Discrete Fourier Transform (DFT)

used for finite-length signals, $\mathbf{x}, \mathbf{X} \in \mathbb{C}^N$, $n, k = 0, 1, 2, \ldots, N-1$

DFT basis vectors:	$w_k[n] = e^{-j\frac{2\pi}{N}nk}$	$\mathbf{w}_k \in \mathbb{C}^N$
DFT matrix:	$W[n,k] = e^{-j\frac{2\pi}{N}nk}$	$\mathbf{W} \in \mathbb{C}^{N\times N}$
analysis formula:	$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$	$\mathbf{X} = \mathbf{W}\mathbf{x}$
synthesis formula:	$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}$	$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$
energy conservation:	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	$\lVert\mathbf{x}\rVert^2 = \frac{1}{N}\lVert\mathbf{X}\rVert^2$
time reversal:	$x[-n \mod N] \overset{\text{DFT}}{\longleftrightarrow} X[-k \mod N]$	$\mathscr{R}\mathbf{x} \overset{\text{DFT}}{\longleftrightarrow} \mathscr{R}\mathbf{X}$
conjugation:	$x^*[n] \overset{\text{DFT}}{\longleftrightarrow} X^*[-k \mod N]$	$\mathbf{x}^* \overset{\text{DFT}}{\longleftrightarrow} \mathscr{R}\mathbf{X}^*$
time shift:	$x[(n-d) \mod N] \overset{\text{DFT}}{\longleftrightarrow} e^{-j\frac{2\pi}{N}dk} X[k]$	$\mathscr{S}^{-d}\mathbf{x} \overset{\text{DFT}}{\longleftrightarrow} \mathbf{w}_d^*\mathbf{X}$
frequency shift:	$e^{j\frac{2\pi}{N}nd} x[n] \overset{\text{DFT}}{\longleftrightarrow} X[(k-d) \mod N]$	$\mathbf{w}_d\mathbf{x} \overset{\text{DFT}}{\longleftrightarrow} \mathscr{S}^{-d}\mathbf{X}$

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notable DFT pairs:

$x[n] = \delta[n-d], \quad d \in \mathbb{Z}$	$X[k] = e^{-j\frac{2\pi}{N}dk}$
$x[n] = 1$	$X[k] = N\delta[k]$
$x[n] = e^{j\frac{2\pi}{N}dn}$	$X[k] = N\delta[k-d]$
$x[n] = \cos\left(\frac{2\pi}{N}dn + \phi\right)$	$X[k] = \frac{N}{2}\left(e^{j\phi}\delta[k-d] + e^{-j\phi}\delta[k-N+d]\right)$
$x[n] = \sin\left(\frac{2\pi}{N}dn + \phi\right)$	$X[k] = -j\frac{N}{2}\left(e^{j\phi}\delta[k-d] - e^{-j\phi}\delta[k-N+d]\right)$
$x[n] = \begin{cases} 1 & n < M \\ 0 & M \le n < N \end{cases}$	$X[k] = \frac{\sin((\pi/N)Mk)}{\sin((\pi/N)k)} e^{-j\frac{\pi}{N}(M-1)k}$

Discrete-Time Fourier Transform (DTFT)

used for infinite-length, finite-energy signals ($\mathbf{x} \in \ell_2(\mathbb{Z}), \mathbf{X} \in L_2([-\pi, \pi])$)

analysis formula:	$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$	$\mathbf{X} = \text{DTFT}\{\mathbf{x}\}$
synthesis formula:	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$	$\mathbf{x} = \text{IDTFT}\{\mathbf{X}\}$
energy conservation:	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert X(\omega)\rvert^2 d\omega$	$\lVert\mathbf{x}\rVert^2 = \lVert\mathbf{X}\rVert^2$
time reversal:	$x[-n] \xleftrightarrow{\text{DTFT}} X(-\omega)$	$\mathscr{R}\mathbf{x} \xleftrightarrow{\text{DTFT}} \mathscr{R}\mathbf{X}$
conjugation:	$x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\omega)$	$\mathbf{x}^* \xleftrightarrow{\text{DTFT}} \mathscr{R}\mathbf{X}^*$
time shift:	$x[n-d] \xleftrightarrow{\text{DTFT}} e^{-j\omega d} X(\omega)$	
frequency modulation:	$e^{j\varphi n} x[n] \xleftrightarrow{\text{DTFT}} X(\omega - \varphi)$	
convolution:	$(x * y)[n] \xleftrightarrow{\text{DTFT}} X(\omega) Y(\omega)$	$\mathbf{x} * \mathbf{y} \xleftrightarrow{\text{DFT}} \mathbf{XY}$
modulation:	$x[n] y[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) Y(\omega - \theta) d\theta$	$\mathbf{xy} \xleftrightarrow{\text{DFT}} \mathbf{X} * \mathbf{Y}$

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notable DTFT pairs:

$x[n] = \delta[n-d], \quad d \in \mathbb{Z}$	$X(\omega) = e^{-j\omega d}$
$x[n] = a^n u[n] \quad \lvert a\rvert < 1$	$X(\omega) = \frac{1}{1 - a e^{-j\omega}}$
$x[n] = \begin{cases} 1 & 0 \le n < M \\ 0 & \text{otherwise} \end{cases}$	$X(\omega) = \frac{\sin((M/2)\omega)}{\sin(\omega/2)} e^{-j\frac{M-1}{2}\omega}$
$x[n] = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right)$	$X(\omega) = \operatorname{rect}\left(\frac{\omega}{2\omega_c}\right)$
$x[n] = 1$	$X(\omega) = \tilde{\delta}(\omega)$
$x[n] = e^{j\omega_0 n}$	$X(\omega) = \tilde{\delta}(\omega - \omega_0)$
$x[n] = \cos(\omega_0 n + \phi)$	$X(\omega) = (1/2)[e^{j\phi} \tilde{\delta}(\omega - \omega_0) + e^{-j\phi} \tilde{\delta}(\omega + \omega_0)]$
$x[n] = \sin(\omega_0 n + \phi)$	$X(\omega) = (-j/2)[e^{j\phi} \tilde{\delta}(\omega - \omega_0) - e^{-j\phi} \tilde{\delta}(\omega + \omega_0)]$

$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ *(periodized Dirac delta)*, $\quad \operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$, $\quad \operatorname{rect}(t) = \begin{cases} 1 & \lvert t\rvert < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

Continuos-Time Fourier Transform (CTFT)

used for infinite-length, finite-energy continuous-time signals ($\mathbf{x}, \mathbf{X} \in L_2(\mathbb{R})$)

analysis formula:	$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$	$\mathbf{X} = \mathrm{CTFT}\{\mathbf{x}\}$
synthesis formula:	$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$	$\mathbf{x} = \mathrm{ICTFT}\{\mathbf{X}\}$
energy conservation:	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$	$\lVert\mathbf{x}\rVert^2 = \lVert\mathbf{X}\rVert^2$
scaling:	$x(at) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} \frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$	
time reversal:	$x(-t) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} X(-f)$	$\mathscr{R}\mathbf{x} \stackrel{\mathrm{CTFT}}{\longleftrightarrow} \mathscr{R}\mathbf{X}$
conjugation:	$x^*(t) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} X^*(-f)$	$\mathbf{x}^* \stackrel{\mathrm{CTFT}}{\longleftrightarrow} \mathscr{R}\mathbf{X}^*$
time shift:	$x(t-t_0) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} e^{-j2\pi f t_0} X(f)$	
frequency modulation:	$e^{j2\pi f_0 t} x(t) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} X(f-f_0)$	
convolution:	$(x * y)(t) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} X(f)Y(f)$	$\mathbf{x} * \mathbf{y} \stackrel{\mathrm{DFT}}{\longleftrightarrow} \mathbf{XY}$
modulation:	$x(t)y(t) \stackrel{\mathrm{CTFT}}{\longleftrightarrow} \int_{-\infty}^{\infty} X(s)Y(f-s) ds$	$\mathbf{xy} \stackrel{\mathrm{DFT}}{\longleftrightarrow} \mathbf{X} * \mathbf{Y}$

notable CTFT pairs:

$x(t) = e^{-at} u(t)$, $\mathrm{Re}\{a\} > 0$	$X(f) = \frac{1}{a + j2\pi f}$
$x(t) = \mathrm{rect}(t)$	$X(f) = \mathrm{sinc}(f)$
$x(t) = \mathrm{sinc}(t)$	$X(f) = \mathrm{rect}(f)$

(all the following deltas are Dirac deltas)

$x(t) = \delta(t)$	$X(f) = 1$
$x(t) = \delta(t - t_0)$	$X(f) = e^{-j2\pi f t_0}$
$x(t) = 1$	$X(f) = \delta(f)$
$x(t) = e^{j2\pi f_0 t}$	$X(f) = \delta(f - f_0)$
$x(t) = \cos(2\pi f_0 t)$	$X(f) = \frac{1}{2}\big(\delta(f-f_0) + \delta(f+f_0)\big)$
$x(t) = \sin(2\pi f_0 t)$	$X(f) = \frac{1}{2j}\big(\delta(f-f_0) - \delta(f+f_0)\big)$

Z-Transform

		ROC
definition:	$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$	R_x
linearity:	$a\,x[n] + b\,y[n] \overset{\mathscr{Z}}{\longleftrightarrow} a X(z) + b\,Y(z)$	$R_x \cap R_y$
time shift:	$x[n-d] \overset{\mathscr{Z}}{\longleftrightarrow} z^{-d} X(z)$	R_x
convolution:	$(x * y)[n] \overset{\mathscr{Z}}{\longleftrightarrow} X(z)Y(z)$	$R_x \cap R_y$
time reversal:	$x[-n] \overset{\mathscr{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right)$	$z \mid z^{-1} \in R_x$
conjugation:	$x^*[n] \overset{\mathscr{Z}}{\longleftrightarrow} X^*(z^*)$	$z \mid z^* \in R_x$

notable z-transform pairs:

$x[n] = \delta[n-d], \quad d \in \mathbb{Z}$	$X(z) = z^{-d}$	$\mathbb{C}$
$x[n] = \begin{cases} a^n & 0 \le n < M \\ 0 & \text{otherwise} \end{cases}$	$X(z) = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$	$\mathbb{C}$
$x[n] = a^n u[n]$	$X(z) = \frac{1}{1 - a z^{-1}}$	$\lvert z \rvert > \lvert a \rvert$
$x[n] = n a^n u[n]$	$X(z) = \frac{a z^{-1}}{(1 - a z^{-1})^2}$	$\lvert z \rvert > \lvert a \rvert$

Multirate

upsampling:	$x_U[n] = \begin{cases} x[k] & n = Nk \\ 0 & \text{otherwise} \end{cases}$	$X_U(z) = X(z^N)$	$X_U(\omega) = X(N\omega)$
downsampling:	$x_D[n] = x[Nn]$	$X_D(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{-j\frac{2\pi}{N}k} z^{\frac{1}{N}})$	$X_D(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{\omega - 2\pi k}{N}\right)$

interpolation: upsampling followed by lowpass filtering, $\omega_c = \pi/N$

decimation: lowpass filtering, $\omega_c = \pi/N$, followed by downsampling

Stochastic and Adaptive

correlation:

$$r_{xy}[k] = \sum_{n=-\infty}^{\infty} x^*[n]y[n+k]$$

$$r_{xy}[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^*[n]y[n+k]$$

$$r_{xy}[k] = \mathrm{E}\big[x^*[n]y[n+k]\big]$$

$$\mathbf{r}_{xy} = \mathscr{R}\mathbf{x}^* * \mathbf{y}$$

$$\mathbf{r}_x = \mathscr{R}\mathbf{x}^* * \mathbf{x}$$

Power Spectral Density (PSD): $P_x(\omega) = \mathrm{DTFT}\{r_x[k]\}$ $\qquad \mathbf{r}_x \overset{\text{DTFT}}{\longleftrightarrow} \mathbf{P}_x$

total power: $W_x = r_x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(\omega)d\omega$ $\qquad W_x = \|\mathbf{x}\|^2 = r_x[0]$

PSD of filtered process:	$\mathbf{y} = \mathbf{h} * \mathbf{x} \quad \Rightarrow \quad P_y(\omega) = \lvert H(\omega)\rvert^2 P_x(\omega)$
PSD of white noise:	$P_\eta(\omega) = \sigma_\eta^2$ (constant)
PSD of quantization error, R bps:	$P_e(\omega) = \Delta^2/12, \quad \Delta = (x_{\max} - x_{\min})/2^R$
SNR of quantized signal, R bps:	$\mathrm{SNR} = 6R$ dB