COM-202 - Signal Processing

Solutions for Homework 3

Exercise 1. DFT of elementary signals

Derive the formula for the DFT of the length-N signal

$$x[n] = \cos((2\pi/N)Ln + \phi).$$

Solution: *We have:*

$$\begin{aligned} x[n] &= \frac{e^{j\phi}}{2} e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2} e^{-j(2\pi/N)Ln} \\ &= \frac{e^{j\phi}}{2} e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2} e^{-j(2\pi/N)Ln} e^{j(2\pi/N)Nn} \\ &= \frac{e^{j\phi}}{2} e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2} e^{j(2\pi/N)(N-L)n}. \end{aligned}$$

Therefore, we can write in vector notation:

$$\mathbf{x} = \frac{e^{j\phi}}{2}\mathbf{w}^{(L)} + \frac{e^{-j\phi}}{2}\mathbf{w}^{(N-L)},$$

and the result follows from the linearity of the expansion formula:

$$\begin{aligned} X[k] &= \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \\ &= \left\langle \mathbf{w}^{(k)}, \frac{e^{j\phi}}{2}\mathbf{w}^{(L)} + \frac{e^{-j\phi}}{2}\mathbf{w}^{(N-L)} \right\rangle = \frac{e^{j\phi}}{2}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(L)} \rangle + \frac{e^{-j\phi}}{2}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(N-L)} \rangle \end{aligned}$$

Now, if $L \neq N - L$, we have:

$$X[k] = \begin{cases} \frac{N}{2} e^{j\phi} & \text{if } k = L \\ \frac{N}{2} e^{-j\phi} & \text{if } k = N - L \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise, if $L = N - L$, we have:

$$X[k] = \begin{cases} \frac{N}{2} e^{j\phi} + \frac{N}{2} e^{-j\phi} & \text{if } k = L = N - L \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2. DFT by inspection

The signal **x** shown at the top of the next page is the sum of the three 64-periodic signals plotted in the bottom panels of the figure, i.e.

$$\mathbf{x} = \mathbf{a} + \mathbf{b} + \mathbf{c}.$$

By looking at the plots, determine the expressions for **a**, **b**, **c** and use the result to compute the DFT coefficients $X[k]$, $k = 0, 1, \ldots, 63$.

Solution:

By simple visual inspection we can determine that

$$\begin{aligned} a[n] &= 3\cos(3(2\pi/64)n) \\ b[n] &= \sin(7(2\pi/64)n) = -\cos(7(2\pi/64)n + \pi/2). \\ c[n] &= 2 \end{aligned}$$

The DFT coefficients are $X[k] = A[k] + B[k] + C[k]$, with

$$\begin{aligned} A[k] &= (3N/2)\delta[k-3] + (3N/2)\delta[k-61] \\ B[k] &= -(jN/2)\delta[k-7] + (jN/2)\delta[k-57] \\ C[k] &= 2N\delta[k] \end{aligned}$$

with period $N = 64$. Note that all DFT functions are defined for $k \in \{0, 1, ..., N-1\}$ and this is sufficient because they are periodic with N. Thus, in the end, we have:

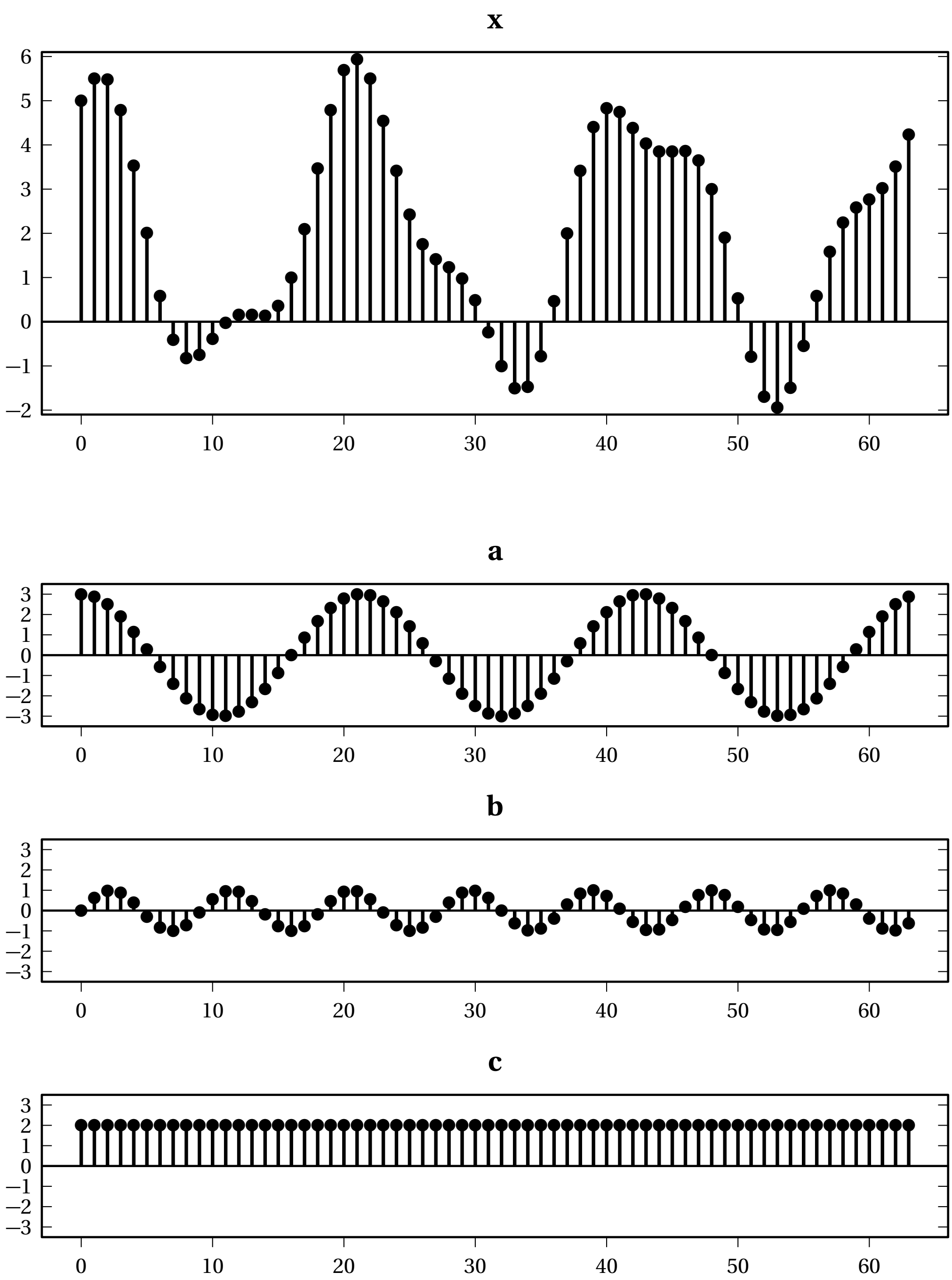
$$\begin{aligned} X[0] &= 128 \\ X[3] &= 96 \\ X[7] &= -32j \\ X[57] &= 32j \\ X[61] &= 96 \end{aligned}$$

and $X[k] = 0$ for all the other values of $k\ in \{0, 1, ..., 63\}$.

Exercise 3. DFT of a short signal

Compute the 8 DFT coefficients of the signal $\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}^T$.

Note: **x** is a periodic signal with period 8 and its values in modulo 8 are given above.


x
6
5
4
3
2
1
0
−1
−2
0 10 20 30 40 50 60
a
3
2
1
0
−1
−2
−3
0 10 20 30 40 50 60
b
3
2
1
0
−1
−2
−3
0 10 20 30 40 50 60
c
3
2
1
0
−1
−2
−3
0 10 20 30 40 50 60

Solution:

There are many ways to solve this problem. A simple method is to observe that we can write $\mathbf{x} = \mathbf{a} + \mathbf{b}$ *with*

$$\mathbf{a} = \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}^T$$
$$\mathbf{b} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}^T$$

which, in signal notation, corresponds to

$$a[n] = \sin((2\pi/8)2n - \pi/2)$$
$$b[n] = \cos((2\pi/8)2n + \pi/2)$$

Using the result from the previous exercise we have

$$A[k] = \begin{cases} -4je^{j\pi/2} = -4 & k = 2 \\ 4je^{-j\pi/2} = -4 & k = 6 \end{cases}$$

and

$$B[k] = \begin{cases} 4e^{j\pi/2} = 4j & k = 2 \\ 4e^{-j\pi/2} = -4j & k = 6 \end{cases}$$

so that

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 4(-1+j) & 0 & 0 & 0 & 4(-1-j) & 0 \end{bmatrix}^T$$

Note that $\mathbf{A}, \mathbf{B}$, *and* $\mathbf{X}$ *are periodic with period 8.*

Exercise 4. Structure of the DFT formulas

The DFT and IDFT formulas are similar, but not identical. Given a length-N signal $\mathbf{x}$, write an expression for the signal $\mathbf{y}$ obtained by applying the DFT twice in a row to $\mathbf{x}$:

$$\mathbf{y} = \text{DFT}\{\text{DFT}\{\mathbf{x}\}\}$$

Solution:

Let $\mathbf{s}[k] = DFT\{\mathbf{x}[i]\}$, *and* $\mathbf{y}[n] = DFT\{\mathbf{s}[k]\}$. *We have:*

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} s[k] e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{k=0}^{N-1} \left(\sum_{i=0}^{N-1} x[i] e^{-j\frac{2\pi}{N}ik} \right) e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{i=0}^{N-1} x[i] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k}. \end{aligned}$$

Because of the orthogonality of the roots of unity, we have

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k} = \begin{cases} N & \textit{when } i+n \textit{ is a multiple of } N \\ 0 & \textit{otherwise} \end{cases} = N\delta[(i+n) \mod N]$$

so that

$$\begin{aligned} y[n] &= \sum_{i=0}^{N-1} x[i] N\delta[(i+n) \mod N] \\ &= \begin{cases} N\,x[0] & \textit{for } n=0 \\ N\,x[N-n] & \textit{otherwise.} \end{cases} \end{aligned}$$

where $\mathbf{x}$ *and* $\mathbf{y}$ *are periodic signals with period* N*. Also, we can write* $\mathbf{y}$ *in compact form using the time-reversal operator:*

$$\mathbf{y} = N\mathscr{R}\mathbf{x}$$

where $(\mathscr{R}\mathbf{x})[n] = x[-n]$.

Exercise 5. DFT of repeated signals

Take $\mathbf{x} \in \mathbb{C}^N$ and its DFT $\mathbf{X}$. Now build a signal of length $2N$ by duplicating each element of $\mathbf{x}$:

$$\mathbf{y} = \begin{bmatrix} x[0] & x[0] & x[1] & x[1] & x[2] & x[2] & \dots & x[N-1] & x[N-1] \end{bmatrix}^T$$

Determine the $2N$-point DFT $\mathbf{Y}$ in terms of the N DFT coefficients in $\mathbf{X}$.

Note: $\mathbf{x}$ and $\mathbf{X}$ are periodic signals with period N. $\mathbf{y}$ and $\mathbf{Y}$ are periodic signals with period $2N$. Our goal is to find the values of $\mathbf{Y}[k]$ for $k \in \{0,1,...,2N-1\}$ in terms of the values of $\mathbf{X}[k]$ for $k \in \{0,1,...,N-1\}$.

Solution:

Consider the auxiliary signal

$$\mathbf{s} = \begin{bmatrix} x[0] & 0 & x[1] & 0 & x[2] & 0 & \dots & x[N-1] & 0 \end{bmatrix}^T$$

which is periodic with period $2N$*. Then, we can express* $\mathbf{y}$ *as*

$$\mathbf{y} = \mathbf{s} + \mathscr{S}^{-1}\mathbf{s}$$

where time shifting operator is defined as $(\mathscr{S}^{-1}\mathbf{s})[n] = s[n-1]$.

Then, the $2N$ DFT coefficients of $\mathbf{s}$ are:

$$\begin{aligned} S[k] &= \sum_{n=0}^{2N-1} s[n] e^{-j\frac{2\pi}{2N}nk} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}2nk} \\ &= X[k] = X[k \mod N] \end{aligned}$$

In the last line, the N-periodicity of a lenght-N DFT guarantees that $X[k]$ is well defined for all k.

A shift in time corresponds to multiplication by a linear phase term in frequency, and so

$$Y[k] = (1 + e^{-j\frac{2\pi}{2N}k}) X[k \mod N]$$

for all $k \in \{0, 1, \ldots, 2N-1\}$.

Exercise 6. DFT in matrix form

Consider the DFT expressed as a as a matrix/vector multiplication and call $\mathbf{W}$ the $N \times N$ DFT matrix. Is $\mathbf{W}$ Hermitian-symmetric for all values of N?

Solution:

Recall the DFT (analysis) formula:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$$

We can define an $N \times N$ square matrix $\boldsymbol{W}$ by stacking the conjugates of the basis vectors $\{\boldsymbol{w}^{(k)}\}_{k=0,\ldots,N-1}$:

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{w}^{*(0)} \\ \boldsymbol{w}^{*(1)} \\ \boldsymbol{w}^{*(2)} \\ \ldots \\ \boldsymbol{w}^{*(N-1)} \end{bmatrix}$$

and get the analysis formula in matrix - vector multiplication form:

$$\boldsymbol{X} = \boldsymbol{W}\boldsymbol{x}.$$

Then, knowing that:

$$\boldsymbol{W}\boldsymbol{W}^H = \boldsymbol{W}^H \boldsymbol{W} = N\boldsymbol{I}$$

we obtain the synthesis formula in matrix-vector multiplication form:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}.$$

A matrix $\mathbf{A}$ *is hermitian when*

$$\mathbf{A} = \mathbf{A}^H$$

Therefore, for $\mathbf{W}$ *to be hermitian, we would need:*

$$\mathbf{W}_{nk} = \mathbf{W}^*_{kn}$$

for all $n, k \in \{0, 1, \dots N-1\}$. *This translates to having:*

$$e^{-j\frac{2\pi}{N}nk} = e^{j\frac{2\pi}{N}kn}.$$

for all $n, k \in \{0, 1, \dots N-1\}$, *which is generally not the case.*

Consider, for example, the case when $n = k = 1$. *We would need to have:*

$$e^{-j\frac{2\pi}{N}} = e^{j\frac{2\pi}{N}}$$

which is equivalent to:

$$e^{j\frac{4\pi}{N}} = 1.$$

Clearly, this can only happen when $N = 1$ *or* $N = 2$. *Moreover, for these values of* N, $e^{-j\frac{2\pi}{N}nk} = e^{j\frac{2\pi}{N}kn}$ *holds for all* $n, k \in \{0, 1, \dots N-1\}$.

Exercise 7. Implementing the DFT

You have been asked to implement a five-point DFT on a microprocessor that can only perform real-valued additions, subtractions and multiplications (in other words, there is no mathematical library with which to compute trigonometric functions, nor native support for complex numbers). In your code, you can store the values of just the two following numerical constants:

$$C = \cos(2\pi/5) \approx 0.309$$
$$S = \sin(2\pi/5) \approx 0.951$$

Write an algorithm that, for a real-valued input data vector $[x_0, x_1, x_2, x_3, x_4]$, computes the real and imaginary parts of its 5-point DFT using only additions, multiplications and the constants C, S. You can use any notation you prefer but try to be as clear as possible in your derivation.

Solution:

Set $W = e^{-j2\pi/5}$; the DFT matrix for a 5-point DFT is

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W^6 & W^8 \\ 1 & W^3 & W^6 & W^9 & W^{12} \\ 1 & W^4 & W^8 & W^{12} & W^{16} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W & W^3 \\ 1 & W^3 & W & W^4 & W^2 \\ 1 & W^4 & W^3 & W^2 & W \end{bmatrix}$$

where we have exploited the fact that $W^n = W^{(n \mod 5)}$. The elements of the matrix that we need to compute are W to W^4. We have:

$$\begin{aligned} W &= C - jS \\ W^2 &= (C - jS)^2 = (C^2 - S^2) - 2jCS \\ W^3 &= e^{-j6\pi/5} = e^{j4\pi/5} = (W^2)^* = (C^2 - S^2) + 2jCS \\ W^4 &= e^{-j8\pi/5} = e^{j2\pi/5} = W^* = C + jS \end{aligned}$$

With this, given a real-valued input vector $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, we have

$$\begin{aligned} X_0 &= x_0 + x_1 + x_2 + x_3 + x_4 \\ X_1 &= x_0 + (x_2 + x_3)(C^2 - S^2) + (x_1 + x_4)C - j[(x_1 - x_4)S + 2(x_2 - x_3)CS] \\ X_2 &= x_0 + (x_1 + x_4)(C^2 - S^2) + (x_2 + x_3)C - j[(x_3 - x_2)S + 2(x_1 - x_4)CS] \\ X_3 &= X_2^* \\ X_4 &= X_1^* \end{aligned}$$

where we used the fact that $\mathbf{x}$ is a real-valued signal in the calculation of the last 2 lines.