COM-202 - Signal Processing

Homework 2

24 February 2025, Monday

Please submit your answer to Exercise 6 by 6 March 2025, Thursday, 23:59

Exercise 1. Energy of complex-valued signals

Compute the energy of the signal defined as

$$x[n] = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^n + j\left(\frac{1}{\sqrt{3}}\right)^n & n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n.$$

From the formula for the geometric sum, valid for $|a| < 1$, we have

$$\sum_{n=1}^{\infty} (1/a)^n = -1 + \sum_{n=0}^{\infty} (1/a)^n = \frac{1}{1 - 1/a} - 1 = \frac{1}{a - 1}$$

so that

$$E_x = \frac{1}{2-1} + \frac{1}{3-1} = \frac{3}{2}$$

Exercise 2. Operators and linearity

A discrete-time signal *operator* is a transformation acting on the entire signal:

$$\mathbf{y} = \mathscr{F}\mathbf{x}$$

A *linear* operator has the following properties (where α is a complex-valued scalar):

$$\mathscr{F}(\alpha\mathbf{x}) = \alpha\mathscr{F}\mathbf{x}$$
$$\mathscr{F}(\mathbf{x} + \mathbf{y}) = \mathscr{F}\mathbf{x} + \mathscr{F}\mathbf{y}$$

(a) Show that the time-shift operator for infinite-length signals, defined by $(\mathscr{S}\mathbf{x})[n] = x[n+1]$, is a linear operator.

(b) Show that the squaring operator, defined by $(\mathscr{Q}\mathbf{x})[n] = (x[n])^2$ is *not* linear.

Solution:

(a) let $\mathbf{p} = \alpha\mathbf{x}$. *Then*

$$(\mathscr{S}(\alpha\mathbf{x}))[n] = (\mathscr{S}\mathbf{p})[n] = p[n+1] = \alpha x[n+1] = \alpha x[n+1] = \alpha(\mathscr{S}\mathbf{x})[n]$$

Similarly, let $\mathbf{p} = \mathbf{x}+\mathbf{y}$, *and*

$$(\mathscr{S}(\mathbf{x}+\mathbf{y}))[n] = (\mathscr{S}\mathbf{p})[n] = p[n+1] = x[n+1] + y[n+1] = (\mathscr{S}\mathbf{x})[n] + (\mathscr{S}\mathbf{y})[n]$$

(b) $(\mathscr{Q}(\alpha\mathbf{x}))[n] = \alpha^2 x^2[n] = \alpha^2(\mathscr{Q}\mathbf{x})[n] \neq \alpha(\mathscr{Q}\mathbf{x})[n]$

Exercise 3. Operators in matrix notation

Linear operators acting on finite-length signals can always be expressed as a matrix-vector product. For example, consider the shift-by-one operator in $\mathbb{C}^N$, which is defined as a right *circular* shift:

$$(\mathscr{S}\mathbf{x})[n] = x[(n-1) \mod N].$$

In vector notation we can write

$$\mathscr{S}\mathbf{x} = \mathbf{S}\mathbf{x}$$

where the matrix $\mathbf{S}$ has the following form (using $N = 4$ for convenience):

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Express in matrix form the following operators in $\mathbb{C}^4$:

(a) the first-difference operator, defined by $(\mathscr{V}\mathbf{x})[n] = x[n] - x[(n-1) \mod N]$

(b) the averaging operator, defined by $(\mathscr{A}\mathbf{x})[n] = (x[n] + x[(n+1) \mod N])/2$

(c) the time reversal operator, defined by $(\mathscr{R}\mathbf{x})[n] = x[-n \mod N]$

Solution:

(a)

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

(b)

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}.$$

(c)

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Exercise 4. Elementary signal operators

Using elementary signal operators

(a) express $\boldsymbol{\delta}$ in terms of $\mathbf{u}$

(b) express $\mathbf{u}$ in terms of $\boldsymbol{\delta}$

(c) express the constant signal $\mathbf{1}$, which is equal to 1 for all $n \in \mathbb{Z}$, in terms of $\mathbf{u}$ and $\boldsymbol{\delta}$

(d) express the constant signal $\mathbf{1}$ terms of $\mathbf{u}$ only

(e) express $\mathbf{x}$, with $x[n] = \cos(2n)$, in terms of the signal $\mathbf{c}$, with $c[n] = \cos(n)$, and of any of the previous signals

As a reminder

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and

$$
\begin{aligned}
(\mathscr{S}^{-1}\mathbf{x})[n] &= x[n-1] \\
(\mathscr{V}\mathbf{x})[n] &= x[n]-x[n-1] \\
(\mathscr{R}\mathbf{x})[n] &= x[-n] \\
(\mathscr{Q}\mathbf{x})[n] &= x^2[n] \\
(\mathscr{E}\mathbf{x})[n] &= \sum_{k=-\infty}^{n} x[k]
\end{aligned}
$$

Solution:

(a) We have $u[n]-u[n-1]=\delta[n]$ *so*

$$\boldsymbol{\delta} = \mathscr{V}\mathbf{u}$$

(b) Integration undoes differentiation also in discrete time so $u[n]=\sum_{k=-\infty}^{n}\delta[k]$*:*

$$\mathbf{u} = \mathscr{E}\boldsymbol{\delta}$$

(c) using time-reversal

$$\mathbf{1} = \mathscr{R}\mathbf{u}+\mathbf{u}-\boldsymbol{\delta}$$

(d) using time-reversal and a shift

$$\mathbf{1} = \mathscr{R}\mathbf{u}+\mathscr{S}^{-1}\mathbf{u}$$

(e) since $\cos(2\alpha)=2\cos^2(\alpha)-1$ *we can write*

$$\mathbf{x} = 2\mathscr{Q}\mathbf{c}-\mathbf{1}$$

Exercise 5. Vector space

For each of the definitions given below, determine whether resulting space is a vector space and, if not, explain why:

(a) the set of vectors $\begin{bmatrix} x_0 & x_1 \end{bmatrix}^T \in \mathbb{R}^2$ for which $x_1 = 3x_0 + 1$ and with the usual definitions of scalar multiplication and vector addition

(b) the set of vectors $\begin{bmatrix} x_0 & x_1 \end{bmatrix}^T \in \mathbb{R}^2$ with the standard definition for vector addition and the following definition for scalar multiplication:

$$\alpha \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \alpha x_0 \\ x_1 \end{bmatrix}$$

Solution:

(a) ***not*** *a vector space. The space does not contain the zero vector, and it is not closed under either vector addition or scalar multiplication, eg.*

$$\alpha \begin{bmatrix} x_0 \\ 3x_0 + 1 \end{bmatrix} = \begin{bmatrix} \alpha x_0 \\ 3\alpha x_0 + \alpha \end{bmatrix}$$

but, except for $\alpha = 1$,

$$3\alpha x_0 + \alpha \neq 3(\alpha x_0) + 1$$

(b) ***not*** *a vector space. With that definition, scalar multiplication is no longer distributive:*

$$(\alpha + \beta)\mathbf{x} = \begin{bmatrix} (\alpha + \beta)x_0 \\ x_1 \end{bmatrix} \neq \alpha\mathbf{x} + \beta\mathbf{x} = \begin{bmatrix} \alpha x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} \beta x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} (\alpha + \beta)x_0 \\ 2x_1 \end{bmatrix}$$

Exercise 6. Bases & Python

Consider the vector space $V \subset \mathbb{C}^8$ spanned by the *rows* of $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) What is an easy way to prove that the rows in $\mathbf{H}$ do indeed form a basis?

(b) Use Python to verify point (a); obviously you can use `numpy`.

The basis described by $\mathbf{H}$ is called the *Haar basis* and it is one of the most celebrated cornerstones of a branch of signal processing called wavelet analysis (which we won't study in this class). To get a feeling for its properties, however, consider the following set of Python experiments:

(c) Verify that $\mathbf{HH}^H$ is a diagonal matrix, which means the vectors are orthogonal.

(d) Consider a constant signal $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ and compute its coefficients in the Haar basis.

(e) Consider an alternating signal $\mathbf{y} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$ and compute its coefficients in the Haar basis.

Solution:

(a) The rows of $\mathbf{H}$ *form a basis if they are linearly independent. A matrix has independent rows if it has full row-rank, i.e.* $\det(\mathbf{H}) \neq 0$.

```
import numpy as np

H = np.array([[1, -1,  0,  0,  0,  0,  0,  0],
              [0,  0,  1, -1,  0,  0,  0,  0],
              [0,  0,  0,  0,  1, -1,  0,  0],
              [0,  0,  0,  0,  0,  0,  1, -1],
              [1,  1, -1, -1,  0,  0,  0,  0],
              [0,  0,  0,  0,  1,  1, -1, -1],
              [1,  1,  1,  1, -1, -1, -1, -1],
              [1,  1,  1,  1,  1,  1,  1,  1],])

### Task (b)
print(f"rank(H) = {np.linalg.matrix_rank(H)}")
print(f"det(H) = {np.linalg.det(H)}")  # alternative; must by != 0

### Task (c)
A = H @ H.T
assert np.allclose(np.diag(np.diag(A)), A)

### Task (d)
x = np.ones((8,))
x_c = H @ x  # computes <H[k, :], x>, all similarity
             # measures between rows of `H` and `x`.
x_c /= np.linalg.norm(H, axis=-1)**2  # but rows of H are not
                                      # unit-norm: need to account
                                      # for this when computing
                                      # coefficients.
print(f"x_c = {x_c}")
print(f"x = H.T @ x_c = {H.T @ x_c}")

### Task (e), {same process as (d)}
y = np.r_[1., -1, 1, -1, 1, -1, 1, -1]
y_c = H @ y
y_c /= np.linalg.norm(H, axis=-1)**2
print(f"y_c = {y_c}")
```

```
print(f"y = H.T @ y_c = {H.T @ y_c}")
```

Exercise 7. Bases

Let $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$ be a basis for a subspace S. Prove that any vector $\mathbf{z} \in S$ is *uniquely* represented in this basis.

Hint: remember that the vectors in a basis are linearly independent and use this to prove the thesis by contradiction.

Solution: *Suppose by contradiction that the vector* $\mathbf{z} \in S$ *admits two distinct representations in the basis* $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$. *In other words, suppose that there exist two set of scalars* $\alpha_0,\ldots,\alpha_{N-1}$ *and* $\beta_0,\ldots,\beta_{N-1}$, *with* $\alpha_i \neq \beta_i$ *for all* i, *such that*

$$\mathbf{z} = \sum_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)}$$

and

$$\mathbf{z} = \sum_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}.$$

In this case we can write

$$\sum_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)} = \sum_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}$$

or, equivalently,

$$\sum_{k=0}^{N-1} (\alpha_k - \beta_k) \mathbf{x}^{(k)} = 0.$$

The above expression is a linear combination of basis vectors that and equals the zero vector. Because $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$ *are linearly independent, the only set of coefficients that satisfies the above equation is a set of null coefficients so that it must be* $\alpha_i \neq \beta_i$ *for all* i, *in contradiction with the hypothesis.*
