COM-202 - Signal Processing

Solutions for Homework 6

Exercise 1. LTI Systems (I)

Consider a discrete-time system described by the following input-output relationship:

$$y[n] = n\,x[n].$$

Is the system linear? Is it time-invariant?

Solution: *The system is linear; for any two input signals $\mathbf{x}_A, \mathbf{x}_B$, let the corresponding outputs be*

$$y_A[n] = n\,x_A[n]$$
$$y_B[n] = n\,x_B[n];$$

for $\mathbf{x} = a\mathbf{x}_A + b\mathbf{x}_B$, the output is

$$y[n] = n\,x[n] = a\,n\,x_A[n] + b\,n\,x_B[n] = a\,y_A[n] + b\,y_B[n].$$

The system however is not time-invariant. To show this, call $\mathbf{h}_k$ the system's response to the shifted delta sequence $\boldsymbol{\delta}_k = \mathscr{S}^{-k}\boldsymbol{\delta}$:

$$h_k[n] = n\delta[n-k] = \begin{cases} k & n = k \\ 0 & n \neq 1 \end{cases} = k\delta[n-k].$$

If the system was time-invariant we should have $h_k[n] = h_0[n-k]$ but, obviously, since $h_0[n] = 0$ for all n, this is not the case.

Exercise 2. LTI Systems (II)

For each of the input-output relationships listed below, determine whether the discrete system they describe is linear, time-invariant, and BIBO stable. If the system is LTI, determine its impulse response.

(a) $y[n] = x[-n]$

(b) $y[n] = e^{-j\omega n} x[n]$

(c) $y[n] = \sum_{k=n-L}^{n+L} x[k]$ for $L \in \mathbb{N}^+$

(d) $y[n] = n\, y[n-1] + x[n]$,
assuming causality and zero initial conditions, that is, all inputs and outputs are zero for $n < 0$.
(Hint: to prove linearity, you can then proceed by induction.)

Solution: *In every case, calling* $\mathscr{H}$ *the system described by the input-output relationship, we will assume the following:*

$$\begin{aligned}
&\mathbf{x}_A, \mathbf{x}_B \in \ell_2(\mathbb{Z})\\
&\mathbf{y}_A = \mathscr{H}\mathbf{x}_A\\
&\mathbf{y}_B = \mathscr{H}\mathbf{x}_B\\
&\mathbf{x} = a\mathbf{x}_A + b\mathbf{x}_B, \quad a, b \in \mathbb{C}\\
&\mathbf{y} = \mathscr{H}\mathbf{x}\\
&\mathbf{x}_d = \mathscr{S}^{-d}\mathbf{x}\\
&\mathbf{y}_d = \mathscr{H}\mathbf{x}_d
\end{aligned}$$

The system will be linear if $\mathbf{y} = a\mathbf{y}_A + b\mathbf{y}_B$ *and time-invariant if* $\mathbf{y}_d = \mathscr{S}^{-d}\mathbf{y}$.

(a) The system is linear:

$$y[n] = x[-n] = a\, x_A[-n] + b\, x_B[-n] = a\, y_A[n] + b\, y_B[n]$$

The system is not time-invariant: first of all notice that the change of sign affects the whole expression for the index:

$$y[n] = x[-n] \Rightarrow y[(n-d)] = x[-(n-d)] = x[-n+d]$$

Now, if $x_d[n] = x[n-d]$ *then* $x_d[(-n)] = x[(-n)-d]$ *and so*

$$y_d[n] = x_d[-n] = x[-n-d] \neq y[n-d] = x[-n+d]$$

The system is BIBO stable since, if $|x[n]| \leq M$ *for all* n, *then obviously* $|x[-n]| \leq M$ *as well.*

(b) The system is linear:

$$y[n] = e^{-j\omega n}(a\, x_A[n] + b\, x_B[n]) = a(e^{-j\omega n} x_A[n]) + b(e^{-j\omega n} x_B[n]) = a\, y_A[n] + b\, y_B[n]$$

The system is not time-invariant:

$$y_d[n] = e^{-j\omega n} x_d[n] = e^{-j\omega n} x[n-d] \neq y[n-d] = e^{-j\omega(n-d)} x[n-d]$$

The system is BIBO stable since, if $|x[n]| \leq M$ *for all* n, *then* $|y[n]| = |e^{-j\omega n} x[n]| = |x[n]| \leq M$.

(c) *The system is linear:*

$$y[n] = \sum_{k=n-L}^{n+L} (a\,x_A[k] + b\,x_B[k]) = a \sum_{k=n-L}^{n+L} x_A[k] + b \sum_{k=n-L}^{n+L} x_B[k] = a\,y_A[n] + b\,y_B[n]$$

The system is time-invariant since

$$\begin{aligned} y_d[n] &= \sum_{k=n-L}^{n+L} x_d[k] = \sum_{k=n-L}^{n+L} x[k-d] \\ y[n-d] &= \sum_{k=n-d-L}^{n-d+L} x[k] = \sum_{m=n-L}^{n+L} x[m-d] \end{aligned}$$

where we have used the change of variable $m = k + d$ *in the second line.*

The system is BIBO stable since, if $|x[n]| \le M$, *then* $|\sum_{k=n-L}^{n+L} x[n]| \le (2L+1)M$

The system's impulse response is

$$h[n] = \begin{cases} 1 & \text{if } |n| \le L \\ 0 & \text{otherwise.} \end{cases}$$

(d) *To show that the system is linear, we need to show that* $y[n] = a\,y_A[n] + b\,y_B[n]$ *for all* n. *For this, we can proceed by induction: if the equality is true for some value* n_0 *then it is true for all* $n \ge n_0$ *since*

$$\begin{aligned} y[n+1] &= (n+1)y[n] + a\,x_A[n+1] + b\,x_B[n+1] \\ &= (n+1)(a\,y_A[n] + b\,y_B[n]) + a\,x_A[n+1] + b\,x_B[n+1] \\ &= a\{(n+1)y_A[n] + x_A[n+1]\} + b\{(n+1)y_B[n] + x_B[n+1]\} \\ &= a\,y_A[n+1] + b\,y_B[n+1]. \end{aligned}$$

Since we assume zero initial conditions, we know that $x_A[n] = x_B[n] = y_A[n] = y_B[n] = y[n] = 0$ *for all* $n < 0$ *and so we can start the recursion at any value* $n_0 < 0$.

The system is not time-invariant; to show this, call $\mathbf{h}_d$ *the output of the system when the input is the shifted delta sequence* $\boldsymbol{\delta}_d = \mathscr{S}^{-d}\boldsymbol{\delta}$; *when the shift is zero we have*

$$h_0[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ 1 & n = 1 \\ 2 & n = 2 \\ 6 & n = 3 \\ \dots \end{cases}$$

and when the shift is one:

$$h_1[n] = \begin{cases} 0 & n < 0 \\ 0 & n = 0 \\ 1 & n = 1 \\ 2 & n = 2 \\ 6 & n = 3 \\ \ldots \end{cases}$$

If the system was time-invariant we should have $h_1[n] = h_0[n-1]$ for all n but $h_1[2] \neq h_0[1]$.

The system is not BIBO stable; again, using the delta sequence as input, it is easy to show that

$$h_0[n] = (n!)u[n]$$

which is not a bounded sequence.

Exercise 3. Convolution

Compute the nonzero values of the sequence $\mathbf{y} = \mathbf{x} * \mathbf{h}$ where

$$x[n] = \begin{cases} 3 & n = -1 \\ 1 & n = 0 \\ -1 & n = 1 \\ 2 & n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} -1 & n = -1 \\ 2 & n = 1 \\ 4 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$y[n] = \begin{cases} -3 & n = -2 \\ -1 & n = -1 \\ 7 & n = 0 \\ 12 & n = 1 \\ 2 & n = 2 \\ 0 & n = 3 \\ 8 & n = 4 \\ 0 & \textit{otherwise} \end{cases}$$

Exercise 4. Triangular sequence

For M an odd positive integer, consider the discrete-time sequence

$$t_M[n] = \begin{cases} M - |n| & |n| < M \\ 0 & \text{otherwise.} \end{cases}$$

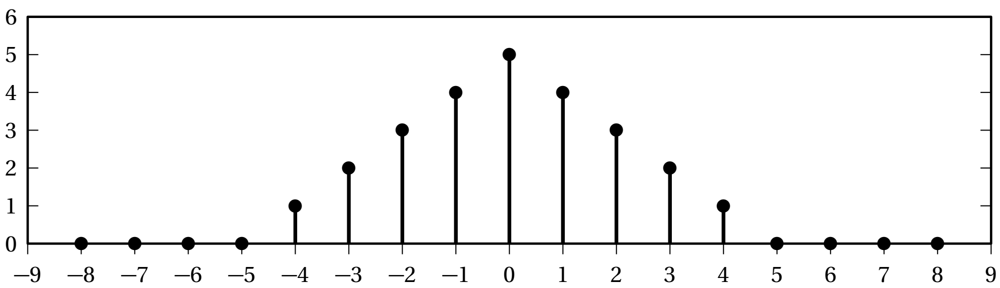
(a) sketch $\mathbf{t}_5$

(b) Find a symmetric sequence $\mathbf{r}_M$ such that $\mathbf{t}_M = \mathbf{r}_M * \mathbf{r}_M$ for any odd positive integer M. As a hint, try to convolve the sequence $r[n] = \delta[n+1] + \delta[n] + \delta[n-1]$ with itself.

(c) Using the results found in the previous question, compute the DTFT of $\mathbf{t}_M$.

Solution:

(a) The sequence $\mathbf{t}_M$ has a symmetric, zero-centered, triangular shape with $2M-1$ nonzero samples:



(b) The discrete-time sequence $\mathbf{t}_M$ can be written as the convolution of a zero-centered rectangular sequence with M nonzero samples:

$$r_M[n] = \begin{cases} 1 & |n| \le (M-1)/2 \\ 0 & \textit{otherwise.} \end{cases}$$

We can verify this by computing the value of the convolution in n; since $\mathbf{r}_M$ is symmetric we have

$$(\mathbf{r}_M * \mathbf{r}_M)[n] = \sum_{k=-(M-1)/2}^{(M-1)/2} r_M[k] r_M[n-k];$$

the value of the sum is equal to the number of overlapping nonzero samples between the original $\mathbf{r}_M$ and a copy of $\mathbf{r}_M$ shifted by n. When $|n| \ge M$ the two sequences do not overlap at all whereas the maximum overlap occurs for $n = 0$, where the convolution sum is equal to M.

Using Python and NumPy, we can easily verify the above result for $M = 5$ using the following code:

```
r = np.ones(5)
t = np.convolve(r, r)
plt.stem(t)
```

(c) The DTFT of the rectangular signal $\mathbf{r}_M$ *is well-known and has been derived in class. Using the convolution theorem, we can write*

$$\begin{aligned} T_M(\omega) &= R_M(\omega)R_M(\omega) \\ &= \left(\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right)^2. \end{aligned}$$

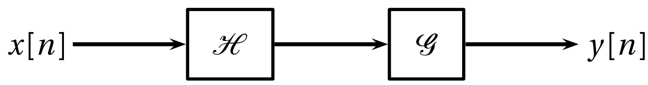
Exercise 5. A nonlinear system

Consider a discrete-time system $\mathscr{H}$ whose input-output relationhip is $y[n] = x^2[n]$.

(a) Show with an example that the system is nonlinear.

(b) Prove that the system is time-invariant.

Now consider the following cascade:



where $\mathscr{G}$ is an ideal highpass filter with frequency response:

$$G(\omega) = \begin{cases} 0 & \text{for } |\omega| < \pi/2, \\ 1/2 & \text{otherwise} \end{cases}$$

(the 2π-periodicity of $G(\omega)$ is implicit).

(c) Compute the output of the cascade when the input is $x[n] = 2\cos(\omega_0 n)$ for $\omega_0 = 3\pi/8$. How would you describe the combined effect of the cascade on the input?

(d) Compute the output once again but using $\omega_0 = 7\pi/8$.

Solution:

(a) let $\mathbf{h}_1 = \mathscr{H}\boldsymbol{\delta}$ *and* $\mathbf{h}_2 = \mathscr{H}(a\boldsymbol{\delta})$*; since* $\delta^2[n] = \delta[n]$*, we have*

$$\begin{aligned} h_1[n] &= \delta[n] \\ h_2[n] &= a^2\delta[n] \neq a h_1[n] = a\delta[n] \end{aligned}$$

(b) Let $\mathbf{y} = \mathscr{H}\mathbf{x}$ *and* $\mathbf{y}_d = \mathscr{H}(\mathscr{S}^{-d}\mathbf{x})$*; we have*

$$y_d[n] = (x[n-d])^2 = y[n-d]$$

(c) *Call* $\mathbf{v}$ *the signal at the output of the first block* $\mathcal{H}$*; using a well-known trigonometric identity we can write*

$$v[n] = 4\cos^2(\omega_0 n) = 2 + 2\cos(2\omega_0 n)$$

so that $\mathbf{v}$ *is the sum of a constant term and of a sinusoid at* double *the original input frequency.*

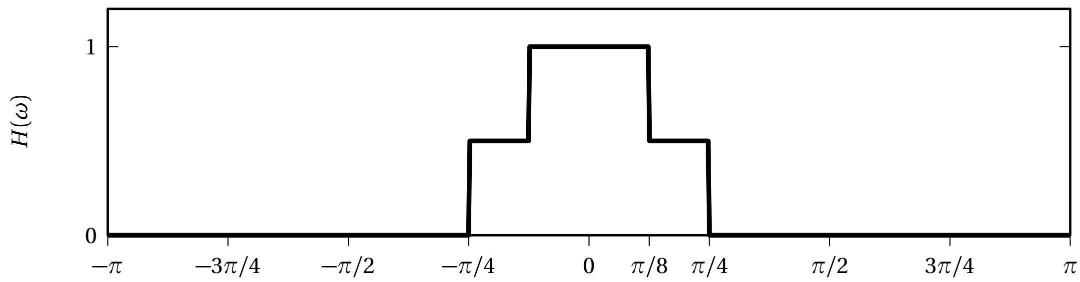
With $\omega_0 = 3\pi/8$, $v[n] = 2 + 2\cos((3\pi/4)n)$*; since* $\mathcal{G}$ *is a highpass with cutoff frequency* $\pi/2$*, the filter will eliminate all frequency components below* $\pi/2$ *and therefore it will remove the constant term in the input but the cosine at frequency* $3\pi/4$ *will not be affected. The final output is therefore* $y[n] = \cos((3\pi/4)n)$.

(d) *When* $\omega_0 = 7\pi/8$*, then* $2\omega_0 = 7\pi/4 > \pi$*. Since we are in discrete time, frequencies are always to be wrapped back over the* $[-\pi,\pi]$ *interval by adding the necessary integer multiple of* 2π*. In this case,* $|7\pi/4 - 2\pi| = |-\pi/4| < \pi$ *and therefore* $\cos((7\pi/4)n) = \cos((-\pi/4)n) = \cos((\pi/4)n)$.

Again, $\mathcal{G}$ *will eliminate all frequency components below* $\pi/2$ *and, in this case, both the constant term and the cosine term in* $\mathbf{v}$ *fall in the filter's stopband so that the output will be* $y[n] = 0$.

Exercise 6. Ideal filters

Compute the impulse response of an ideal filter whose real-valued frequency response $H(\omega)$ is shown in the following figure:



Solution:

By looking at the frequency response we can write

$$H(\omega) = \frac{1}{2}\,\text{rect}\left(\frac{\omega}{2\omega_A}\right) + \frac{1}{2}\,\text{rect}\left(\frac{\omega}{2\omega_B}\right)$$

with $\omega_A = \pi/4$ *and* $\omega_B = \pi/8$.

We know that a frequency response of the form $\text{rect}(\omega/(2\omega_c))$ *corresponds to the impulse response*

$$\frac{\omega_c}{\pi}\,\text{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

and therefore

$$h[n] = \frac{1}{8}\,\text{sinc}\left(\frac{n}{4}\right) + \frac{1}{16}\,\text{sinc}\left(\frac{n}{8}\right)$$

Exercise 7. The Gibbs Phenomenon

In this exercise you will verify the existence of the Gibbs phenomenon using Python and NumPy. Consider a filter whose impulse response is the symmetrically truncated impulse response of an ideal lowpass with cutoff $\omega_c = \pi/2$:

$$h_M[n] = \begin{cases} (1/2)\,\text{sinc}(n/2) & |n| \le M \\ 0 & \text{otherwise} \end{cases}$$

Let $H_M(\omega)$ be the DTFT of $\mathbf{h}_M$:

(a) Plot $H_{20}(\omega)$ for $1.4 \le \omega \le 1.7$ using at least 2000 uniformly spaced points over the range for ω.

(b) Now plot $H_{100}(\omega)$ and $H_{200}(\omega)$ and verify that the maximum value of the DTFTs remains approximately the same in both cases.

Solution: *The frequency response of the truncated ideal lowpass is*

$$H_M(\omega) = \frac{1}{2}\sum_{n=-M}^{M} \text{sinc}(n/2)\, e^{-j\omega n}$$

First let's define a function to compute $H_M(\omega)$ *over a given set of* K *freqency values* $\{\omega_0, \ldots, \omega_{K-1}\}$. *An efficient way to do so is to define a* $K \times (2M+1)$ *matrix* $\mathbf{W}$ *with entries* $W[k,m] = e^{-j\omega_k(m-M)}$, *with* $0 \le k < K$ *and* $0 \le m < 2M+1$; *the desired DTFT values can thus be computed in one go via the matrix-vector multiplication*

$$\mathbf{W}\begin{bmatrix} h[-M] \\ h[-M+1] \\ \vdots \\ h[0] \\ \vdots \\ h[M-1] \\ h[M] \end{bmatrix}$$

```
def H(omega, M):
    n = np.arange(-M, M+1)
    h = 0.5 * np.sinc(0.5 * n)
    W = np.exp(-1j * np.outer(omega, n))
    return omega, np.real(W @ h)
```

The following code can be use to plot the required charts:

```
for M in (20, 100, 200):
    plt.plot(*H(np.linspace(1.4, 1.7, 2000), M), label=f"M={M}")
plt.axhline(1.09, c='r', ls=':')
plt.legend()
```

Interpretation:
$H_M(\omega)$ *can be considered an approximation of an ideal lowpass with cutoff* $\omega_c = \pi/2$. *The time-domain approximation error caused by truncating the impulse response appears in the frequency response as an oscillatory error near the cutoff frequency; this type of error is known as the Gibbs phenomenon. When* M *grows larger, that is when the truncation error in the impulse response becomes smaller, the frequency response becomes progressively flatter and closer to the ideal response almost everywhere but, interestingly enough, the maximum value of the oscillatory error does not decrease, as we can see on the plots. The peaks however gets thinner and the total area under the oscillations decreases.*