COM-202 - Signal Processing

Solutions for Homework 7

Exercise 1. A simple model for a bank

A simple discrete-time feedback loop with a single delay can be used to describe an elementary banking model where compound interest accrues yearly. Assume the following:

- you can only deposit (or withdraw) funds from your account on January 1st of each year; call the yearly deposit $x[n]$, with $n = 0$ when you open the account;
- on December 31st each year, the bank looks at your total assets and adds R percent of it to your account the next day; R is the interest rate;
- let's assume for simplicity that you never take any money out, so your balance is always positive.

With these assumptions we can model $y[n]$, the amount of money in your account in year n, via the recursive equation

$$y[n] = \alpha y[n-1] + x[n]$$

where $\alpha = 1 + R$.

Compute the closed-form expression for $y[n]$ when $x[n] = \beta u[n]$, that is, when you deposit β units of currency in your account every year.

Solution:

You can solve this problem in two ways:

- **Using the z-transform:** *From*

  $$y[n] = \alpha y[n-1] + \beta u[n]$$

  we have

  $$\begin{aligned} Y(z) &= \alpha z^{-1} Y(z) + \beta \frac{1}{1-z^{-1}} \\ &= \frac{\beta}{(1-\alpha z^{-1})(1-z^{-1})}. \end{aligned}$$

Using partial fraction expansion we can write

$$Y(z) = \frac{A}{(1-\alpha z^{-1})} + \frac{B}{(1-z^{-1})}$$

with

$$\begin{aligned} A + B &= \beta \\ A + \alpha B &= 0 \end{aligned}$$

This yields

$$Y(z) = \frac{\alpha\beta}{\alpha-1}\frac{1}{(1-\alpha z^{-1})} - \frac{\beta}{\alpha-1}\frac{1}{(1-z^{-1})}.$$

The first term on the right-hand side is the z-transform of a causal exponential sequence with base α whereas the second term is the z-transform of the unit step, with both terms multiplied by scalars. Therefore we have:

$$y[n] = \frac{\beta}{\alpha-1}\left(\alpha^{n+1} - 1\right) u[n].$$

- **By induction:** *With zero initial conditions, $y[n] = 0$ for $n < 0$; then*

$$\begin{aligned} y[0] &= \alpha y[-1] + \beta = \beta \\ y[1] &= \alpha y[0] + \beta = \alpha\beta + \beta \\ y[2] &= \alpha y[1] + \beta = \alpha^2\beta + \alpha\beta + \beta \\ &\dots \\ y[n] &= \beta(\alpha^n + \dots + 1) = \beta \sum_{k=0}^{n} \alpha^k \\ &= \beta \frac{\alpha^{n+1} - 1}{\alpha - 1}. \end{aligned}$$

Exercise 2. Allpass filters

An allpass filter is a filter whose magnitude response is a constant. Allpass filters are useful when we need to modify only the phase of an input signal.

(a) Consider a discrete-time LTI system with transfer function

$$H(z) = \frac{1-(1/a)z^{-1}}{1-az^{-1}}$$

where a is a real-valued constant. Show that the system is allpass, i.e., show that $|H(\omega)| = d$ for all values of the frequency ω.

(b) Determine the magnitude response of a filter with transfer function

$$G(z) = \frac{1-(1/a)z^{-1}}{1-az^{-1}} \; \frac{1-(1/b)z^{-1}}{1-bz^{-1}} \; \frac{1-(1/c)z^{-1}}{1-cz^{-1}}$$

where a, b, and c are real-valued constants.

Solution:

(a) The magnitude response of the filter is

$$\begin{aligned}
|H(\omega)| &= \left|\frac{1-(1/a)e^{-j\omega}}{1-ae^{-j\omega}}\right| \\
&= \left|\frac{(1/a)e^{-j\omega}\,(ae^{j\omega}-1)}{1-ae^{-j\omega}}\right| \\
&= \frac{1}{|a|}\,\left|e^{-j\omega}\right|\left|\frac{1-ae^{j\omega}}{1-ae^{-j\omega}}\right| \\
&= \frac{1}{|a|}
\end{aligned}$$

The last line follows from the fact that $(1-ae^{-j\omega})$ and $(1-ae^{-j\omega})$ are complex conjugate (since a is real) and, for any $s \in \mathbb{C}$, it is always $|s/s^*| = 1$. To see why, simply express s in polar coordinates:

$$\left|\frac{s}{s^*}\right| = \left|\frac{\rho e^{j\theta}}{\rho e^{-j\theta}}\right| = |e^{2j\theta}| = 1.$$

An alternative, if more laborious, way to solve the problem is

$$\begin{aligned}
|H(e^{j\omega})|^2 &= \left|\frac{1-(1/a)e^{-j\omega}}{1-ae^{-j\omega}}\right|^2 \\
&= \left|\frac{ae^{-j\omega}-1}{ae^{-j\omega}-a^2}\right|^2 \\
&= \frac{|a\cos(\omega)-1-ja\sin(\omega)|^2}{|a\cos(\omega)-a^2-ja\sin(\omega)|^2} \\
&= \frac{(a\cos(\omega)-1)^2+(a\sin(\omega))^2}{(a\cos(\omega)-a^2)^2+(a\sin(\omega))^2} \\
&= \frac{a^2+1-2a\cos(\omega)}{a^2+a^4-2a^3\cos(\omega)} = \frac{1}{|a|^2}.
\end{aligned}$$

(b) The filter $G(z)$ is the cascade of 3 allpass filters for which we know the magnitude response. We therefore have

$$|G(\omega)| = \frac{1}{|abc|}.$$

Exercise 3. Linear phase

Show that a causal, odd-length, antisymmetric FIR filter has a linear phase response. Recall that, for an FIR of length M, symmetry (or antisymmetry) is around the midpoint of the impulse response so that, in this case, the antisymmetry condition translates to $h[n] = -h[M-1-n]$ for $n = 0, \ldots, M-1$.

Solution:

If M is odd, we can write $M = 2C + 1$ with $C \in \mathbb{N}^+$ and the filter's midpoint coincides with the sample $h[C]$; the antisymmetry condition becomes

$$h[n] = -h[2C-n] \quad n = 0, \ldots, 2C.$$

For $n = C$, the expression yields $h[C] = -h[C]$, which implies that $h[C] = 0$ (i.e. the midpoint of an antisymmetric impulse response is necessarily equal to zero).

Consider now a noncausal shifted version of the original impulse response centered around the midpoint, $h_c[n] = h[n+C]$; the antisymmetry condition for this filter is simply $h_c[-n] = -h_c[n]$ and its transfer function is

$$\begin{aligned} H_c(z) &= \sum_{n=-C}^{C} h[n] z^{-n} \\ &= \sum_{n=-C}^{-1} h[n] z^{-n} + \sum_{n=1}^{C} h[n] z^{-n} \\ &= \sum_{n=1}^{C} h[n](z^{-n} - z^{n}). \end{aligned}$$

To find the frequency response, let's evaluate $H_c(z)$ on the unit circle:

$$\begin{aligned} H_c(\omega) &= \sum_{n=1}^{C} h[n](e^{-j\omega n} - e^{j\omega n}) \\ &= -2j \sum_{n=1}^{C} h[n] \sin(\omega n) \\ &= \left[2 \sum_{n=1}^{C} h[n] \sin(\omega n) \right] e^{-j\frac{\pi}{2}}. \end{aligned}$$

Since the term in brackets is real-valued, the phase response of $H_c(\omega)$ is constant and equal to $-\pi/2$.

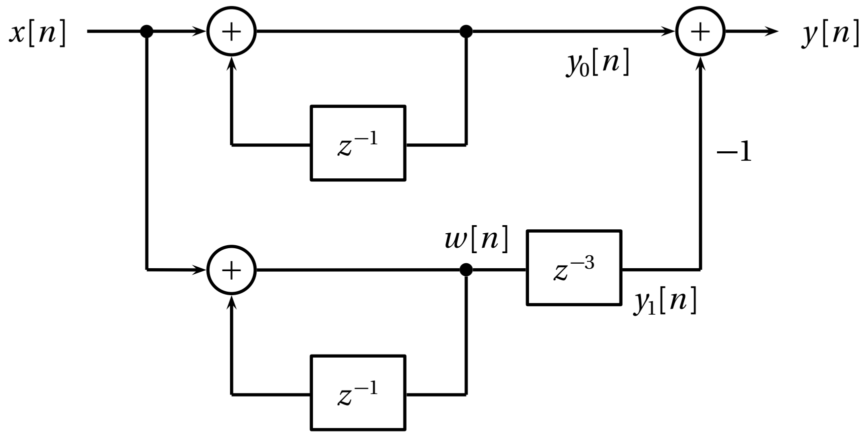
To find the phase response of the original filter, we simply shift the impulse response back to its causal formulation and obtain

$$H(\omega) = e^{-j\omega C} H_c(\omega) = \left[2 \sum_{n=1}^{C} h[n] \sin(\omega n) \right] e^{-j(\omega C + \frac{\pi}{2})}$$

whose phase is linear in ω.

Exercise 4. Discrete-time system diagram

Consider the causal system described by the following block diagram. Assume a causal input ($x[n] = 0$ for $n < 0$) and zero initial conditions.



(a) Find the three constant-coefficients difference equations that describe the relationship between the input $x[n]$ and $y_0[n]$, $y_1[n]$, $y[n]$.

(b) Find $H_0(z)$, $H_1(z)$ and $H(z)$, the transfer functions associated to the three CCDEs you found in the previous point.

(c) Is the whole system stable?

(d) Consider the input $x[n] = u[n]$, where, as usual, $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ for $n < 0$. How do $y_0[n], y_1[n]$ and $y[n]$ evolve over time? Sketch their values.

Solution:

(a) *(i) The upper branch is a integrator:*

$$y_0[n] = x[n] + y_0[n-1].$$

(ii) The lower branch is the cascade of an integrator followed by a delay. The easiest way to find the CCDE is by remembering that the order of the elements in a

cascade of filters does not matter. We can therefore swap the delay and the integrator to obtain the CCDE describing the output of an integrator when the input is delayed by 3

$$y_1[n] = x[n-3] + y_1[n-1].$$

Alternatively, call $w[n]$ *the signal before the delay so that* $y_1[n] = w[n-3]$*; this relationship is time-invariant and therefore it is also*

$$\begin{aligned} w[n] &= y_1[n+3] \\ w[n-1] &= y_1[n+2] \end{aligned}$$

Since $w[n]$ *is the output of an integrator, we have*

$$w[n] = x[n] + w[n-1]$$

which we can write as

$$y_1[n+3] = x[n] + y_1[n+2].$$

By shifting the above CCDE in time we obtain once again

$$y_1[n] = x[n-3] + y_1[n-1].$$

(iii) Since $y[n] = y_0[n] - y_1[n]$*, the global input-output CCDE is*

$$\begin{aligned} y[n] &= y_0[n] - y_1[n] \\ &= x[n] + y_0[n-1] - x[n-3] - y_1[n-1] \\ &= x[n] - x[n-3] + (y_0[n-1] - y_1[n-1]) \\ &= x[n] - x[n-3] + y[n-1]. \end{aligned}$$

(b) The transfer functions are easily derived:

$$\begin{aligned} H_0(z) &= \frac{1}{1-z^{-1}} \\ H_1(z) &= \frac{z^{-3}}{1-z^{-1}} \\ H(z) &= \frac{1-z^{-3}}{1-z^{-1}} = \frac{(1-z^{-1})(1+z^{-1}+z^{-2})}{1-z^{-1}} = 1 + z^{-1} + z^{-2}. \end{aligned}$$

(c) The system is BIBO stable since the overall transfer function has only zeros and therefore the system implements an FIR filter.

(d) The first subsystem is a simple integrator: at each step the previous output is summed to the current input. Since the input is a step sequence, the output for $n \geq 0$ *will be:*

$$\begin{aligned} y[0] &= x[0] = 1 \\ y[1] &= y[0] + x[1] = 2 \\ y[2] &= y[1] + x[2] = 3 \\ &\dots \\ y[n] &= n + 1 \end{aligned}$$

or, concisely,

$$y_0[n] = (n+1)\,u[n].$$

The lower subsystem is identical to the first, except it is followed by a delay by three; therefore

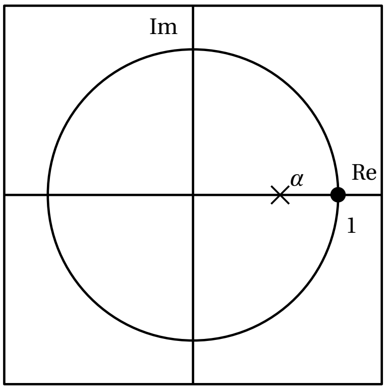
$$y_1[n] = y_0[n-3] = (n-2)u[n-3].$$

Finally, since the overall transfer function is FIR, the system's ouput is simply $y[n] = u[n] + u[n-1] + u[n-2]$; this is the superposition of three unit step sequences delayed by one each; since $y[0] = 1$, $y[1] = 2$, and $y[n] = 3$ for $n \geq 2$, we can write

$$y[n] = 3 - 2\delta[n] - \delta[n-1].$$

Exercise 5. Impulse response from poles and zeros

Compute the impulse response of the causal filter with the following pole-zero plot:



Solution:

The system has a pole in $z = \alpha$ and a zero in $z = 1$. We can write the transfer function of the system as

$$H(z) = \frac{1 - z^{-1}}{1 - \alpha z^{-1}} = \frac{1}{1 - \alpha z^{-1}} - z^{-1}\frac{1}{1 - \alpha z^{-1}}.$$

A first-order section with a pole in $z = \alpha$ has a transfer function $G(z) = 1/(1 - \alpha z^{-1})$ and impulse response $g[n] = \alpha^n\, u[n]$. Therefore the impulse response of the above system is

$$h[n] = g[n] - g[n-1] = \alpha^n\, u[n] - \alpha^{n-1}\, u[n-1] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \alpha^{n-1}(\alpha - 1) & n > 0 \end{cases}.$$

Exercise 6.

A causal LTI system is described by the following difference equation, where b is a real number.

$$y[n]+(b+1)^2 y[n-1]+(2b^3+b^2)y[n-2]=x[n].$$

(a) Find the transfer function $H(z)$ and the range of b such that the system is stable.

(b) Sketch a block diagram implementing this system.

Solution:

(a) Taking the z-Transform on both sides of the difference equation, we have

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1+(b+1)^2z^{-1}+(2b^3+b^2)z^{-2}}=\frac{1}{(1+b^2z^{-1})(1+(2b+1)z^{-1})},$$

whose two poles are at $z=-b^2$ and $z=-2b-1$.

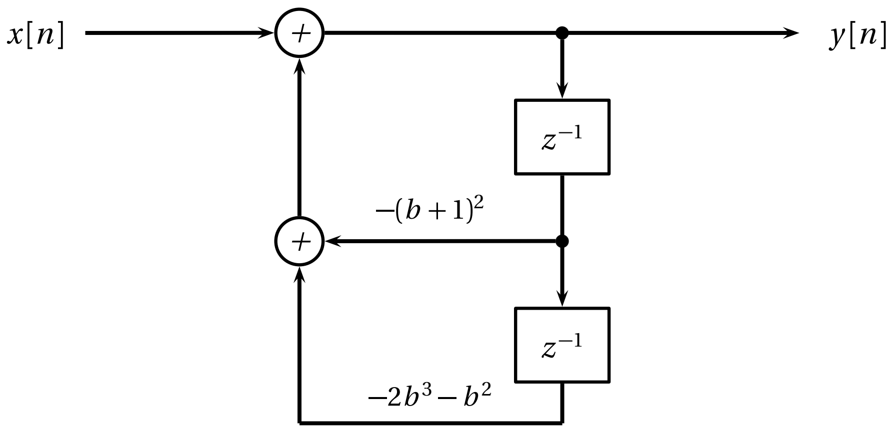
Since the system is causal, the ROC extends outwards from a circle passing through the pole with largest magnitude. For the system to be stable the ROC must contain the unit circle and for this to happen we must have

$$\max\{b^2,|2b+1|\}<1$$

that is,

$$-1<b<0.$$

(b) A straightforward implementation of this filter is via a standard second-order section where only the feedback branches have nonzero weights



Alternately, since we already have the transfer function in factored form and since the two poles are real-valued, we can implement the system as a cascade of two first-order feedback loops as shown in the following figure; the advantage of this implementation is that it uses the actual values of the poles and therefore any potential issues due to limited numerical precision are mitigated.