COM-202 - Signal Processing

Solutions for Homework 4

Exercise 1. DFT as matrix-vector multiplication

Compute the DFT of the $\mathbb{C}^4$ vector $\mathbf{x} = \begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix}^T$ as a matrix-vector multiplication.

Solution: *the DFT matrix for* $\mathbb{C}^4$ *is*

$$\mathbf{W_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

which can be seen by considering that (n,m)*–th entry of* $\mathbf{W_4}$ *as* $\mathbf{W_4}[n,m] = e^{-\frac{j2\pi nm}{4}}$. *Thus, we have:*

$$\mathbf{X} = \mathbf{W_4}\mathbf{x} = \begin{bmatrix} 0 \\ (2-2j) \\ 0 \\ (2+2j) \end{bmatrix}$$

Exercise 2. Signals with imaginary DFT

In this exercise we will consider sets of finite-length signals whose DFT coefficients are purely imaginary, that is, their real part is equal to zero.

(a) consider the set of length-3 signals

$$A = \{\mathbf{x} \in \mathbb{C}^3 \big| \text{Re}\{x_0\} = 0, x_2 = -x_1^*\}$$

Show that if $\mathbf{x} \in A$ and $\mathbf{X} = \text{DFT}\{\mathbf{x}\}$ then $\text{Re}\{X[k]\} = 0$ for $k = 0, 1, 2$.

(b) Describe the set of signals in $\mathbb{C}^4$ for which the DFT coefficients are all purely imaginary.

Solution: *In the solution we will use the following facts: if* $a, b \in \mathbb{C}$ *then*

$$\begin{aligned} a - a^* &= 2j\,\mathrm{Im}\{a\} \\ \mathrm{Re}\{ab\} &= \mathrm{Re}\{a\}\mathrm{Re}\{b\} - \mathrm{Im}\{a\}\mathrm{Im}\{b\} \end{aligned}$$

(a) *The Fourier matrix in* $\mathbb{C}^3$ *is*

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -w & -w^* \\ 1 & -w^* & -w \end{bmatrix}, \qquad w = \frac{1 + j\sqrt{3}}{2}$$

Given $\mathbf{x} \in A$ *we have*

$$\mathbf{X} = \mathbf{W}_3\mathbf{x} = \mathbf{W}_3 \begin{bmatrix} x_0 \\ x_1 \\ -x_1^* \end{bmatrix} = \begin{bmatrix} x_0 + x_1 - x_1^* \\ x_0 - w x_1 + (w x_1)^* \\ x_0 - w^* x_1 + w x_1^* \end{bmatrix}$$

Since $w x_1^* = (w^* x_1)^*$ *we have*

$$\begin{aligned} X[0] &= x_0 + 2j\mathrm{Im}\{x_1\} \\ X[1] &= x_0 - 2j\mathrm{Im}\{w x_1\} \\ X[2] &= x_0 + 2j\mathrm{Im}\{w x_1^*\} \end{aligned}$$

and all three values are purely imaginary since $\mathrm{Re}\{x_0\} = 0$ *by definition.*

(b) *If* $\mathbf{W}_4$ *the Fourier matrix for* $\mathbb{C}^4$*, we need to determine the set of vectors for which*

$$\mathrm{Re}\{\mathbf{W}_4\mathbf{x}\} = \mathrm{Re}\{\mathbf{W}_4\}\mathrm{Re}\{\mathbf{x}\} - \mathrm{Im}\{\mathbf{W}_4\}\mathrm{Im}\{\mathbf{x}\} = 0$$

Let's write $\mathbf{x} = \mathbf{a} + j\mathbf{b}$ *with* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$*; since*

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

we need to find the solutions to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0$$

This results in the following undetermined system of equations

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 0 \\ a_0 - a_2 + b_1 - b_3 = 0 \\ a_0 - a_1 + a_2 - a_3 = 0 \\ a_0 - a_2 - b_1 + b_3 = 0 = 0 \end{cases}$$

from which we have $a_0 = a_2 = 0$, $a_1 = -a_3$ and $b_1 = b_3$. The set of vector is therefore

$$\{\mathbf{x} \in \mathbb{C}^4 \big| \mathrm{Re}\{x_0\} = \mathrm{Re}\{x_2\} = 0, x_3 = -x_1^*\}$$

or, alternatively, each vector can be expressed as

$$\mathbf{x} = \begin{bmatrix} ja \\ b + jc \\ jd \\ -b + jc \end{bmatrix} \quad a, b, c, d \in \mathbb{R}$$

Exercise 3. DFT and time reversal

Show that $\mathrm{DFT}\{\mathscr{R}\mathbf{x}\} = \mathscr{R}\{\mathrm{DFT}\{\mathbf{x}\}\}$.

Here are some pointers that you may find useful in your derivations:

- if $\mathbf{x} \in \mathbb{C}^N$ and $\mathbf{x}_r = \mathscr{R}\mathbf{x}$, then $x_r[n] = x[-n \mod N]$
- all harmonic complex exponential in $\mathbb{C}^N$ are N-periodic and so they contain an "implicit" modulo operation:

  $$e^{-j\frac{2\pi}{N}nk} = e^{-j\frac{2\pi}{N}(nk \mod N)}$$

- in modular arithmetic the following equivalences always hold

  $$n \mod N = (n + kN) \mod N \quad \forall k \in \mathbb{Z}$$

  $$(n \mod N) \mod N = n \mod N$$

- in modular arithmetic the distributive property for multiplication is

  $$(n_1 n_2) \mod N = [(n_1 \mod N)(n_2 \mod N)] \mod N$$

Solution: *Let* $\mathbf{x}_r = \mathscr{R}\mathbf{x}$*, so that* $x_r[n] = x[-n \mod N]$*, and let* $\mathbf{X}_r = DFT\{\mathbf{x}_r\}$*. Then*

$$\begin{aligned}
X_r[k] &= \sum_{n=0}^{N-1} x_r[n]\, e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N-1} x[-n \mod N]\, e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N-1} x[N-n \mod N]\, e^{-j\frac{2\pi}{N}nk} \qquad (1)\\
&= \sum_{n=0}^{N-1} x[N-n]\, e^{-j\frac{2\pi}{N}nk} \qquad (2)\\
&= \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi}{N}(N-m)k} \qquad (3)\\
&= \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi}{N}(-mk)} \qquad (4)
\end{aligned}$$

where in (1) we applied the modulo equivalence property, in (2) we removed the modulo since $0 \le N-n < N$ *over the range of the summation index* n*, in (3) we operated the change of variable* $m = N-n$*, and in (4) we used the* N*-periodicity of the complex exponential term. Let's now focus on the latter; we have*

$$\begin{aligned}
e^{-j\frac{2\pi}{N}(-mk)} &= e^{-j\frac{2\pi}{N}((m(-k)) \mod N)} \qquad (5)\\
&= e^{-j\frac{2\pi}{N}[(m \mod N)(-k \mod N) \mod N]} \qquad (6)\\
&= e^{-j\frac{2\pi}{N}m(-k \mod N)} \qquad (7)
\end{aligned}$$

where in (5) we made the modulo operation in the complex exponential explicit, in (6) we used the modular distributivity for multiplication and in (7) we removed the modulo for the summation index m *since its value is between zero and* $N-1$*, and we removed the last modulo operation since it is already implicit in the complex exponential. In the end we have*

$$X_r[k] = \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi}{N}m(-k \mod N)}$$

and since

$$X[k] = \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi}{N}mk}$$

we can see that $X_r[k] = X[-k \mod N]$*, that is,* $DFT\{\mathscr{R}\mathbf{x}\} = \mathscr{R}\mathbf{X} = \mathscr{R}\{DFT\{\mathbf{x}\}\}$.
