COM-202 - Signal Processing

Solutions for Homework 5

Exercise 1. DTFTs

Compute the DTFTs of the following sequences:

(a) $x[n] = \frac{1}{2^n} u[n] - \frac{1}{4^n} u[n-1]$

(b) $x[n] = a^n \cos(\omega_0 n) u[n], \qquad |a| < 1$

Solution:

(a) By linearity the DTFT will be the sum of the DTFTs of both terms. For the first term (a decaying exponential sequence starting at $n = 0$) we know that the DTFT is simply $\frac{1}{1-(1/2)e^{-j\omega}}$. For the second term, we have

$$\begin{aligned}\sum_{n=1}^{\infty} 4^{-n} e^{-j\omega n} &= \sum_{n=0}^{\infty} 4^{-n} e^{-j\omega n} - 1 \\ &= \frac{1}{1-(1/4)e^{-j\omega}} - 1\end{aligned}$$

so that in the end

$$X(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{1}{1-\frac{1}{4}e^{-j\omega}} + 1.$$

(b) Let $a[n] = a^n u[n]$, so that $A(\omega) = 1/(1-a e^{-j\omega})$. Using Euler's formula we can write $\cos(\omega_0 n) = (1/2)(e^{j\omega_0 n} + e^{-j\omega_0 n})$ so that, by linearity,

$$X(\omega) = \frac{1}{2}\text{DTFT}\{e^{j\omega_0 n} a[n]\} + \frac{1}{2}\text{DTFT}\{e^{-j\omega_0 n} a[n]\}.$$

Using the time-shift property of the DTFT we finally have

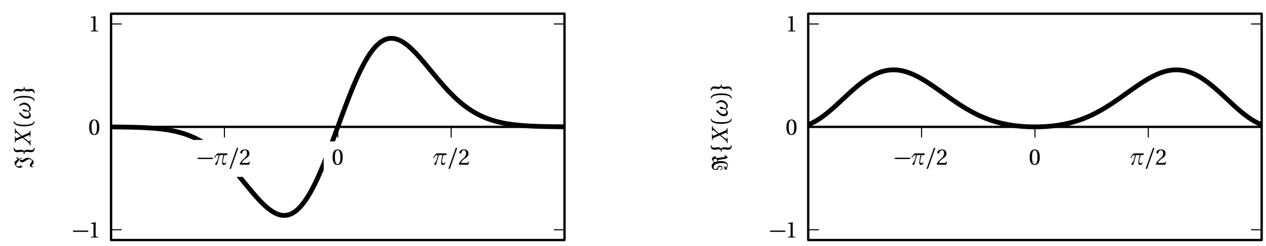
$$\begin{aligned}X(\omega) &= \frac{1}{2}\left[A(\omega-\omega_0) + A(\omega+\omega_0)\right] \\ &= \frac{1 - a\cos\omega_0 e^{-j\omega}}{1 - 2a\cos\omega_0 e^{-j\omega} + a^2 e^{-2j\omega}}.\end{aligned}$$

Exercise 2. DTFT visual inspection

The real and imaginary parts of $X(\omega)$ are shown in the figure below. By visual inspection of the plots, prove that:

(a) $x[n]$ is 0-mean, i.e., $\sum_{n\in\mathbb{Z}} x[n] = 0$;

(b) $x[n]$ is real valued.



Solution:

(a) From the plots one can see that $X(\omega)$ is 0 for $\omega = 0$,

$$X(0) = \sum_{n\in\mathbb{Z}} x[n] = 0,$$

so $x[n]$ is 0-mean.

(b) From the plots one can see that the real part of $X(\omega)$ is symmetric around $\omega = 0$, and its imaginary part is antisymmetric. Since

$$x^*[n] \longleftrightarrow X^*(-\omega),$$

then

$$X^*(-\omega) = \Re\{X(-\omega)\} - j\Im\{X(-\omega)\} = \Re\{X(\omega)\} + j\Im\{X(\omega)\} = X(\omega).$$

Therefore, $x[n] = x^*[n]$.

Exercise 3. DTFT of a symmetric sequence

Compute the DTFT of $x[n] = a^{|n|}$ for $n \in \mathbb{Z}$ and $|a| < 1$, and sketch its magnitude for a close to 1.

Solution:

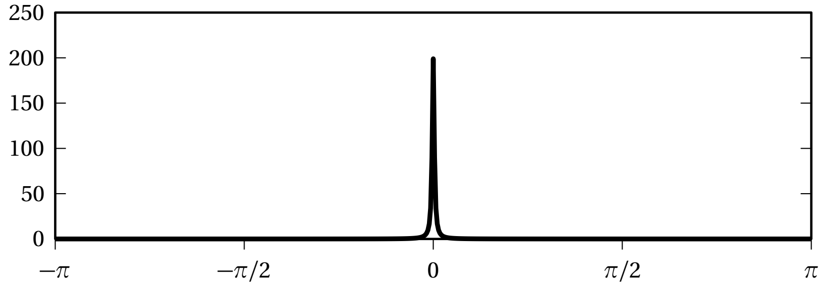
Set $t[n] = a^n\, u[n]$*; then*

$$x[n] = t[n] + t[-n] - \delta[n].$$

Since $T(\omega) = (1 - a e^{-j\omega})^{-1}$*, it is*

$$\begin{aligned} X(\omega) &= \frac{1}{1 - a e^{-j\omega}} + \frac{1}{1 - a e^{j\omega}} - 1 \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

Here is a plot for $a = 0.99$*:*



Exercise 4. DTFT properties

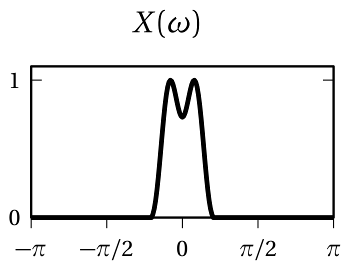
Given a finite-energy sequence $x[n]$ and its DTFT $X(\omega)$, express the DTFT of the sequence $y[n] = (-1)^n x[n]$ in terms of $X(\omega)$.

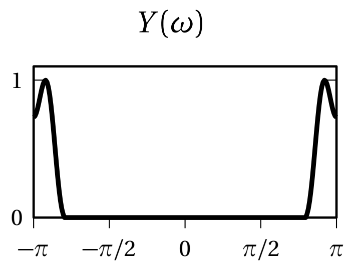
[Hint: remember that $(-1)^n = e^{j\pi n}$*]*

Solution:

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{j\pi n} e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\pi)n} \\ &= X(\omega - \pi). \end{aligned}$$

This results in a translation of the DTFT plot by π. Because of the 2π-periodicity of the DTFT over the $[-\pi,\pi]$ interval, this is visually equivalent to a circular shift, e.g.:





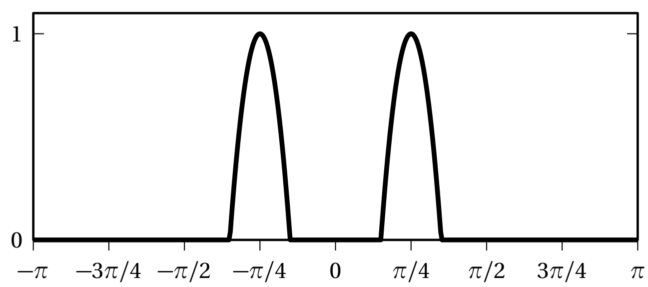
Exercise 5. Alternate notation for the DTFT

In many signal processing textbooks and publications (including the material used in previous editions of this class) DTFTs are represented using the notation

$$X(e^{j\omega}).$$

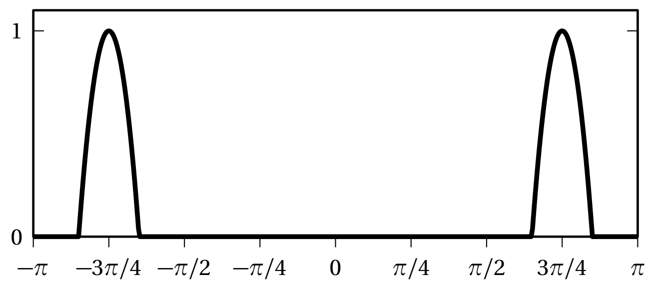
The main advantage of that choice is clarity since any expression of the form $A(e^{jb})$ is immediately interpreted as a Fourier transform; additionally, the expression automatically "encodes" the 2π-periodicity of the DTFT since obviously $X(e^{j(\omega+2k\pi)}) = X(e^{j\omega})$ with no need of knowing the actual expression for $X(e^{j\omega})$. On the other hand, the notation becomes very cumbersome when needing to apply shifts and scalings to the frequency variable, leading to hard-to-read expressions such as $X(e^{j\frac{\omega-2\pi m}{N}})$ which are very hard to read.

It's nevertheless important to become familiar with this alternate notation because it's very common. Here's a question to test your knowledge: for the DTFT $X(e^{j\omega})$ shown below, sketch the magnitude of $X(-e^{j\omega})$.



Solution: *The "tricky" thing is to remember that the free variable is ω:*

$$X(-e^{j\omega}) = X(e^{-j\pi}e^{j\omega}) = X(e^{j(\omega-\pi)}):$$



Exercise 6. DTFT of a modified sequence

Consider a causal sequence $x[n] \in \ell_2(\mathbb{Z})$ (i.e. an integer valued energy signal) with DTFT $X(\omega)$ where $x[n] = 0 for n < 0$. A new sequence $y[n]$ is defined as

$$y[n] = \begin{cases} 0 & \text{for } n < 0 \\ x[n] & \text{for } n \text{ even} \\ a^n & \text{for } n \text{ odd} \end{cases}$$

with $|a| < 1$. Derive the expression for $Y(\omega)$ in terms of $X(\omega)$.

[*Hint: the sequence* $(1 + e^{j\pi n})/2$ *may prove useful.*]

Solution:

The sequence $t[n] = (1+e^{j\pi n})/2$ *is equal to 1 for n even and to 0 for n odd. Let* $a[n] = a^n\, u[n]$*; we can write:*

$$\begin{aligned} y[n] &= x[n]\, t[n] + a[n]\, t[n-1] \\ &= \frac{1}{2}\left(x[n] + e^{j\pi n} x[n] + a[n] + a[n]e^{j\pi(n-1)}\right) \\ &= \frac{1}{2}\left(x[n] + e^{j\pi n} x[n] + a[n] - a[n]e^{j\pi n}\right). \end{aligned}$$

Using linearity and the shift property for the DTFT we have:

$$\begin{aligned} Y(\omega) &= \frac{1}{2}[X(\omega) + X(\omega-\pi) + A(\omega) - A(\omega-\pi)] \\ &= \frac{1}{2}\left[X(\omega) + X(\omega-\pi) + \frac{1}{1-ae^{-j\omega}} - \frac{1}{1-ae^{-j(\omega-\pi)}}\right] \\ &= \frac{1}{2}\left[X(\omega) + X(\omega-\pi) + \frac{1}{1-ae^{-j\omega}} - \frac{1}{1+ae^{-j\omega}}\right] \\ &= \frac{1}{2}[X(\omega) + X(\omega-\pi)] + \frac{ae^{-j\omega}}{1-a^2e^{-j2\omega}}. \end{aligned}$$

Exercise 7. DTFT, DFT, and numerical computations

Consider the following finite-support signal, where $M \in \mathbb{N}$:

$$x[n] = \begin{cases} 1 & 0 \le n < M, \\ 0 & \text{otherwise.} \end{cases}$$

In this exercise you will need to write some Python/NumPy code to compare the theoretical value of the DTFT $X(\omega)$ to its numerical approximation.

(a) Derive the closed-form expression for $X(\omega)$.

(b) Using Python, plot the values of $|X(\omega)|$ over a set of uniformly spaced frequency values between $-\pi$ and π; set $M = 20$ and use 10,000 frequency values.

(c) Now generate an N-point finite-length sequence $x_N[n]$ where $x_N[n] = x[n]$ for $n = 0, 1, \ldots, N-1$. Compute its DFT using NumPy's FFT package and plot the magnitude of the coefficients for $N = 31, 51, 101$. Align the DFT plots so that you can visually compare the DFT coefficients to the analytical values of the DTFT obtained in the previous step.

Solution: *The analytical expression for the DTFT of the signal is*

$$X(\omega) = \frac{\sin((M/2)\omega)}{\sin(\omega/2)} e^{-j\frac{M-1}{2}\omega}.$$

Here is a Python Notebook code snippet that provides the plots required by the exercise:

```
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (14,4)

M = 20

# compute the value of the theoretical DTFT over 10,000 points
w = np.linspace(-np.pi, np.pi, 10000)
X = np.sin((M / 2.0) * w) / np.sin(w / 2.0)

# now compute the three FFT-based approximations
for N in [31, 51, 101]:
    x = np.zeros(N)
    x[0:M] = 1
    X_a = np.fft.fftshift(np.fft.fft(x))
    w_a = np.fft.fftshift((2 * np.pi / N) * np.arange(N))
    w_a[:N // 2] -= 2 * np.pi
    plt.figure();
    plt.plot(w, np.abs(X), 'C2', lw=4, label='analytical');
    plt.plot(w_a, np.abs(X_a), 'C3', label=f'{N}-point DFT');
    plt.legend();
```