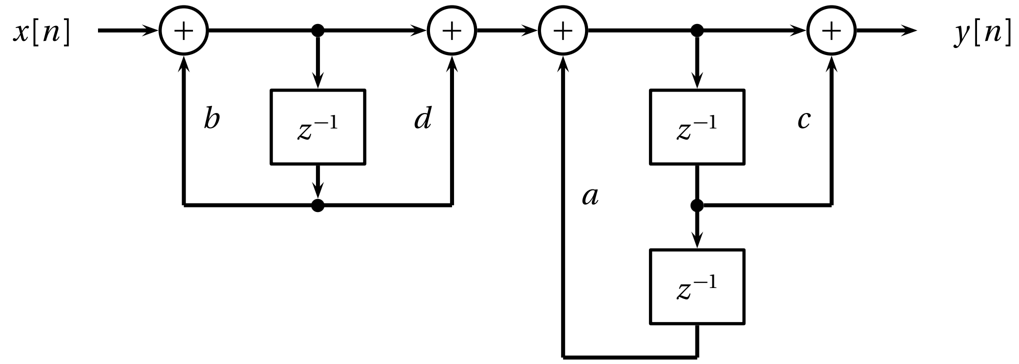
By plugging in the expressions for $H_1(z)$ and $H_2(z)$ we have

$$\begin{aligned} H_C(z) &= \left[\frac{1-(1/2)z^{-1}}{3} - z^{-1}(1-(2/3)z^{-1})(1-(1/2)z^{-1})\right]^{-1} \\ &= 3\left[(1-(1/2)z^{-1})[1-3z^{-1}(1-(2/3)z^{-1})]\right]^{-1} \\ &= 3\left[(1-(1/2)z^{-1})(1-3z^{-1}+2z^{-2}))\right]^{-1}. \end{aligned}$$

By finding the roots of the expression in brackets we can determine that the poles of $H_C(z)$ are in $z = 1/2, z = 1$, and $z = 2$; therefore the system is not stable.

Exercise 18. System analysis

Consider the following block diagram, where a, b, c, d are real-valued coefficients:



(a) [10p] Determine the values for a, b, c, d so that the diagram implements the causal CCDE

$$y[n] = x[n] + 3x[n-1] + 2x[n-2] + 2y[n-1] + \frac{1}{10}y[n-2] - \frac{1}{5}y[n-3].$$

(b) [5p] Is the resulting filter stable?

Solution:

(a) the block diagram shows the cascade of two incomplete second-order sections; the transfer function is therefore

$$H(z) = \frac{1+dz^{-1}}{1-bz^{-1}}\,\frac{1+cz^{-1}}{1-az^{-2}} = \frac{1+(c+d)z^{-1}+cdz^{-2}}{1-bz^{-1}-az^{-2}+abz^{-3}}$$

and the associated CCDE is

$$y[n] = x[n] + (c+d)x[n-1] + (cd)x[n-2] + b\,y[n-1] + a\,y[n-2] - (ab)y[n-3]$$

from which

$$\begin{aligned} a &= 1/10 \\ b &= 2 \\ c &= 1 \\ d &= 2 \end{aligned}$$

(alternatively, $c = 2, d = 1$ is also a valid choice).

(b) *The system has a pole in $z = 2$ so it is not stable.*

Exercise 19. CCDE

Consider the noncausal constant-coefficient difference equation

$$y[n] = y[n+1] + b\,y[n-1] + x[n]$$

(a) make it causal, i.e., write a CCDE in causal form that implements the same transfer function as described by the equation above

(b) for which values of $b \in \mathbb{R}$ is the causal system stable?

Solution:

(a) *by delaying all terms:*

$$y[n-1] = y[n] + b\,y[n-2] + x[n-1]$$

from which

$$H(z) = \frac{-z^{-1}}{1 - z^{-1} + b z^{-2}}$$

(b) *the poles of the transfer function are*

$$p_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - b}$$

and we want $|p_{1,2}| < 1$.

For $b < 1/4$ the square root is real and therefore:

$$|p_1| < 1 \text{ for } \sqrt{1/4 - b} < 1/2 \Rightarrow 0 < b < 1/4$$

$$|p_2| < 1 \; for \; \sqrt{1/4-b} < -3/2 \Rightarrow -2 < b < 1/4$$

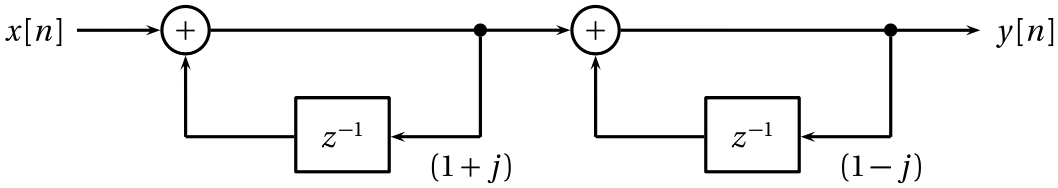
For $b > 1/4$ the square root is imaginary and $|p_{1,2}|^2 = b$ so that:

$$|p_{1,2}| < 1 \; for \; 1/4 < b < 1$$

so that in the end the system is stable for $0 < b < 1$

Exercise 20. System analysis

Consider the following causal system:



Compute its transfer function, determine its stability and, if possible, sketch its magnitude response.

Solution:

The transfer function of the system is $H(z) = H_1(z)H_2(z)$ where

$$H_{1,2}(z) = \frac{1}{1-(1\pm j)z^{-1}}.$$

The system has therefore no zeros and two poles at $z = (1 \pm j)$ or, in polar coordinates, at $z = e^{\pm j\frac{\pi}{4}}$; since the poles are greater than on in magnitude, the filter is not BIBO stable.

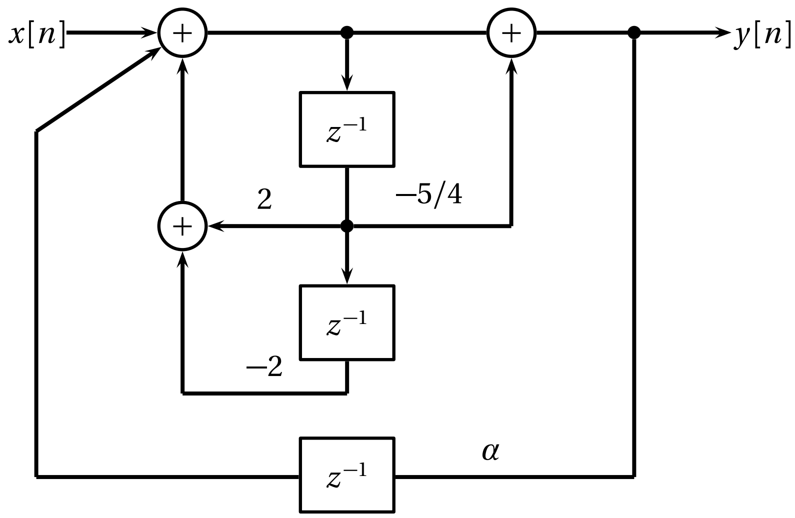
It is easy to show by induction that its impulse response is

$$h[n] = \begin{cases} (-1)^k 2^{2k} & n = 4k \\ (-1)^k 2^{2k+1} & n = 4k+1, 4k+2 \\ 0 & n = 4k+3 \end{cases}$$

which grows exponentially with n. Since the impulse response has infinite energy and power, it does not admit a DTFT and therefore the frequency response of the system is not defined.

Exercise 21. System analysis

Consider the causal LTI system shown in the following block diagram and consider $\alpha = -2$:



(a) Compute the system's transfer function $H(z)$

(b) Plot the system's poles and zeros on the complex plane

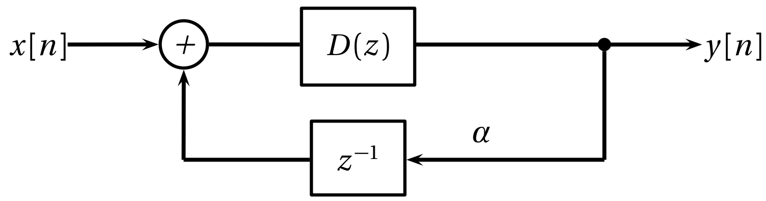
(c) Sketch the magnitude of the system's frequency response $|H(\omega)|$

(d) Find the transfer function of a *stable* filter $G(z)$ so that $|H(z)G(z)| = 1$

(e) Would the system be stable if we removed the lowest branch (i.e. if we set $\alpha = 0$)?

Solution:

The system can be simplified as



From this we can write the input/output relation in the z domain

$$Y(z) = D(z)[X(z) + \alpha z^{-1} Y(z)]$$

from which we obtain the transfer function

$$H(z) = \frac{D(z)}{1 - \alpha z^{-1} D(z)}$$

If we write $D(z)$ *as a ratio of polynomials, i.e.* $D(z) = B(z)/A(z)$, *we finally obtain*

$$H(z) = \frac{B(z)}{B(z) - \alpha z^{-1} A(z)}$$

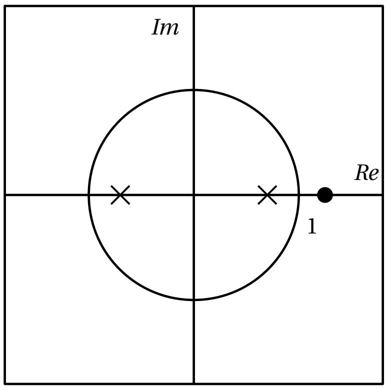
From the figure, it is immediate to see that $D(z)$ is an (incomplete) second order section in direct form II, incomplete since it has a single zero. Its transfer function is therefore

$$D(z) = \frac{1-(5/4)z^{-1}}{1-2z^{-1}+2z^{-2}} = \frac{1-(5/4)z^{-1}}{(1-(1+j)z^{-1})(1-(1-j)z^{-1})}$$

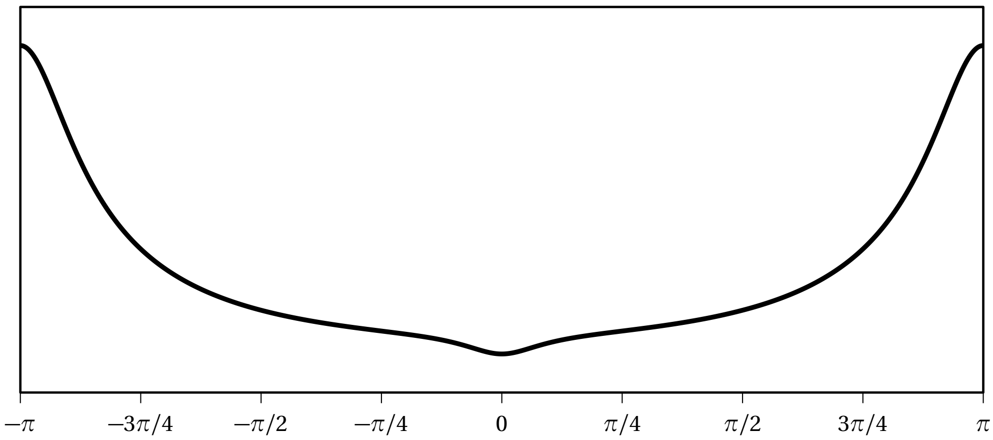
(a) *By letting $B(z) = 1-(5/4)z^{-1}$ and $A(z) = 1-2z^{-1}+2z^{-2}$, the transfer function with $\alpha = -2$ becomes*

$$H(z) = \frac{1-(5/4)z^{-1}}{1-2z^{-1}+2z^{-2}+2z^{-1}(1-(5/4)z^{-1})} = \frac{1-(5/4)z^{-1}}{1-(1/2)z^{-2}}$$

(b) *There is a zero in $z = 5/4$ and two poles in $z = \pm\sqrt{1/2}$*



(c) *since pole and zero on the positive real axis are almost equidistant from one, their effects cancel each other out; the pole in $z = -\sqrt{1/2}$ brings the magnitude of the frequency response up to create a highpass characteristic:*



(d) *The inverse transfer function is not stable because the zero of $H(z)$ is outside the unit circle. By choosing*

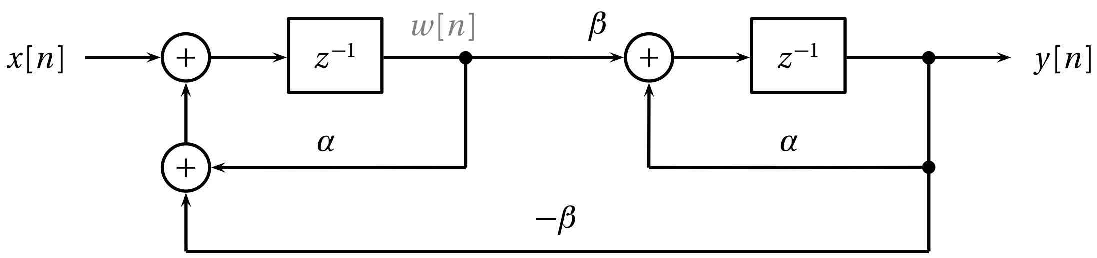
$$G(z) = \frac{1-(1/2)z^{-2}}{(5/4)-z^{-1}}$$

the product G(z)H(z) is the allpass term $(1-(5/4)z^{-1})/((5/4)-z^{-1})$ *whose frequency response magnitude is one.*

(e) If we remove the feedback branch, the transfer function becomes $H(z) = D(z)$*. The poles of* $D(z)$ *are larger than one in magnitude (*$|z_{1,2}| = |1 \pm j| = \sqrt{2}$*) and so the system would not be stable.*

Exercise 22. System analysis

Consider the causal system described by the following block diagram:



(a) Compute its transfer function $H(z) = Y(z)/X(z)$.

(b) Assume now that

$$\alpha = r\cos\theta$$
$$\beta = r\sin\theta$$

for $0 < r < 1$ and $0 < \theta < \pi/2$. Describe the type of filter implemented by the block diagram for this choice of coefficients and sketch its pole-zero plot.

Solution:

(a) Consider the auxiliary signal $w[n]$*, coming out of the first delay block. Using z-transforms we can write*

$$W(z) = z^{-1}(X(z) + \alpha W(z) - \beta Y(z))$$
$$Y(z) = z^{-1}(\beta W(z) + \alpha Y(z))$$

From this

$$W(z) = \frac{z^{-1}}{1-\alpha z^{-1}}\big(X(z) - \beta Y(z)\big)$$

and, substituting back,

$$Y(z) = \frac{\beta z^{-2}}{1-\alpha z^{-1}} X(z) - \frac{\beta^2 z^{-2}}{1-\alpha z^{-1}} Y(z) + \alpha z^{-1} Y(z)$$

so that, finally,

$$H(z) = \frac{\beta z^{-2}}{1 - 2\alpha z^{-1} + (\alpha^2 + \beta^2) z^{-2}}.$$

(b) *When* $\alpha = r\cos\theta$ *and* $\beta = r\sin\theta$ *the transfer function simplifies to*

$$H(z) = \frac{r\sin\theta\, z^{-2}}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}};$$

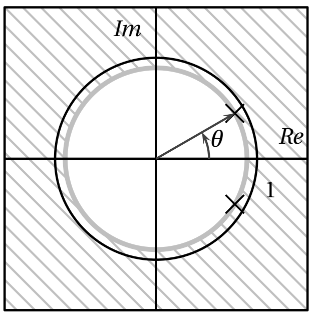
except for the scalar factor $\beta = r\sin\theta$ *and the delay* z^{-2}*, which do not affect the shape of the magnitude response, this is the transfer function of a simple* ***resonator*** *with a single pair of complex conjugate poles at* $re^{\pm j\theta}$*. Even if you do not remember the formula for the resonator, you can easily compute the poles of this filter by finding the roots of the denominator; for this multiply by* z^2 *and solve*

$$z^2 - 2r\cos\theta\, z + r^2 z^2 = 0$$

The solutions are

$$\begin{aligned} z_{1,2} &= (2r\cos\theta \pm \sqrt{4r^2(\cos^2\theta - 1)})/2 \\ &= r(\cos\theta \pm \sqrt{-\sin^2\theta}) \\ &= re^{\pm j\theta} \end{aligned}$$

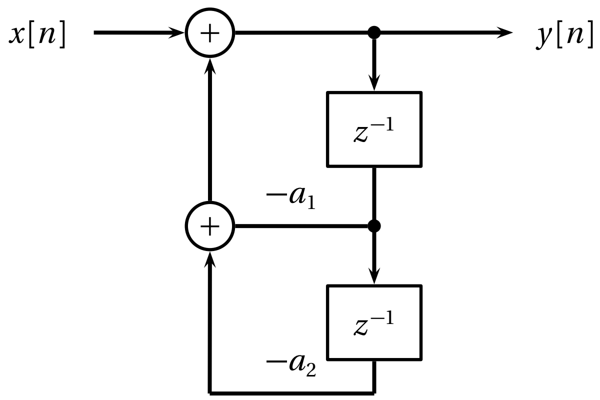
The pole-zero plot is as follows:



Exercise 23. Numerical precision

In all digital architecture numbers are encoded and stored using a finite number of bits; because of this unavoidable *finite precision*, care must be taken in the implementation of digital filters to prevent undesirable numerical effects that are not apparent in the basic theoretical derivation.

In this exercise we will study the effect of finite precision on the coefficients of a second order IIR filter that has a single pair of complex-conjugate poles at $p_{1,2} = re^{\pm j\theta}$. Assume the filter is implemented using a standard second-order section:



(a) find the values of the coefficients a_1 and a_2 as a function of r and θ

Assume that you want to design the filter so that p_1 is in the first quadrant of the complex plane. Clearly, when $a_{1,2}$ have infinite precision, you can place p_1 anywhere in the quadrant.

Consider however the case in which $a_{1,2}$ are expressed in signed fixed point notation using 3 bits. In practice this means that there are only seven possible values for each coefficient , namely 0, ± 0.25, ± 0.5, and ± 0.75.

(b) sketch all the possible locations for p_1 on the first quadrant of the complex plane when this fixed-point representation is used for $a_{1,2}$; to do so, you may want to concentrate on the real part and the magnitude of the pole with respect to the value of the coefficients. You can use the first chart at the end of the exercise for the drawing and, if you don't have a calculator, know that $\sqrt{0.25} = 0.5$, $\sqrt{0.5} \approx 0.7$, and $\sqrt{0.75} \approx 0.87$.

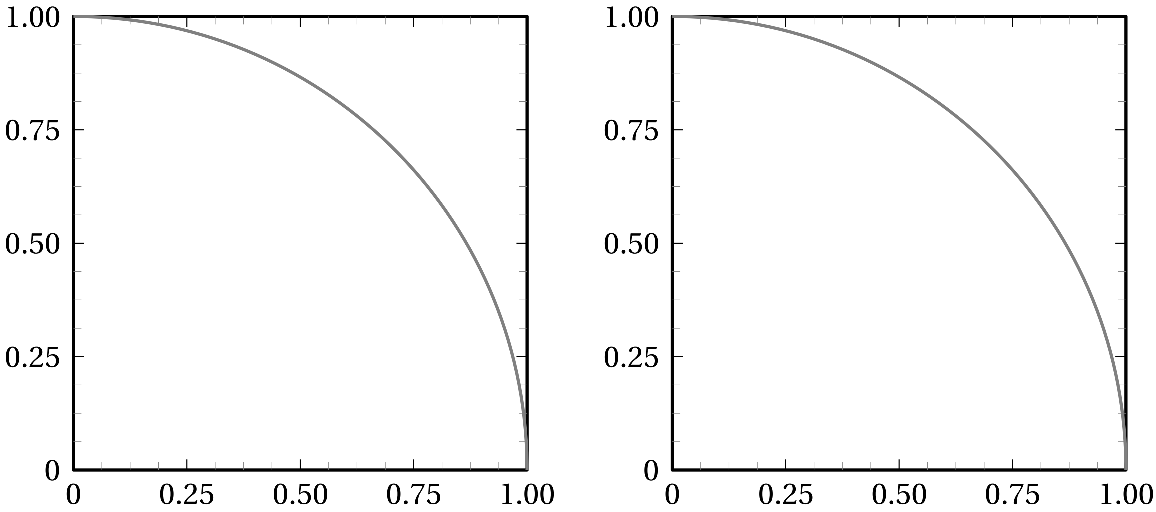
(c) suppose you need to implement a filter with $p_1 = 0.9e^{j\pi/10}$. What is the problem that you will encounter with the current fixed-point precision coefficients?

Consider now implementing the filter using the structure shown in **Exercise 22**, with $\alpha = r\cos\theta$ and $\beta = r\sin\theta$ both encoded in signed fixed point notation using 3 bits.

(d) sketch all the possible locations for p_1 on the first quadrant of the complex plane in this new implementation. You can use the second chart at the end of the exercise for the drawing.

(e) is this implementation better when it comes to implementing the filter with $p_1 = 0.9e^{j\pi/10}$?

(f) is this implementation more expensive computationally? Explain.



Solution:

(a) *The transfer function of the filter is*

$$\begin{aligned} H(z) &= \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} \\ &= \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{1}{(1 - r e^{j\theta} z^{-1})(1 - r e^{-j\theta} z^{-1})} \\ &= \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}} \end{aligned}$$

so that the filter coefficients in the direct form implementation are

$$\begin{aligned} a_1 &= -2r\cos\theta \\ a_2 &= r^2 \end{aligned}$$

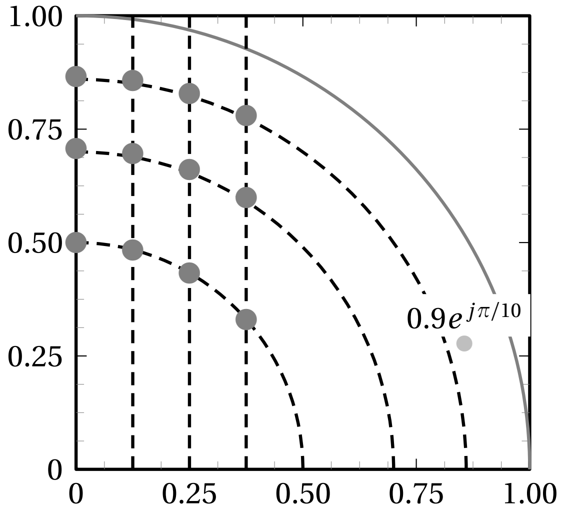
(b) *To find all the possible locations in the first quadrant for p_1 you can proceed in two ways.*

- **Option one: proceed geometrically:** *simply note that:*
  - *the real part of the pole is* $\Re(p_1) = r\cos\theta = -a_1/2$
  - *the pole's magnitude is* $|p_1| = r = \sqrt{a_2}$.

  As a consequence, the only possible pole locations will be at the intersections between

  - *vertical lines going through the real axis at the three possible positive values for* $-a_1/2$
  - *circles centered in zero and with radius equal to one of the three possible values of* $\sqrt{a_2}$.

The resulting 12 possible pole locations are shown below; we do not consider the case $a_2 = 0$, for which the second-order system degenerates into a first-order one.



- **Option two: proceed analytically:** *although this is way more tedious, you can express the real and imaginary parts of the poles as a function of $a_{1,2}$. By setting*

$$1 + a_1 z^{-1} + a_2 z^{-2} = 0$$

you can find that the pole in the first quadrant will have the following form:

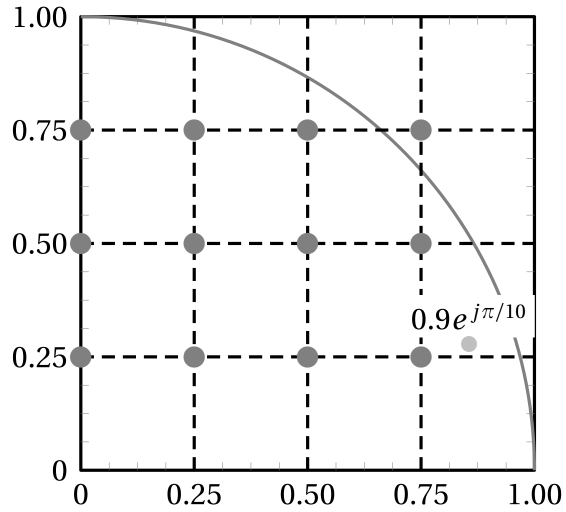
$$p_1 = -\frac{a_1}{2} + j\frac{\sqrt{4a_2 - a_1^2}}{2}.$$

You can now build a table for all valid values of a_1 and a_2 and compute the approximate pole locations (again, we neglect the case $a_2 = 0$):

	0.25	*0.5*	*0.75*
0	*0.5j*	*0.7j*	*0.87j*
-0.5	*0.125 + 0.48j*	*0.125 + 0.7j*	*0.25 + 0.86j*
-0.25	*0.25 + 0.43j*	*0.25 + 0.66j*	*0.25 + 0.82j*
-0.75	*0.375 + 0.33j*	*0.375 + 0.6j*	*0.375 + 0.78j*

(c) *Not only the pole at $0.9e^{j\pi/10}$ cannot be implemented exactly, but the closest pole location is very far from the intended position. This shows the heavy limits of an architecture using fixed-point with a small number of bits per coefficient.*

(d) *If we use the structure in Exercise 22, the real and imaginary parts of the poles are encoded directly in the filter coefficients. Therefore the pole locations will be distributed on a uniform grid like so*



(e) *Although the pole at $0.9e^{j\pi/10}$ still cannot be implemented exactly, there is now a reachable pole location that is a much closer approximation to the required value, so this implementation provides a much better result.*

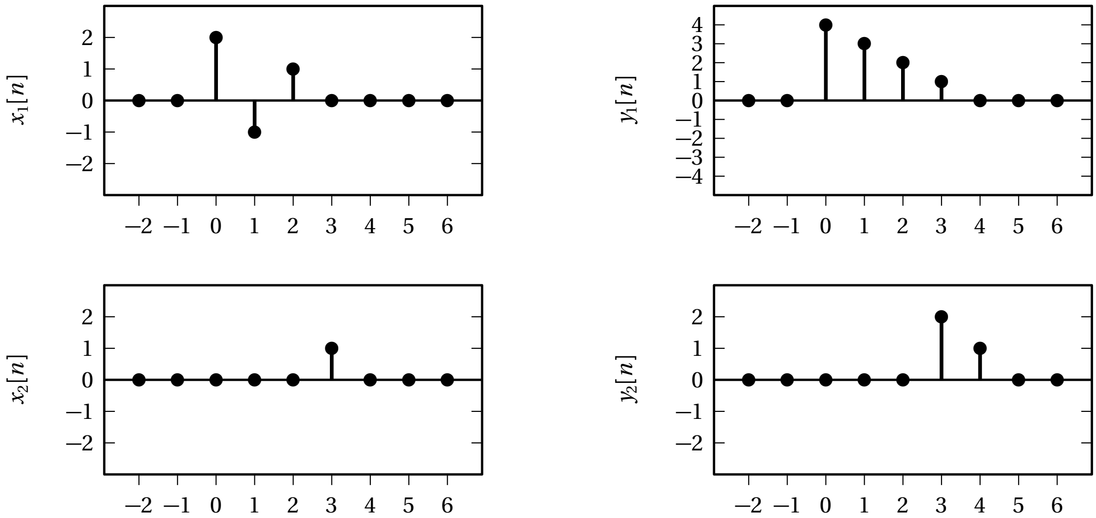
(f) *The direct form implementation only requires two multiplications per output sample whereas the current implementation requires four, for a doubling of the computational cost.*

Exercise 24. Linearity

Suppose you know that a system $\mathscr{H}$ is time-invariant. Below you can see the system's outputs $y_1[n]$ and $y_2[n]$ when the inputs are $x_1[n]$ and $x_2[n]$ respectively (all signals are infinite-length and all samples not shown in the figure are equal to zero). Based on the plots:

(a) Can you tell if the system $\mathscr{H}$ is linear?

(b) Compute $\mathscr{H}\{\delta[n]\}$.



Solution:

(a) By simple inspection we can write:

$$x_1[n] = 2x_2[n+3] - x_2[n+2] + x_2[n+1];$$

We know that the system is time invariant, therefore $\mathscr{H}\{x_2[n+N]\} = y_2[n+N]$. *If the system were linear we would therefore have:*

$$\mathscr{H}\{x_1[n]\} = \hat{y}[n] = 2y_2[n+3] - y_2[n+2] + y_2[n+1]$$

However it is easy to see that $\hat{y}[n] \neq y_1[n]$ *(in fact,* $\hat{y}[n] = \ldots, 0, 0, \underline{4}, 0, 1, 1, 0, 0, \ldots$*). As a consequence,* $\mathscr{H}$ *is not linear.*

(b) $\delta[n] = x_2[n+3]$ *so that, because of time invariance,* $\mathscr{H}\{\delta[n]\} = y_2[n+3] = \ldots, 0, 0, \underline{2}, 1, 0, 0, \ldots$

Exercise 25. FIR filtering

Consider a finite-support sequence $x[n]$ and an LTI filter $\mathscr{H}$.

(a) Prove that if $\mathscr{H}$ is FIR then $y[n] = \mathscr{H}\{x[n]\}$ is a finite-support sequence as well.

(b) Show with a counterexample that the converse is not true, i.e. show that, for a filter $\mathscr{H}$, if the the input $x[n]$ is finite-support and the output $y[n]$ is also finite-support, this does not imply that $\mathscr{H}$ is FIR.

Solution:

(a) *The easiest way to prove the result is to invoke the convolution theorem. In the z-transform domain, the filter's output can be expressed as $Y(z) = H(z)X(z)$. If $x[n]$ is finite-support and $h[n]$ is FIR, then both $X(z)$ and $H(z)$ are finite-degree polynomials in z and so their product is also finite-degree.*

Alternatively, you can consider the convolution sum for a FIR with support over $[M_1, M_2]$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=M_1}^{M_2} h[k]x[n-k]$$

If $x[n]$ is zero outside of $[N_1, N_2]$, then all the terms in the sum will be zero for $n > N_2 + M_2$ and for $n < N_1 + M_1$.

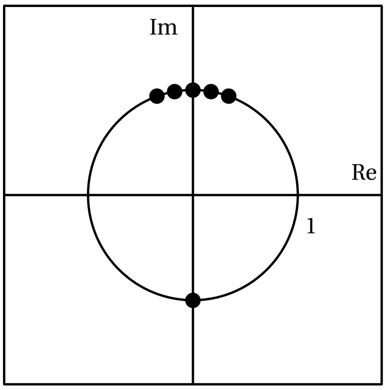
(b) *The simplest example is to have an input that exactly cancels the filter's poles. As an IIR filter, take a simple leaky integrator with transfer function*

$$H(z) = \frac{1}{1 - \lambda z^{-1}}$$

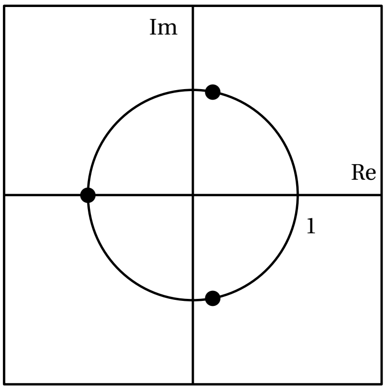
and consider the finite-support input $x[n] = \ldots, 0, 0, \underline{1}, -\lambda, 0, 0, \ldots$. We have that $Y(z) = H(z)X(z) = 1$ so that $y[n] = \delta[n]$, which is finite-support.

Exercise 26. Optimal FIR

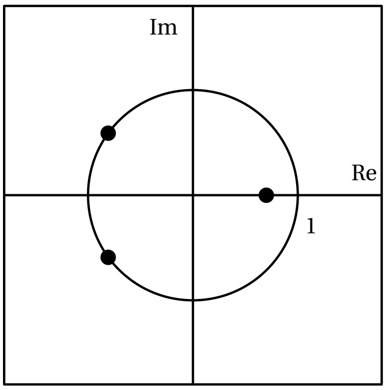
Below are the pole-zero plots of four different transfer functions, where zeros are indicated by a dot and poles are indicated by a cross; all poles and zeros have multiplicity one. Only one of the plots corresponds to a real-valued linear-phase FIR filter; find which one it is and explain your rejection criteria for the three plots you eliminated.



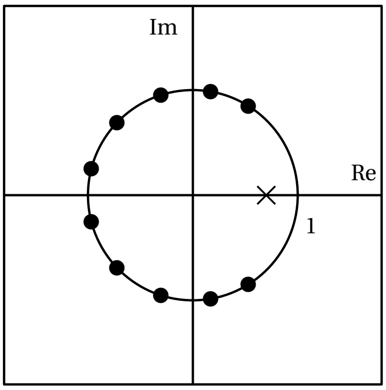
$H_1(z)$



$H_2(z)$



$H_3(z)$



$H_4(z)$

Solution:

(a) NO. The zeros are not in complex-conjugate pairs, so the filter's impulse response cannot be real-valued.

(b) YES. There are three zeros, one real and two complex-conjugate. The number of taps is four so the FIR is either type II or type IV; since there is no zero in 1, it is a type II

(c) NO. The zero z_0 on the positive axis does not have a reciprocal zero in $1/z_0$ so the filter cannot be linear-phase.

(d) NO. The filter has a pole, so it cannot be FIR linear phase.

Exercise 27. Linear phase

Given a real-valued constant $\beta > 1$, consider the FIR filter with impulse response

$$h[n] = \begin{cases} 1 & n = -1 \\ \beta & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $[-\omega_0, \omega_0]$ be the frequency interval over which, for our practical purposes, the small-angle approximation $\tan\omega \approx \omega$ is sufficiently accurate. Show that the filter's phase response is approximately linear over $[-\omega_0, \omega_0]$, independently of the value of β.

Solution:

The frequency response of the filter is

$$\begin{aligned} H(\omega) &= e^{j\omega} + \beta e^{-j\omega} \\ &= \cos\omega + j\sin\omega + \beta\cos\omega - j\beta\sin\omega \\ &= (\beta+1)\cos\omega - j(\beta-1)\sin\omega. \end{aligned}$$

The phase response is therefore

$$\angle H(\omega) = \arctan\left(-\frac{\beta-1}{\beta+1}\frac{\sin\omega}{\cos\omega}\right) = \arctan(\lambda\tan\omega).$$

Since $\beta > 1$ by definition, then $-1 < \lambda < 0$ so, if $\omega \in [-\omega_0, \omega_0]$, then $\lambda\omega \in [-\omega_0, \omega_0]$ as well. Consequently

$$\lambda\tan\omega \approx \lambda\omega \approx \tan\lambda\omega.$$

With this, for $|\omega| < \omega_0$,

$$H(\omega) = \arctan(\lambda\tan\omega) \approx \arctan(\tan\lambda\omega) = \lambda\omega$$

which is linear in ω.

Exercise 28. Minimum phase

For a causal system, stability requires that all the poles be inside the unit circle; stability does not impose any condition on a system's zeros, but the location of the zeros becomes important if we want to build the system's inverse. Consider the causal system described by:

$$3y[n] = 3x[n] - 8x[n-1] + 4x[n-2] - y[n-2].$$

(a) show that the system is stable

(b) verify that the inverse system $1/H(z)$ is *not* stable.

Systems whose zeros are inside the unit circle are called *minimum phase* and they always admit a stable inverse. If a system is not minimum phase, a common approach is the following: for every zero $z = a$ outside the unit circle, we cascade a section of the form

$$\frac{a - z^{-1}}{a - z^{-1}}$$

to the transfer function. Although the factors cancel each other out, by collecting the terms appropriately we can instead refactor $H(z)$ as

$$H(z) = H_m(z)H_a(z)$$

where $H_m(z)$ is minimum phase and $H_a(z)$ is all-pass. For the $H(z)$ in the first part of the exercise:

(a) factor $H(z)$ as a minimum phase term times an allpass term

(b) although $H(z)$ does not have a stable inverse, determine a stable $G(z)$ so that

$$|H(z)G(z)| = 1 \quad \text{for } z = e^{j\omega}$$

Solution:

(a)

$$H(z) = \frac{3 - 8z^{-1} + 4z^{-2}}{3 + z^{-2}} = \frac{(1-(2/3)z^{-1})(1-2z^{-1})}{(1-(j/\sqrt{3})z^{-1})(1+(j/\sqrt{3})z^{-1})}$$

poles are inside the unit circle but one zero is not.

(b) Since in the inverse systems zeros become poles, the pole in $z = 2$ makes the system unstable.

(c)

$$H(z) = \frac{(1-(2/3)z^{-1})(1-2z^{-1})}{3+z^{-2}}\frac{2-z^{-1}}{2-z^{-1}} = \frac{(1-(2/3)z^{-1})(2-z^{-1})}{3+z^{-2}}\frac{1-2z^{-1}}{2-z^{-1}}$$

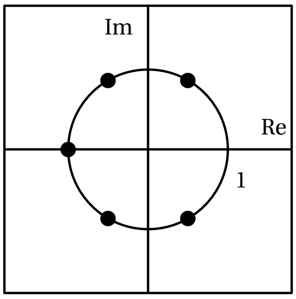
where $(1-2z^{-1})/(2-z^{-1})$ *is a classic allpass term.*

(d)

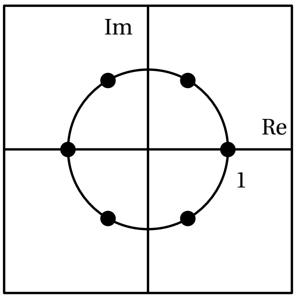
$$G(z) = 1/H_m(z) = \frac{3+z^{-2}}{(1-(2/3)z^{-1})(2-z^{-1})}$$

Exercise 29. Pole-zero plots

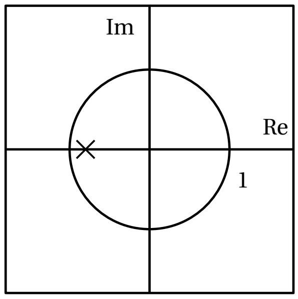
Associate each pole-zero plot to the corresponding impulse response.



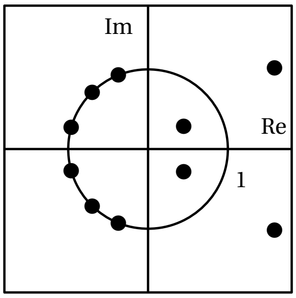
(a)



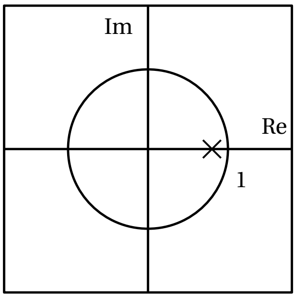
(b)



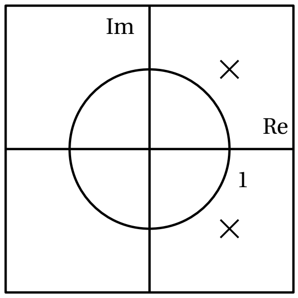
(c)



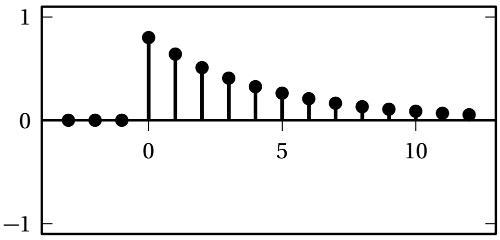
(d)



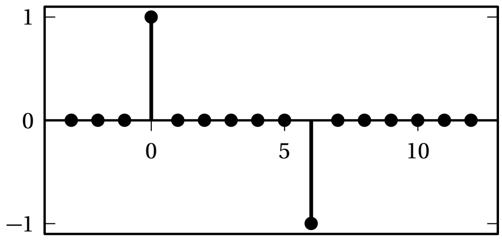
(e)



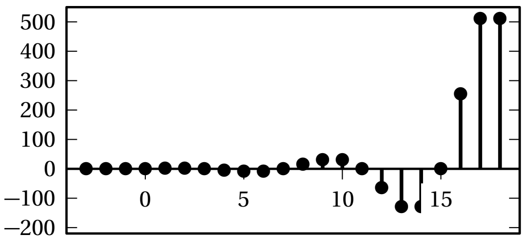
(f)



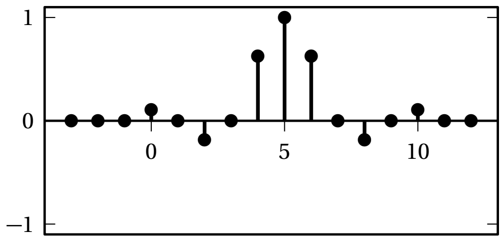
(1)



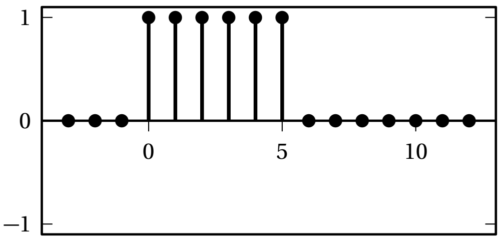
(2)



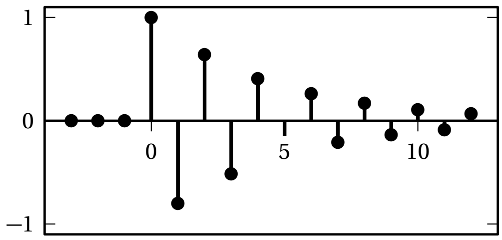
(3)



(4)



(5)



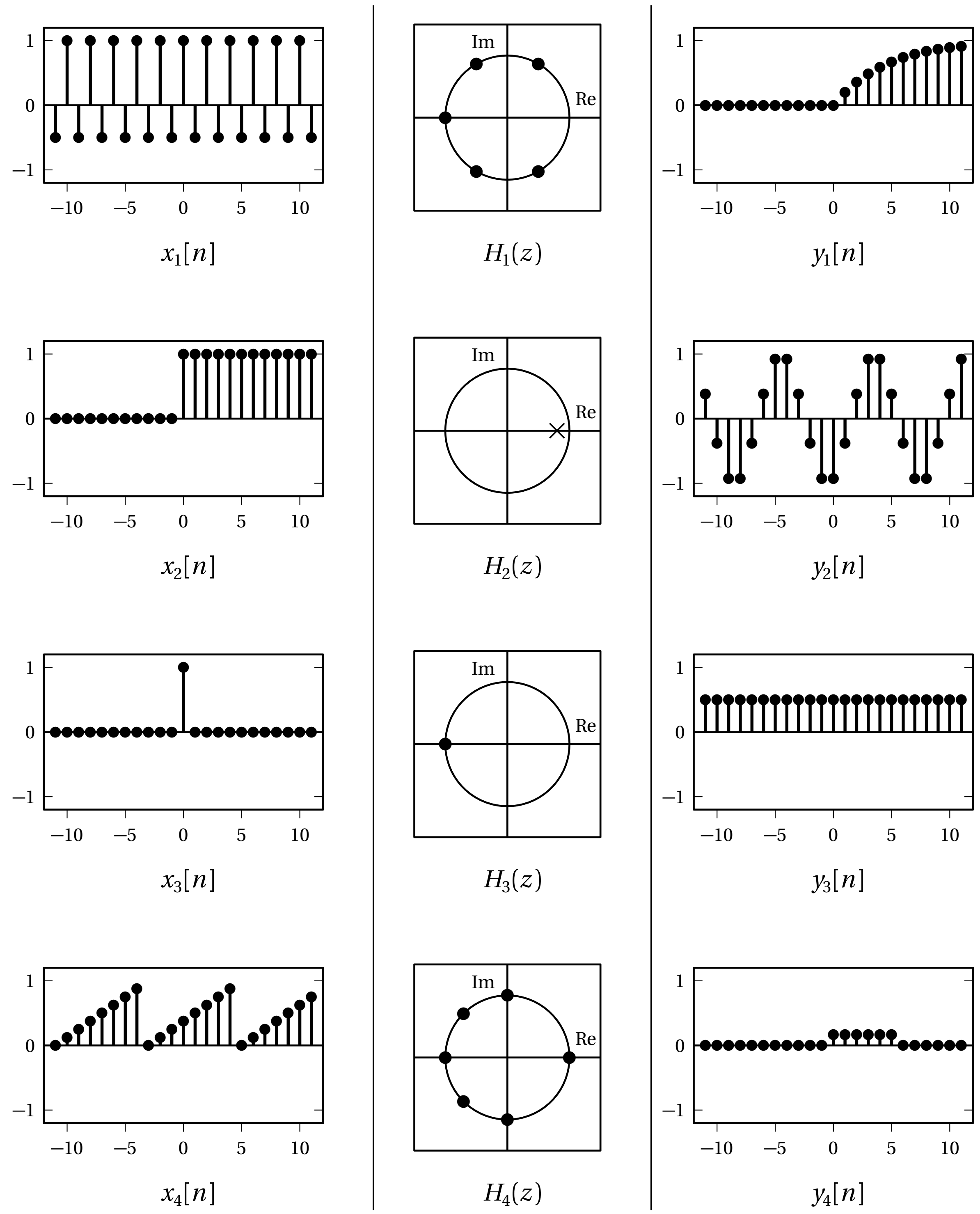
(6)

Solution:

(a) - (5)
(b) - (2)
(c) - (6)
(d) - (4)
(e) - (1)
(f) - (3)

Exercise 30. Pole-zero plots

In the following figure there are four input signals in the left column, four pole-zero plots in the middle, and four output signals on the right. For each input signal $x_i[n]$, $i = 1, 2, 3, 4$, find the filter $h_j[n]$ and the output $y_k[n]$ so that $y_k[n] = (x_i * h_j)[n]$. Assume that all filters are causal and all signals are infinite support (continuing in the obvious way outside of the plotted range). In your answer simply list four triples of the form $i \to j \to k$.



Solution:

$1 \to 3 \to 3, \quad 2 \to 2 \to 1, \quad 3 \to 1 \to 4, \quad 4 \to 4 \to 2$

To establish each triple, we can look more in detail at each item in the figure.

(a) *Observations about the inputs:*

- *$x_1[n]$ is a periodic signal with period 2 (i.e. with the fastest possible frequency); it will therefore contain a term of the form $(-1)^n$ and, from the plot, it is easy to see that $x_1[n] = 0.75(-1)^n + 0.25$;*

- *$x_2[n] = u[n]$ and so the corresponding output will be the integrated impulse response of the associated filter, i.e., $y[n] = \sum_{k=0}^{n} h[k]$;*

- *$x_3[n] = \delta[n]$ and so the corresponding output will be the impulse response of the associated filter;*

- *$x_4[n]$ is a periodic signal with period 8 and so it can be expressed as $x_4[n] = \sum_{k=0}^{7} \alpha_k e^{j\pi kn/4}$.*

(b) *Observations about the filters: note that a pole-zero plot determine the transfer function of a filter up to an unknown scaling factor c; with this,*

- *$H_1(z)$ is the pole-zero plot of a moving average filter of length 6, with impulse response $h_1[n] = c(u[n] - u[n-6])$;*

- *$H_2(z)$ is the pole-zero plot of a leaky integrator with impulse response $h_2[n] = c\lambda^n, 0 < \lambda < 1$;*

- *$H_3(z)$ is an FIR with a single zero in $z = -1$ and thus it implements the CCDE $y[n] = c(x[n] + x[n-1])$;*

- *$H_4(z)$ is an FIR with zeros at $e^{j\pi k/4}$ for $k = 0, \pm 2, \pm 3, 4$ and so it will cancel out any sinusoidal input component at frequencies $\omega_k = \pi k/4$ except for $k = \pm 1$.*

(c) *Observations about the outputs:*

- *$y_1[n]$ starts at zero and grows exponentially and asymptotically; the shape is compatible with an expression of the form $y_1[n] = \sum_{k=0}^{n} \lambda^n$ for $0 < \lambda < 1$ so $y_1[n] = (x_2 * h_2)[n]$*

- *$y_2[n]$ is a sinusoid of the form $\cos((\pi/4)n + \theta)$; since $H_4(z)$ cancels all frequencies at multiples of $\pi/4$ except $\pm\pi/4$ and since $x_4[n]$ contains only frequencies at multiples of $\pi/4$, $y_2[n] = (x_4 * h_4)[n]$*

- *$y_3[n]$ is a noncausal signal with infinite support and therefore the input can only be $x_3[n]$ or $x_4[n]$. By applying $H_3(z)$ to $x_3[n]$ we obtain $y[n] = x_3[n] + x_3[n-1] = 0.5 = y_3[n]$*

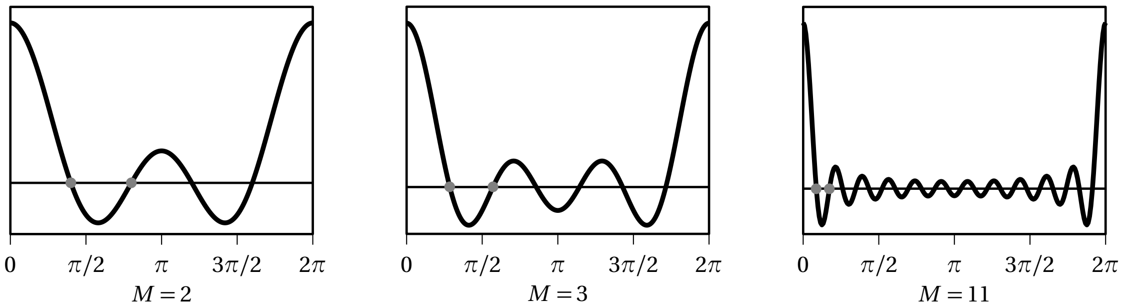
- *$y_4[n]$ is the impulse response of a moving average of length 6 so $y_4[n] = (x_3 * h_1)[n]$;*

Exercise 31. Dirichlet kernel

Consider the 2π-periodic function

$$C_M(\omega) = 1 + 2\sum_{n=1}^{M} \cos(\omega n)$$

for $M \geq 1$; some examples for different values of M are plotted here (note that the vertical scale is *not* the same in each plot):



(a) (5p) what is the maximum value of $C_M(\omega)$ as a function of M?

(b) (15p) the exact minimum value of $C_M(\omega)$ is surprisingly hard to find in closed form but a reasonable estimate is given by $C_M(\omega_0)$, where ω_0 is the midpoint between the coordinates on the frequency axis corresponding to the function's first and second zero crossings (shown by gray dots in the above plots). Provide an approximate expression for $C_M(\omega_0)$ as a function of M, assuming M is sufficiently large.

Hint: by recalling that $2\cos\omega = e^{j\omega} + e^{-j\omega}$, it will be clear that $C(\omega)$ is the DTFT of a very simple signal, a DTFT for which we know a closed-form expression. (Or, if we don't know it, we can find it easily enough using the formula for a geometric sum.) Oh, also remember that $\sin\omega \approx \omega$ for ω small; the engineer's mantra is: "everything is a straight line if you look close enough!"

Solution:

The first thing to notice is that, by using Euler's formula, we can write

$$C_M(\omega) = \sum_{n=-M}^{M} e^{j\omega n}$$

that is, $C_M(\omega)$ is the DTFT of a finite support signal equal to one for $|n| \leq M$ and zero otherwise. This expression also immediately gives us a value for the maximum: first of all

$$|C_M(\omega)| \leq \sum_{n=-M}^{M} |e^{j\omega n}| = 2M + 1$$

and then it is obvious to see that this upper bound is achieved in zero, so $C_M(0) = 2M + 1$ is the maximum value of the function.

Either by direct computation or by remembering the DTFT of a rectangular signal, we can write

$$C_M(\omega) = \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)}.$$

The function is equal to zero for $\omega = 2\pi k/(2M+1)$, $\quad k = 1,\ldots,2M$, *so that the midpoint between the first two zero crossings is*

$$\omega_0 = \frac{1}{2}\left[\frac{2\pi}{2M+1} + \frac{4\pi}{2M+1}\right] = \frac{3\pi}{2M+1}.$$

Now, $\sin(\omega_0(2M+1)/2) = \sin(3\pi/2) = -1$ *and, if* M *is large,* $\sin(\omega_0/2) \approx \omega_0/2$*; therefore*

$$C_M(\omega_0) \approx -\frac{2}{\omega_0} = -\frac{2(2M+1)}{3\pi} \approx -0.4M.$$

(A more rigorous mathematical analysis would show that the true minimum is a bit smaller, i.e., $-0.434467M$*.)*

Exercise 32. Periodization

Consider the signal

$$x[n] = \left(\frac{1}{2}\right)^n u[n], \qquad n \in \mathbb{Z}$$

where $u[n]$ is the unit step sequence.

(a) Write out the expression for the signal

$$x_N[n] = \sum_{p=-\infty}^{\infty} x[n-pN]$$

where N is an integer greater than 1.

(b) Roughly sketch $x_N[n]$ for $N = 10$.

Solution:

The signal $x_N[n]$ *is the periodization of* $x[n]$ *with period* N*, which is guaranteed to exist for all* N *since* $x[n]$ *is absolutely summable. Since* $x_N[n]$ *is* N*-periodic, we can write*

$$x_N[n] = y[n \mod N]$$

where $y[n]$ *is the finite-length signal formed by the samples of* $x_N[n]$ *for* $0 \le n < N-1$.

To find the N *values of* $y[n]$ *we need to compute the sum*

$$y[n] = \sum_{p=-\infty}^{\infty} x[n-pN]$$

for $0 \le n < N-1$. *We have*

$$y[n] = \sum_{p=-\infty}^{\infty} 2^{-(n-pN)}\, u[n-pN] = 2^{-n} \sum_{p=-\infty}^{\infty} 2^{Np}\, u[n-pN]$$

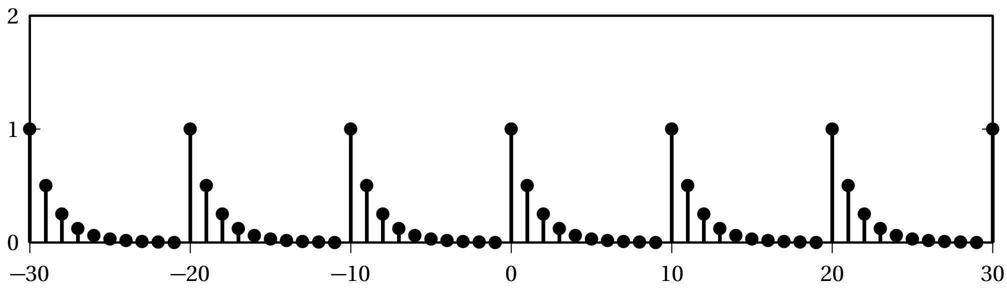
and, for $0 \le n < N-1$, *the expression* $n - pN$ *is greater or equal to* 0 *only for* $p \le 0$. *Therefore*

$$y[n] = 2^{-n} \sum_{p=-\infty}^{0} 2^{Np} = 2^{-n} \sum_{p=0}^{\infty} 2^{-Np} = \frac{2^{-n}}{1-2^{-N}}$$

so that finally

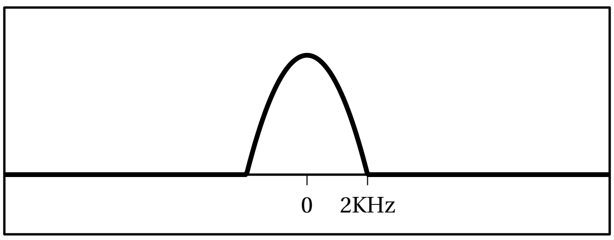
$$x_N[n] = \frac{2^{-(n \mod N)}}{1-2^{-N}}$$

For $N = 10$ *the signal looks like so:*

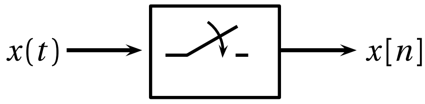


Exercise 33. Sampling

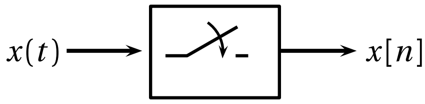
The magnitude spectrum $|X(f)|$ of the bandlimited continuous-time signal $x(t)$ sketched in the following figure:



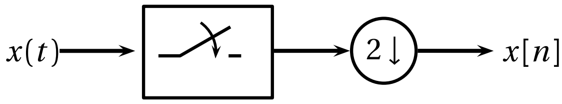
For each of the following systems, sketch the DTFT of the discrete-tiem output signal.

a) 

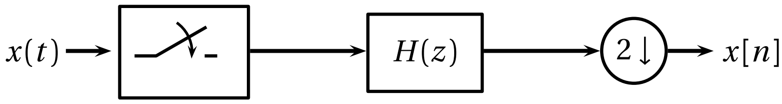
$F_s = 4\text{kHz}$

b) 

$F_s = 2\text{kHz}$

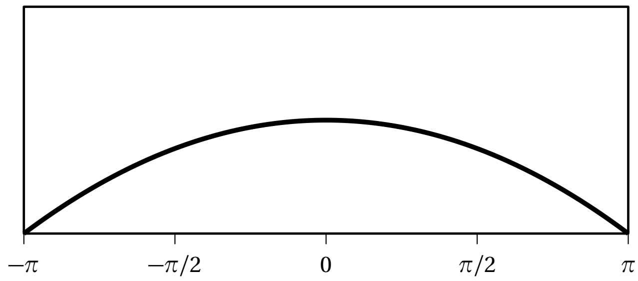
c) 

$F_s = 4\text{kHz}$

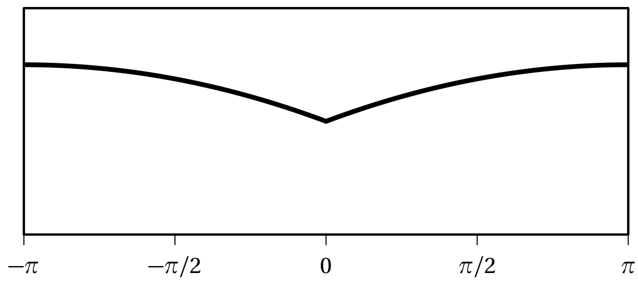
d) 

$F_s = 4\text{kHz}$ lowpass, cutoff $\pi/2$

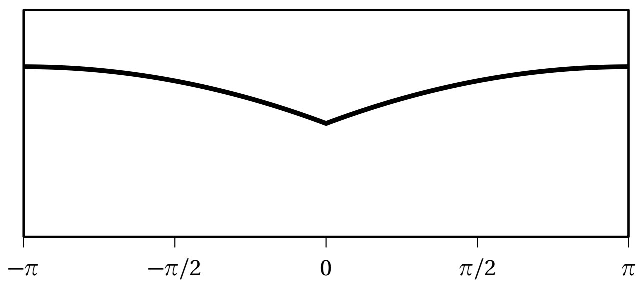
Solution:



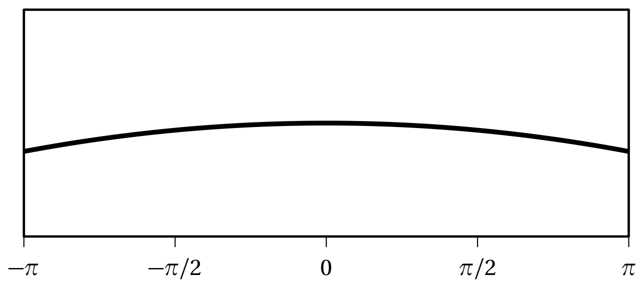
(a)



(b)



(c)



(d)

Exercise 34. Interpolation

The discrete-time signal

$$x[n] = \begin{cases} 1 & \text{for } 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

is interpolated to a continuous-time signal $x_c(t)$ using an ideal sinc interpolator with interpolation period $T_s = 1$; because the sinc interpolator has infinite support, the resulting continuous-time signal will also have infinite support, although it will decay to zero. Given a small positive constant $\delta \ll 1$, we want to find an estimate for a time value t_0, dependent on δ and N, so that

$$|x_c(t)| < \delta \quad \text{for } t > t_0.$$

[Hint: remember that $\left|\sum a_n\right| \le \sum |a_n|$.]

Solution:

We have

$$\begin{aligned}|x_c(t)| &= \left|\sum_{n=0}^{N-1} \text{sinc}(t-n)\right| = \left|\sum_{n=0}^{N-1} \frac{\sin \pi(t-n)}{\pi(t-n)}\right| \\ &\le \sum_{n=0}^{N-1} \frac{|\sin \pi(t-n)|}{|\pi(t-n)|} \\ &\le \sum_{n=0}^{N-1} \frac{1}{|\pi(t-n)|}\end{aligned}$$

Since δ is small, we can safely assume $t_0 > N$, so that, for $t > t_0$, $t-n > 0$ for all n; therefore

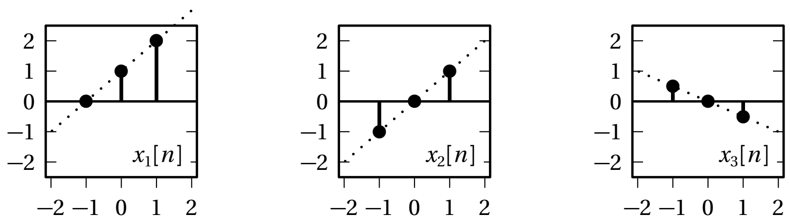
$$|x_c(t)| \le \sum_{n=0}^{N-1} \frac{1}{\pi(t-n)} < \sum_{n=0}^{N-1} \frac{1}{\pi(t-N)} = \frac{N}{\pi(t-N)}.$$

A loose bound for t_0 is therefore

$$t_0 > N\left(\frac{1}{\pi\delta} + 1\right).$$

Exercise 35. Slope filter

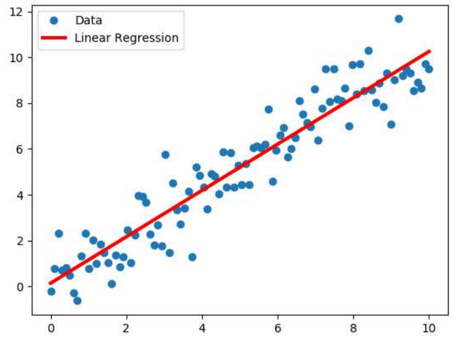
In communication systems it is often useful to be able to estimate the local "slope" of a discrete-time signal; for instance, in the figures below, the local slope around $n = 0$ appears to be equal to 1 for the first two signals and equal to $-1/2$ for the third.



In theory, the local slope at n is the value of the derivative of the sinc-interpolated signal, computed at nT_s; we have seen that this value can be computed directly in discrete time by using an ideal differentiator, that is, a filter with impulse response $h[n] = (-1)^n/n$. The ideal differentiator can be approximated by impulse truncation; consider for instance the 3-tap approximation

$$\hat{h}[n] = \begin{cases} 1 & n = -1 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

If we compute the values $(x_k * \hat{h})[0]$ for the signals above, we can easily find that the result is twice the value of the slope in $n = 0$; for instance $(x_2 * \hat{h})[0] = 1 + 1 = 2$. In other words, the



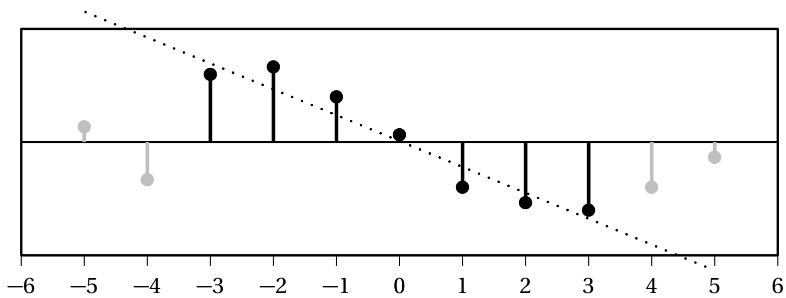
slope in n can be estimated as the value in n of the convolution between the signal and the differentiation filter: the longer the impulse response, the better the approximation.

Unfortunately the ideal differentiator has a highpass frequency response and so, if the input signal is noisy, its performance is not good. We will now study an alternative approach based on *linear regression*. In linear regression, given a set of data points, we want to approximate the data using a straight line; the optimal line is the one that minimizes the mean squared error of the approximation. An example of linear regression as commonly used in statistics is shown in the picture here on the right; in that case the data points are represented by arbitrary pairs of coordinates on the plane. In our case, the problem is simplified by the fact that the abscissae are given by the regularly spaced discrete time index n.

The goal of the exercise is to build an FIR filter that, at each index n, computes the slope of the optimal local linear regressor at n. To formalize the problem, consider the symmetric interval around the origin $-N \leq n \leq N$; given a signal $x[n]$, we approximate the signal around $n = 0$ with a straight line described by the equation $y[n] = \alpha + \beta n$. The mean squared error is given by

$$J(\alpha, \beta) = \sum_{n=-N}^{N} \big(x[n] - (\alpha + \beta n)\big)^2$$

and the optimal values for α and β can be found my minimizing the MSE. For example, in the following figure, minimizing the MSE for $N = 3$ yields the linear regression around $n = 0$ shown by the dotted line.



(a) Derive the expression for β that minimizes J, as a function of $x[n]$ for $-N \leq n \leq N$; this provides the local slope of the signal in $n = 0$.

The expression you found for β can be rearranged to look as the convolution, computed in $n=0$, between $x[n]$ and a $(2N+1)$-tap FIR filter $r_N[n]$:

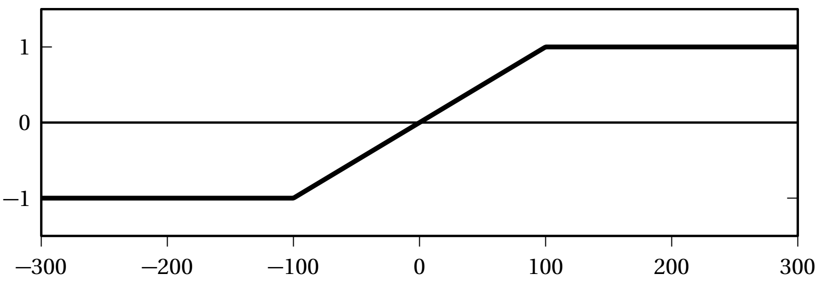
$$\beta = \sum_{k=-N}^{N} r_N[k]x[n-k]\Big|_{n=0};$$

this means that the local slope of $x[n]$ for *any* index n can be obtained simply by filtering $x[n]$ with $r_N[n]$, which is called a *slope filter*.

(b) Compute the five values of the slope filter $r_2[n]$.

(c) Are slope filters linear phase?

Consider now the following discrete-time signal:



(d) Sketch the result of filtering the signal with the slope filter $r_{10}[n]$

(e) Repeat the sketch but this time for the filter $r_{100}[n]$

[Note: a useful formula for the exercise is $\sum_{n=1}^{N} n^2 = N(N+1)(2N+1)/6$.]

Solution:

(a) To find the optimal value of β we set to zero the partial derivative of J with respect to β:

$$\begin{aligned}
-\frac{1}{2}\frac{\partial J}{\partial \beta} &= \sum_{n=-N}^{N} \big(x[n]-\alpha-\beta n\big)n \\
&= \sum_{n=-N}^{N} n x[n] - \alpha \sum_{n=-N}^{N} n - \beta \sum_{n=-N}^{N} n^2 \\
&= \sum_{n=-N}^{N} n x[n] - \beta \frac{N(N+1)(2N+1)}{3} = 0
\end{aligned}$$

so that the optimal value of β is

$$\beta = 3\frac{\sum_{n=-N}^{N} n x[n]}{N(N+1)(2N+1)}$$

(b) *The value of β we just found can be expressed as*

$$\beta = \sum_{k=-\infty}^{\infty} r_N[k]x[n-k]\bigg|_{n=0}$$

with

$$r_N[k] = \begin{cases} \dfrac{-3k}{N(N+1)(2N+1)} & -N \le n \le N \\ 0 & \textit{otherwise} \end{cases}$$

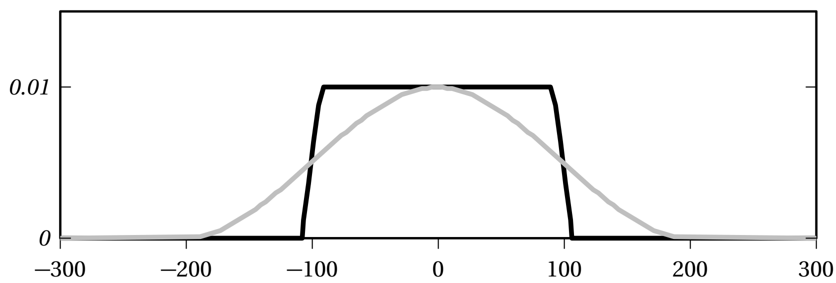
(note the sign change in the numerator because we're flipping the impulse response in the convolution). For $N = 2$, therefore, we have

$$r_2[n] = \begin{cases} 0.2 & n = -2 \\ 0.1 & n = -1 \\ -0.1 & n = 1 \\ -0.2 & n = 2 \\ 0 & \textit{otherwise} \end{cases}$$

(c) *The filter $r_N[n]$ is an odd-length, antisymmetric FIR (Type III) for all N and therefore it has linear phase.*

(d) *The signal has a flat slope for $n < -100$ and for $n > 100$ and a linear slope with $\beta = 2/200 = 0.01$ in between. The result of filtering the signal with $r_{10}[n]$ is shown below in black.*

(e) *If we use a long filter ($r_{100}[n]$ has 201 taps) then the longer impulse response make the slope estimation less sharp and the estimation gets "smeared" in time. The result is sketched in gray below.*



Exercise 36. Communications

In this exercise we will study a data transmission scheme known as *phase modulation* (PM). Consider a discrete-time signal $x[n]$, with the following properties:

- $|x[n]| < 1$ for all n

- $X(\omega) = 0$ for $|\omega| > \alpha$, with α small (that is, $x[n]$ is a baseband signal whose spectrum is nonzero only over a small interval around $\omega = 0$)

A PM transmitter with carrier frequency ω_c works by producing the signal

$$y[n] = \mathscr{P}_{\omega_c}\{x[n]\} = \cos(\omega_c n + k\,x[n])$$

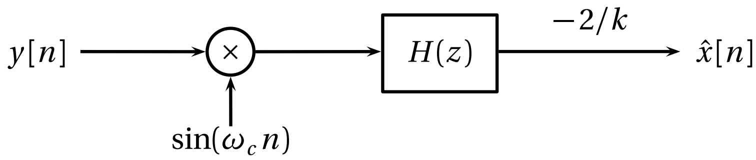
where k is a small positive constant; in other words, the data signal $x[n]$ is used to modify the instantaneous *phase* of a sinusoidal carrier. The advantage of this modulation technique is that it builds a signal with constant envelope (namely, a sinusoid with fixed amplitude) which results in a greater immunity to noise; this is the same principle behind the better quality of FM radio versus AM radio. However phase modulation is less "user friendly" than standard amplitude modulation because it is nonlinear.

(a) Show that phase modulation is *not* a linear operation.

Because of nonlinearity, the spectrum of the signal produced by a PM transmitter cannot be expressed in simple mathematical form. For the purpose of this exercise you can simply assume that the PM signal occupies the frequency band $[\omega_c - \gamma, \omega_c + \gamma]$ (and, obviously, the symmetric interval $[-\omega_c - \gamma, -\omega_c + \gamma]$) with

$$\gamma \approx 2(k+1)\alpha.$$

To demodulate a PM signal the following scheme is proposed, in which $H(z)$ is a lowpass filter with cutoff frequency equal to α:



(b) Show that $\hat{x}[n] \approx x[n]$. Assume that $\omega_c \gg \alpha$ and that k is small, say $k = 0.2$. (You may find it useful to express trigonometric functions in terms of complex exponentials if you don't recall the classic trigonometric identities. Also, remember that $\sin x \approx x$ for x sufficiently small).

Solution:

(a) Given a signal $x[n]$ fulfilling the magnitude and bandwidth requirements, if PM was a linear operation, for any scalar $\beta \in \mathbb{R}$ we should have

$$\mathscr{P}_{\omega_c}\{\beta\,x[n]\} = \beta\,\mathscr{P}_{\omega_c}\{x[n]\}.$$

However, irrespective of $x[n]$, $|\mathscr{P}_{\omega_c}\{\cdot\}| \leq 1$. Since we can always pick a value for β so that the right-hand side of the equality takes values larger than one, the equality cannot hold in general.

(b) *Nonlinear operators make it impossible to proceed analytically in the frequency domain. In the time domain, however, the signal after the multiplier is*

$$\begin{aligned} d[n] &= y[n]\sin(\omega_c n) \\ &= \cos(\omega_c n + kx[n])\sin(\omega_c n) \\ &= (1/2)\sin(\omega_c n + kx[n] + \omega_c n) - (1/2)\sin(\omega_c n + kx[n] - \omega_c n) \\ &= (1/2)\sin(2\omega_c n + kx[n]) - (1/2)\sin(kx[n]) \\ &\approx (1/2)\sin(2\omega_c n + kx[n]) - (k/2)x[n] \end{aligned}$$

where we have used the small-angle approximation for the sine since $|kx[n]| < 0.2$. *The signal* $d[n]$ *now contain a baseband component and a PM component at twice the carrier frequency, which is eliminated by the lowpass filter:*

$$\hat{x}[n] = (-2/k)h[n] * d[n] \approx x[n].$$

[Note: we used the trigonometric identity $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$. *This can be easily derived by developing the product* $(e^{j\alpha} + e^{-j\alpha})(e^{j\beta} - e^{-j\beta})/(2j)$.*]*

Exercise 37. Algorithms

Assyme that your numerical package of choice (Python, Matlab, C++, etc.) implements a function `dfct(x, M, k)`, with `k,M` integers and `x` an array of complex values; the function computes the `k`-th DFT coefficient of the data vector `x` using an `M`-point DFT (with zero-padding if `M` > `len(x)` or with truncation otherwise):

$$\texttt{dfct(x, M, k)} = \sum_{n=0}^{\min\{M-1, \texttt{len(x)}\}} \texttt{x[n]}\, e^{-j\frac{2\pi}{M}kn}$$

Consider now a finite-support discrete-time signal $x[n]$ that is nonzero only for $0 \leq n < N$. Given a frequency value $\omega_0 = 2\pi(A/B)$, with $A, B \in \mathbb{N}$, show how you can compute the DTFT value $X(\omega_0)$ using the `dfct()` function.

Solution:

Choose an integer p so that $pB = M \geq N$; using $\omega_0 = 2\pi(pA/pB)$:

$$\begin{aligned}
X(\omega_0) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega_0 n} \\
&= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{M}pAn} \\
&= \sum_{n=0}^{M-1} x[n]\, e^{-j\frac{2\pi}{M}pAn} \qquad \textit{since } x[n]=0 \textit{ for } n \geq N \\
&= \texttt{dfct}\,(\,[x[0], x[1], \ldots, x[N-1]]\,,\ \ pB\,,\ \ pA)
\end{aligned}$$

Exercise 38. Linear Algebra

Consider the following basis for $\mathbb{C}^4$:

$$\begin{aligned}
\mathbf{x}_0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T \\
\mathbf{x}_1 &= \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \\
\mathbf{x}_2 &= \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T \\
\mathbf{x}_3 &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T
\end{aligned}$$

(a) how would you quickly check in Python/NumPy that the above four vectors do indeed form a basis?

(b) is the basis orthogonal?

(c) compute the inner products of the vector $\mathbf{y} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$ with each basis vector.

Solution:

(a) simply check that `det([x0'; x1'; x2'; x3'])` *is nonzero*

(b) no, $\langle \mathbf{x}_0, \mathbf{x}_1 \rangle = -1 \neq 0$

(c) $\langle \mathbf{y}, \mathbf{x}_1 \rangle = 2$*, the other products are zero*

Exercise 39. Infinite sums

Consider the discrete-time signal $x[n] = \mathrm{sinc}(an)$ with $0 < a < 1$; compute the following sums:

(a) $\sum_{n=-\infty}^{\infty} x[n]$

(b) $\sum_{n=-\infty}^{\infty} x^2[n]$

Solution:

the impulse response of an ideal lowpass filter with cutoff frequency ω_c is

$$h[n] = (\omega_c/\pi)\,\mathrm{sinc}((\omega_c/\pi)n)$$

therefore $x[n]$ is the impulse response of an ideal lowpass filter with cutoff frequency $a\pi$, scaled by $1/a$ so that $X(\omega) = (1/a)\,\mathrm{rect}(\omega/(2a\pi))$. From this:

(a) $\sum_{n=-\infty}^{\infty} x[n] = X(\omega)|_{\omega=0} = 1/a$

(b) $\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 = \frac{1}{2\pi}\int_{-a\pi}^{a\pi} a^{-2} = 1/a$ (by using Parseval's theorem)

Exercise 40. Infinite sum

Compute the sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin(2\pi n/3)}{4\pi n}\cos(\pi n/3)$$

Solution:

Whenever we have a sum of the form $\sum_{n=-\infty}^{\infty} x[n]$ it is always useful to see if we can compute the DTFT of $x[n]$ since the sum is equal to the value of the DTFT in $\omega = 0$

$$\sum_{n=-\infty}^{\infty} x[n] = X(\omega)|_{\omega=0}.$$

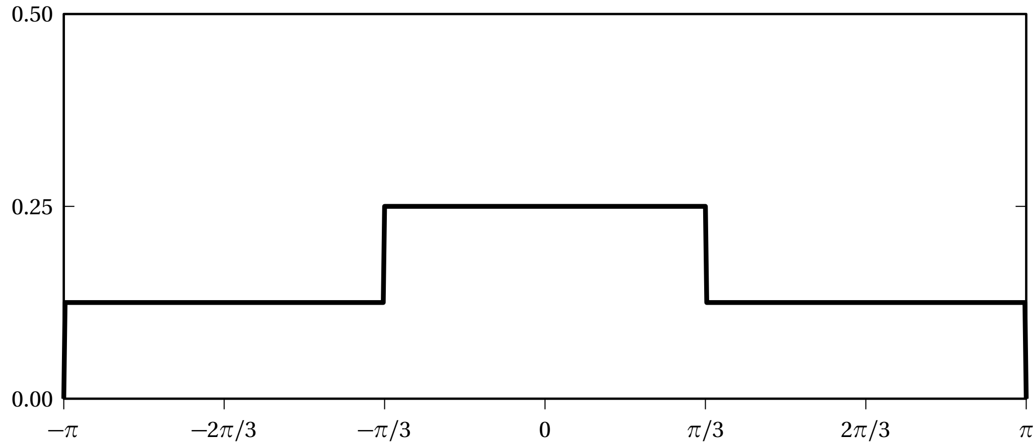
In this case we have

$$\begin{aligned} x[n] &= \frac{\sin(2\pi n/3)}{4\pi n}\cos(\pi n/3) \\ &= \frac{1}{4}\left[\frac{2}{3}\,\mathrm{sinc}\left(\frac{2}{3}n\right)\right]\cos\left(\frac{\pi}{3}n\right) \end{aligned}$$

By using the modulation theorem, the DTFT of $x[n]$ is

$$X(\omega) = \frac{1}{2}\left[\frac{1}{4}\operatorname{rect}\left(\frac{\omega-(\pi/3)}{2\pi/3}\right) + \frac{1}{4}\operatorname{rect}\left(\frac{\omega+(\pi/3)}{2\pi/3}\right)\right]$$

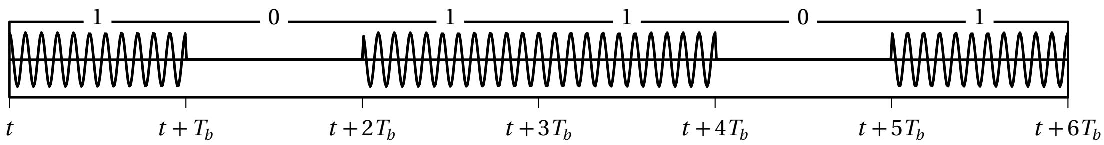
Graphically:



so that $X(\omega)|_{\omega=0} = 1/4$

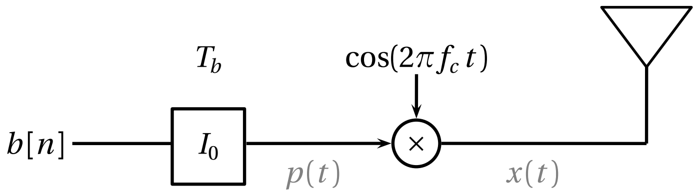
Exercise 41. Communications

Most of the small, battery-operated remote controls that we use to unlock a car or to open a garage door employ a data communication technique called On-Off-Keying (OOK). When you push the button, the transmitter sends a string of binary digits (the secret password) by switching on or off an analog sinusoidal oscillator according to whether the bit to transmit is one or zero. As an example, here is a sketch of what an idealized OOK signal looks like for the binary sequence $[1, 0, 1, 1, 0, 1]$:



In the figure, T_b is the time (in seconds) used to transmit one bit, so that the data rate is $1/T_b$ bits per second; in most commercial systems the data rate is low, about 1000 bps, while the oscillator's frequency is usually around 430 MHz. The main advantages of OOK are the simplicity of the modulation and demodulation circuits, together with the inherent low power consumption of the transmitter: since the oscillator is switched off roughly half of the time, battery life is prolonged.

The implementation for the transmitter can be modeled by the following block diagram:

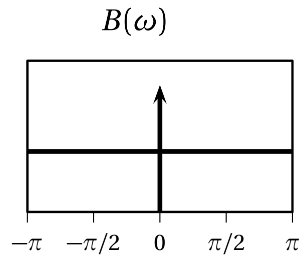


where $b[n]$ is the discrete-time sequence of binary digits to transmit, I_0 is a zero-order-hold D/A interpolator working at a rate T_b, and f_c is the oscillator's frequency; the signal sent to the antenna is thus the continuous-time signal $x(t) = p(t)\cos(2\pi f_c t)$ where

$$p(t) = \sum_{n=-\infty}^{\infty} b[n]\,\mathrm{rect}\left(\frac{t - nT_b}{T_b}\right).$$

In the following exercise questions, assume the following:

- $b[n]$, the binary sequence to transmit, is an infinite-length signal whose DTFT looks as in the picture here on the right; formally, the DTFT can be expressed as $B(\omega) = a + b\tilde{\delta}(\omega)$ for some real-valued constants a and b but note that the actual values of a and b are not relevant to the solution and that you do not need to use the explicit expression for $B(\omega)$ anywhere in your derivations;



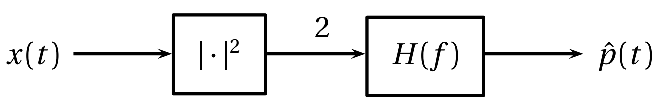
- the frequency of the oscillator in the transmitter is $f_c = 3/T_b$.

With these assumptions:

(a) [5p] Sketch the magnitude of $X(f)$, the spectrum of the transmitted analog signal; the exact amplitude of the spectrum is not important but indicate *precisely* the frequencies at which it is equal to zero and try to be the accurate with the *relative* magnitudes of the peaks. [*Advice: start first by working out the shape of $P(f)$, the spectrum of the baseband signal produced by the interpolator*[1].]

(b) [5p] Do you see any problems with this implementation of the OOK transmitter? How would you fix them?

In spite of the issues inherent to this elementary modulation scheme, the design illustrated above is in fact the one used in practical realizations, especially because the simplicity of the transmitter is matched by that of the receiver. Indeed, an estimate of the signal $p(t)$ can be easily obtained using a simple squaring nonlinearity followed by an analog lowpass filter $H(f)$:



(c) [5p] Show that $\hat{p}(t) \approx p(t)$.

[1]...and, if you've forgotten to write the relevant formula in your notes, here it is: given a discrete-time signal $x[n]$ whose DTFT is $X(\omega)$, and given an interpolator with kernel $i(t)$ and Fourier transform $I(f)$, the spectrum of the continuous-time interpolation of $x[n]$ at a rate of F_s samples per second is $X(f) = (1/F_s)\,I\big(f/F_s\big)\,X(2\pi f/F_s)$

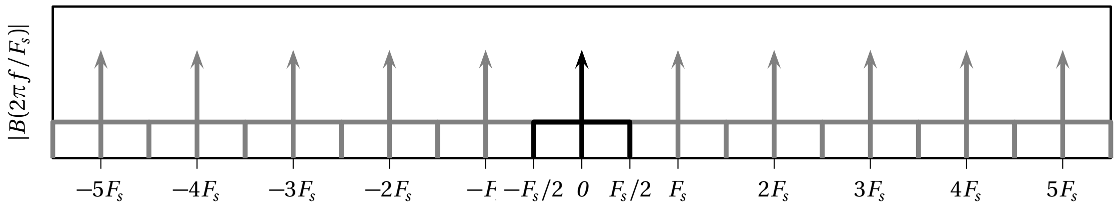
(d) [10p] Determine the analog lowpass filter's cutoff frequency f_L that minimizes the mean square error between $\hat{p}(t)$ and $p(t)$.

Solution:

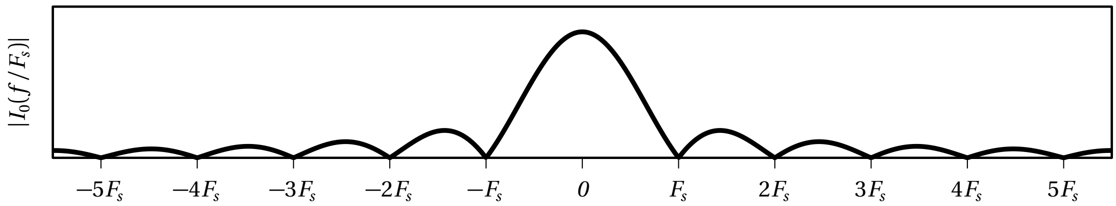
The continuous-time signal $p(t)$ is obtained by interpolating $b[n]$ with a zero-order hold at a frequency $F_s = 1/T_b$; the resulting spectrum is

$$P(f) = (1/F_s)\, I_0\big(f/F_s\big)\; B(2\pi f/F_s)$$

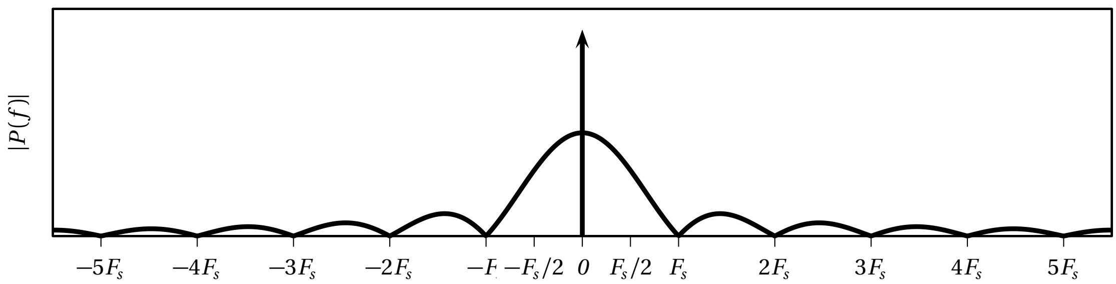
where $I_0(f)$ is the Fourier transform of the zero-order interpolation kernel. The last term in the equation is a rescaled version of the 2π-periodic $B(\omega)$; since $B(\omega)$ is full band, we obtain a constant value plus a series of Dirac deltas at multiples of F_s:



The interpolation kernel is $i_0(t) = \mathrm{rect}(t)$ and therefore its Fourier transform is $I_0(f) = \mathrm{sinc}(f)$; rescaling with respect to F_s we obtain the following plot:



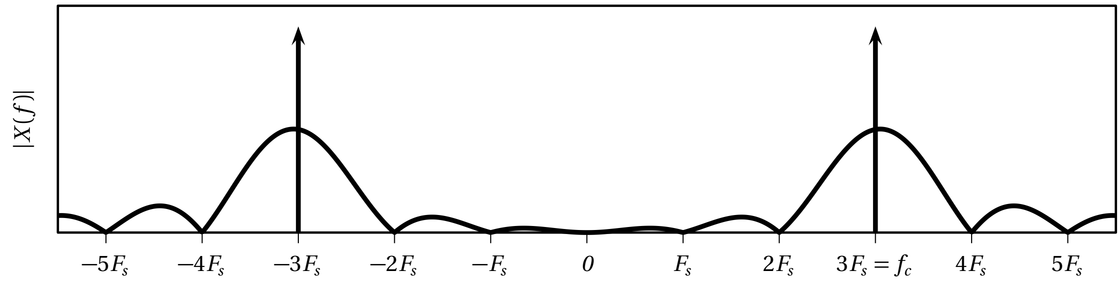
Finally, the magnitude spectrum $|P(f)|$ is obtained from the product of the two spectra above; note that the zeros of $I_0(f/F_s)$ at multiples of F_s cancel out the Diracs in $B(f)$ except in $f = 0$:



(a) *since $x(t) = p(t)\cos(2\pi f_c t)$, by the modulation theorem:*

$$X(f) = [P(f - f_c) + P(f + f_c)]/2$$

and therefore the magnitude spectrum of the transmitted signal looks like so:



(b) *The problems are due to the fact that $P(f)$ is not bandlimited because of the zero-order interpolation; as a consequence:*

  (a) *the two spectral copies created by the modulation overlap with each other, which leads to distortion*

  (b) *the tails of the modulated copies spill out on neighboring frequency bands, which could create interference with other radio devices*

These problems could be reduced by using a higher-order interpolator and, in the limit, a sinc interpolator[2].

(c) *First of all notice that the signal $p(t)$ is either equal to one or to zero for all values of t and therefore $p^2(t) = p(t)$. Next, recall the basic trigonometric identity $\cos^2\alpha = (1+\cos 2\alpha)/2$. With this, the signal entering the lowpass filter is*

$$\begin{aligned} y(t) &= 2x^2(t) = p^2(t)\cdot 2\cos^2(2\pi f_c t) \\ &= p(t) + p(t)\cos(2\pi(2f_c)t) \end{aligned}$$

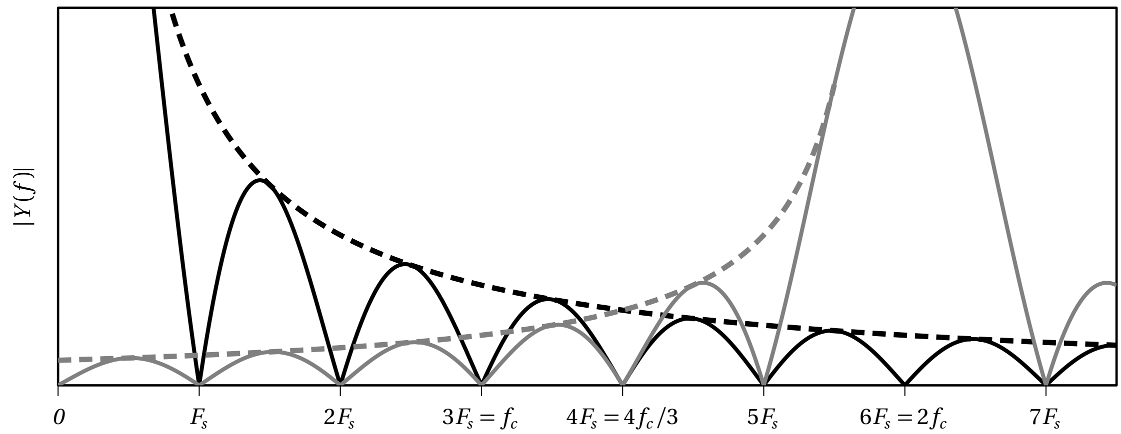
that is, the original $p(t)$ plus a a copy of $p(t)$ modulated at twice the original frequency. In the frequency domain, we thus have

$$Y(f) = P(f) + (1/2)P(f - 2f_c) + (1/2)P(f + 2f_c)$$

If $P(f)$ was bandlimited so that $P(f) = 0$ for $|f| < f_N$ (and $f_N < f_c$), then we could exactly recover $p(t)$ via a lowpass filter with cutoff $f_N \le f_L < f_c - f_N$. Since $P(f)$ is not banlimited, the tails of the cross-modulation copies $P(f \pm 2f_c)/2$ will "spill over" in the baseband and this will cause a certain amount of distortion. Since usually f_c is much much larger than $F_s = 1/T_b$, this distortion will be very small in practical implementations.

[2]This is not done in practice because the resulting design would be much more complex and the energy savings obtained by switching off the oscillator during a zero bit would no longer be there.

(d) *Because of the spectral overlap between $P(f)$ and $P(f \pm 2f_c)/2$ we want a cutoff frequency that preserves most of the energy in $P(f)$ and eliminates most of the energy from the cross-modulated component. To find the optimal point, remember that $P(f)$ has the shape of a sinc function, whose amplitude is upper bounded by $|\pi f|^{-1}$, while the overlapping $P(f-2f_c)/2$ decays as $|2\pi(f-2f_c)|^{-1}$, as illustrated in the following figure:*



The ideal cutoff is therefore the crossover frequency after which the magnitude of $P(f)$ becomes asymptotically smaller than the magnitude of $P(f-2f_c)/2$; this happens for

$$\frac{1}{f} = \frac{1/2}{2f_c - f}$$

that is for

$$f_L = (4/3)f_c;$$

For our particular choice of $f_c = 3F_s$, this corresponds to $f_L = 4F_s = 4/T_b$.

Exercise 42. Multirate

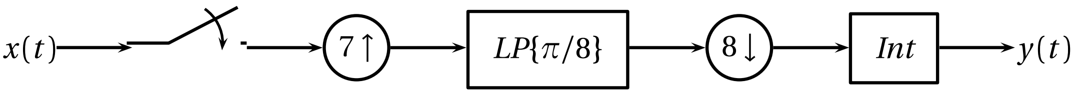
You have been hired as the DJ in residence by the coolest club in town. For the upcoming Saturday night bash you're preparing a great mix of dance songs, all of which are at 128 BPM (beats per minute). There is one song that you absolutely want to include in the mix but unfortunately it is at 112 BPM. Since you're not just a DJ but also a cool digital signal processing expert, you're going to put your hard-earned DSP knowledge to use:

(a) describe a full signal processing scheme that brings up the song to 128 BPM; you can assume the analog signals are bandlimited to 20KHz. Don't forget to exactly specify the parameters of the components (sampling rate, cutoff frequencies, etc.)

(b) if the pitch (i.e. the fundamental frequency) of the song is 440Hz, what will be the pitch of the processed song?

Solution:

(a) *The song, converted to digital, must be "accelerated" by a factor of* $8/7$*, i.e., upsampled by 7 and downsampled by 8. The following scheme can be used:*

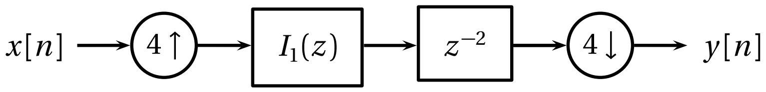


where sampling and interpolator work at least at $F_s = 40KHz$

(b) $440 \cdot (8/7) \approx 503Hz$*, almost a full tone up.*

Exercise 43. Multirate

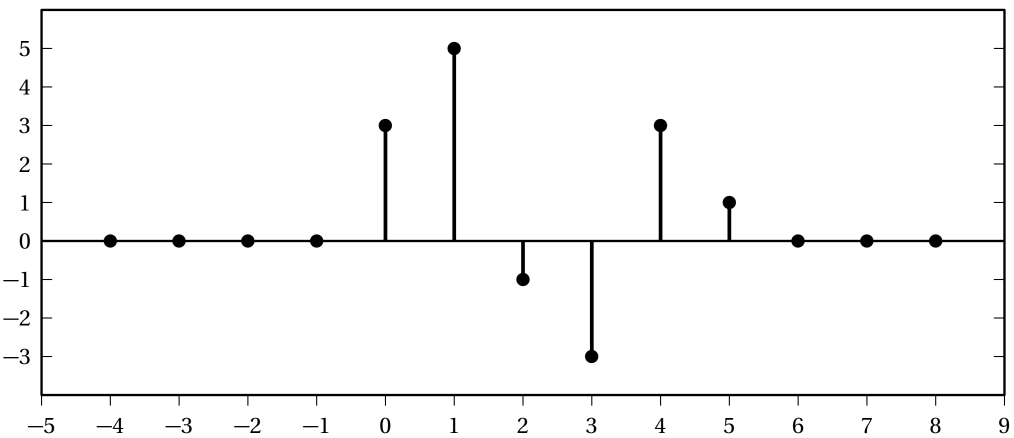
Consider the following multirate processing system:



where $I_1(z)$ is the first-order discrete-time interpolator with impulse response

$$i_1[n] = \begin{cases} 1 - |n|/4 & \text{for } |n| < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Assume $x[n]$ is the finite-support signal shown here:



Compute the values of $y[n]$ for $0 \le n \le 6$, showing your calculation method.

Solution:

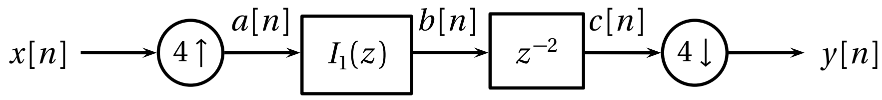
Intuitively, the upsampling by four followed by $I_1(z)$ *creates a linear interpolation over three "extra" samples between the original values the ("connect-the-dots" strategy). The delay by* *two followed by the downsampler selects the midpoint of each interpolation interval. As a whole, the chain implements a fractional delay of half a sample using a linear interpolator so that*

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

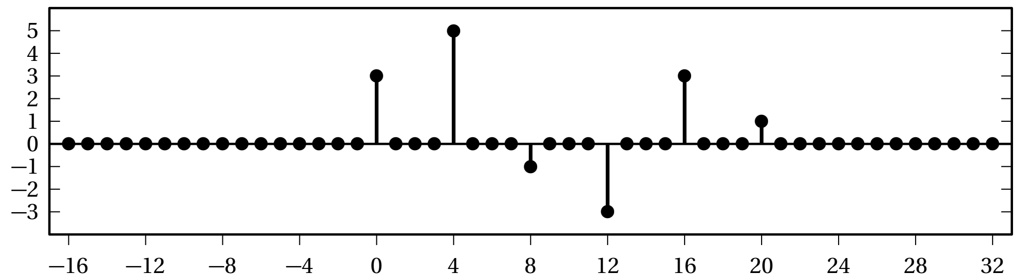
The required values are therefore:

$$y[0] = 1.5, \quad y[1] = 4, \quad y[2] = 2, \quad y[3] = -2, \quad y[4] = 0, \quad y[5] = 2, \quad y[6] = 0.5$$

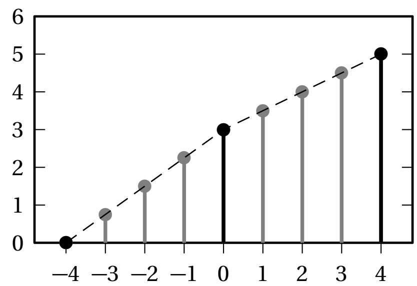
For a proof, label the intermediate signals in the processing chain like so:



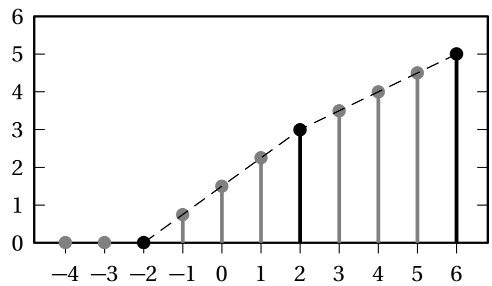
We can either proceed graphically or analytically. Graphically, which is the easiest way, we can start by plotting $a[n]$*:*



Since the interpolator $I(z)$ *has finite support of length 7, we can concentrate on the interval* $[-4, 4]$ *and extend the result to the other points. Linear interpolation fills in the gaps while the delay shifts the interpolated signal by two towards the right:*



$b[n]$



$c[n]$

The downsampler selects the points in $c[n]$ *where* n *is a multiple of four, which are the mid-points between original data values:*

$$y[0] = c[0] = b[-2] = (x[0] + x[-1])/2 = 1.5$$
$$y[1] = c[4] = b[2] = (x[1] + x[0])/2 = 4$$
$$\ldots$$

Alternatively, we can proceed analytically as follows. The z-transform of $c[n]$ is

$$C(z) = X(z^4)z^{-2}I(z)$$

and, after the downsampler, we have

$$\begin{aligned} Y(z) &= \frac{1}{4}\sum_{m=0}^{3} C\left(e^{-j\frac{2\pi}{4}m}\, z^{\frac{1}{4}}\right) \\ &= \frac{1}{4}\sum_{m=0}^{3} X(z)\, e^{-j\frac{2\pi}{4}2m}\, z^{-\frac{1}{2}}\, I\left(e^{-j\frac{\pi}{2}m}\, z^{\frac{1}{4}}\right) \\ &= X(z)\frac{1}{4}z^{-\frac{1}{2}}\left[I\left(z^{\frac{1}{4}}\right) - I\left(-jz^{\frac{1}{4}}\right) + I\left(-z^{\frac{1}{4}}\right) - I\left(jz^{\frac{1}{4}}\right)\right] \end{aligned}$$

The transfer function of the interpolator is

$$I(z) = 1 + (1/4)(z + z^{-1}) + (1/2)(z^2 + z^{-2}) + (3/4)(z^3 + z^{-3})$$

and therefore

$$\begin{aligned} I\left(z^{\frac{1}{4}}\right) &= 1 + (1/4)(z^{1/4} + z^{-1/4}) + (1/2)(z^{1/2} + z^{-1/2}) + (3/4)(z^{3/4} + z^{-3/4}) \\ I\left(-z^{\frac{1}{4}}\right) &= 1 - (1/4)(z^{1/4} + z^{-1/4}) + (1/2)(z^{1/2} + z^{-1/2}) - (3/4)(z^{3/4} + z^{-3/4}) \\ I\left(-jz^{\frac{1}{4}}\right) &= 1 + (j/4)(z^{1/4} + z^{-1/4}) - (1/2)(z^{1/2} + z^{-1/2}) - (3j/4)(z^{3/4} + z^{-3/4}) \\ I\left(jz^{\frac{1}{4}}\right) &= 1 - (j/4)(z^{1/4} + z^{-1/4}) - (1/2)(z^{1/2} + z^{-1/2}) + (3j/4)(z^{3/4} + z^{-3/4}) \end{aligned}$$

Finally,

$$I\left(z^{\frac{1}{4}}\right) + I\left(-z^{\frac{1}{4}}\right) - I\left(-jz^{\frac{1}{4}}\right) - I\left(jz^{\frac{1}{4}}\right) = 2(z^{1/2} + z^{-1/2})$$

so that

$$Y(z) = X(z)\frac{1}{4}z^{-\frac{1}{2}}\left[2(z^{1/2} + z^{-1/2})\right] = \frac{1 + z^{-1}}{2}X(z).$$

Exercise 44. Quantization

Consider a i.i.d. discrete-time random signal whose samples are uniformly distributed over the $[-5, 5]$ interval. Consider a 2-bit quantizer with the following characteristic:

$$\hat{x}[n] = \mathcal{Q}\{x[n]\} = \begin{cases} +1 & \text{if} \quad 0 \le x[n] \le 2 \\ +3 & \text{if} \quad x[n] > 2 \\ -1 & \text{if} \quad -2 \le x[n] < 0 \\ -3 & \text{if} \quad x[n] < -2 \end{cases}$$

Compute the SNR at the output of the quantizer.

Solution:

The power of the signal is simply the variance of the input distribution:

$$P_x = \frac{10^2}{12}$$

The power of the quantization error is

$$P_e = E[(x - \hat{x})^2] = \int_{-5}^{5} p(x)(x - \mathcal{Q}\{x\})^2 dx$$

Exploiting symmetries

$$P_e = \frac{1}{5}\int_0^2 (x-1)^2 dx + \frac{1}{5}\int_2^3 (x-3)^2 dx = \frac{11}{15}$$

so that

$$SNR = \frac{125}{11}$$
