COM-202 - Signal Processing

Solutions for Homework 8

Exercise 1. IIR Filter properties

Consider a causal, stable IIR filter with impulse response $g[n]$ and transfer function $G(z)$. Which of the following statements are always true?

(a) The DTFT of $g[n]$ exists.

(b) The ROC of $G(z)$ includes the curve $|z| = 0.5$.

(c) The inverse filter $H(z) = 1/G(z)$ is FIR.

(d) The inverse filter $H(z) = 1/G(z)$ is stable.

(e) The system $D(z) = (1-3z^{-1})G(z)$ is stable.

(f) The filter with transfer function $C(z) = G(z)G(z)$ is stable.

Solution:

- *True. The DTFT of $g[n]$ is $G(\omega)$, that is, it is the transfer function $G(z)$ computed over the unit circle. Since the filter is stable, the ROC of $G(z)$ includes the unit circle and therefore the DTFT exists.*
- *False. Since the filter is stable and causal, we know that the ROC includes the unit circle and all circles of radius greater than one; however, a stable causal filter may still have a pole with magnitude greater than 0.5 and in this case the ROC will not include a circle with radius 0.5.*
- *False: since the zeros of $G(z)$ become the poles of the inverse filter, the inverse filter will be FIR only if $G(z)$ has no zeros, which is not true in general for a stable IIR filter.*
- *False. since the zeros of $G(z)$ become the poles of the inverse filter, the inverse filter will be stable only if all the zeros of $G(z)$ are inside the unit circle; however, all we know about $G(z)$ is that it is a stable IIR filter, and so the zeros of $G(z)$ can be anywhere without affecting its stability.*
- *True. $D(z)$ is the cascade of two stable filters, $G(z)$ and a stable FIR filter with transfer function $1-3z^{-1}$.*

- *True. The poles of $C(z)$ are in the same positions as the poles of $G(z)$ and only increase their multiplicity (i.e. if $G(z)$ has a single pole in z_0, then $C(z)$ has a double pole in z_0). If the poles of $G(z)$ are inside the unit circle, so are the poles of $C(z)$.*

Exercise 2. FIR Filter properties

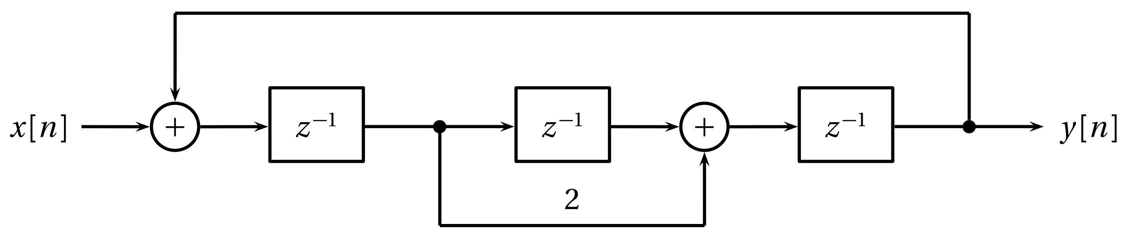
Answer the questions in the previous exercise once again, but this time assume that the filter $G(z)$ is FIR.

Solution:

- *True. The impulse response $g[n]$ has only a finite number of nonzero elements and the DTFT of a finite-support sequence always exists.*
- *True. The ROC for FIR filters includes the entire complex plane, as a FIR transfer function has no pole.*
- *False: since the zeros of $G(z)$ become the poles of the inverse filter, the inverse filter will actually always be IIR.*
- *False. An FIR can have zeros anywhere and so its inverse filter will not be stable in general.*
- *True. $D(z)$ is still FIR.*
- *True. $C(z)$ is still FIR.*

Exercise 3. Block diagram analysis

Consider the causal system implemented by the following block diagram:



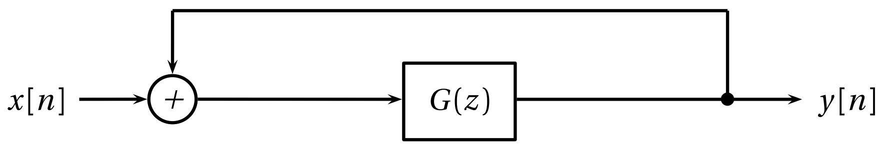
(a) Compute the system's transfer function $H(z)$.

(b) Plot the system's poles and zeros on the complex plane.

(c) Determine if the system is stable.

Solution:

(a) *First, notice that we can simplify the block diagram as*



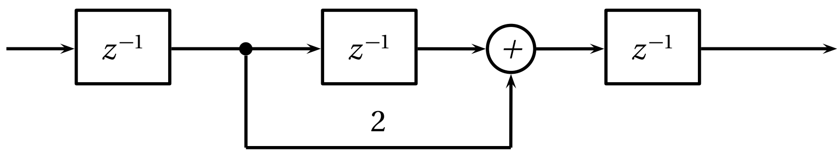
where $G(z)$ *represents the subsystem inside the feedback loop. The input-output relation in the* z*-domain can be written as*

$$Y(z) = G(z)[X(z) + Y(z)]$$

yielding the transfer function

$$H(z) = \frac{G(z)}{1 - G(z)}.$$

The subsystem $G(z)$ *is described by the block diagram:*



Since there are no feedback paths, the subsystem is an FIR filter and, from simple inspection, we can write

$$G(z) = 2z^{-2} + z^{-3}.$$

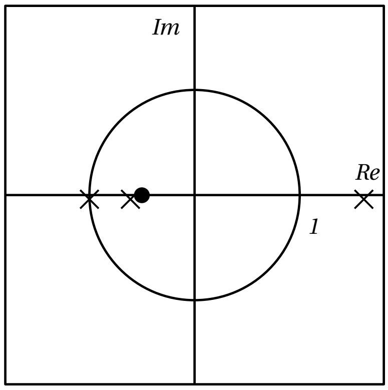
Plugging back this value in $H(z)$ *we have*

$$H(z) = \frac{G(z)}{1 - G(z)} = \frac{2z^{-2} + z^{-3}}{1 - 2z^{-2} - z^{-3}}.$$

(b) *To find the poles, we need to factor the denominator; it is easy to see that* $z = -1$ *is a root and so*

$$H(z) = 2z^{-2} \frac{1 + (1/2)z^{-1}}{(1 + z^{-1})(1 - z^{-1} - z^{-2})}.$$

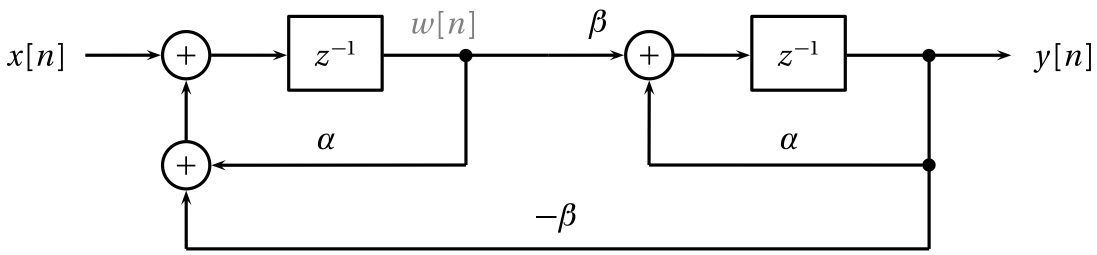
Finally, the transfer function has a zero in $z = -1/2$ *and poles in* -1 *and* $(1 \pm \sqrt{5})/2$*:*



(c) Since one of the poles is outside the unit circle, the system is not stable.

Exercise 4. Block diagram 1

Consider the causal system described by the following block diagram:



(a) Compute its transfer function $H(z) = Y(z)/X(z)$.

(b) Assume now that

$$\alpha = r\cos\theta$$
$$\beta = r\sin\theta$$

for $0 < r < 1$ and $0 < \theta < \pi/2$. Describe the type of filter implemented by the block diagram for this choice of coefficients and sketch its pole-zero plot.

Solution:

(a) Consider the auxiliary signal $w[n]$ coming out of the first delay block. Using z-transforms we can write

$$W(z) = z^{-1}(X(z) + \alpha W(z) - \beta Y(z))$$
$$Y(z) = z^{-1}(\beta W(z) + \alpha Y(z))$$

From this

$$W(z) = \frac{z^{-1}}{1-\alpha z^{-1}}\big(X(z) - \beta Y(z)\big)$$

and, substituting back,

$$Y(z) = \frac{\beta z^{-2}}{1-\alpha z^{-1}}X(z) - \frac{\beta^2 z^{-2}}{1-\alpha z^{-1}}Y(z) + \alpha z^{-1}Y(z)$$

so that, finally,

$$H(z) = \frac{\beta z^{-2}}{1-2\alpha z^{-1} + (\alpha^2+\beta^2)z^{-2}}.$$

(b) *When $\alpha = r\cos\theta$ and $\beta = r\sin\theta$ the transfer function simplifies to*

$$H(z) = \frac{r\sin\theta\, z^{-2}}{1-2r\cos\theta\, z^{-1} + r^2 z^{-2}}.$$

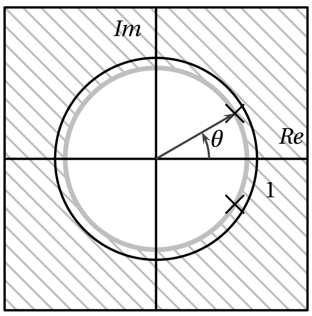
*Except for the scalar factor $\beta = r\sin\theta$ and the delay z^{-2}, which do not affect the shape of the magnitude response, this is the transfer function of a simple **resonator** with a single pair of complex conjugate poles at $re^{\pm j\theta}$. Even if you do not remember the formula for the resonator, you can easily compute the poles of this filter by finding the roots of the denominator; for this multiply by z^2 and solve*

$$z^2 - 2r\cos\theta\, z + r^2 = 0$$

The solutions are

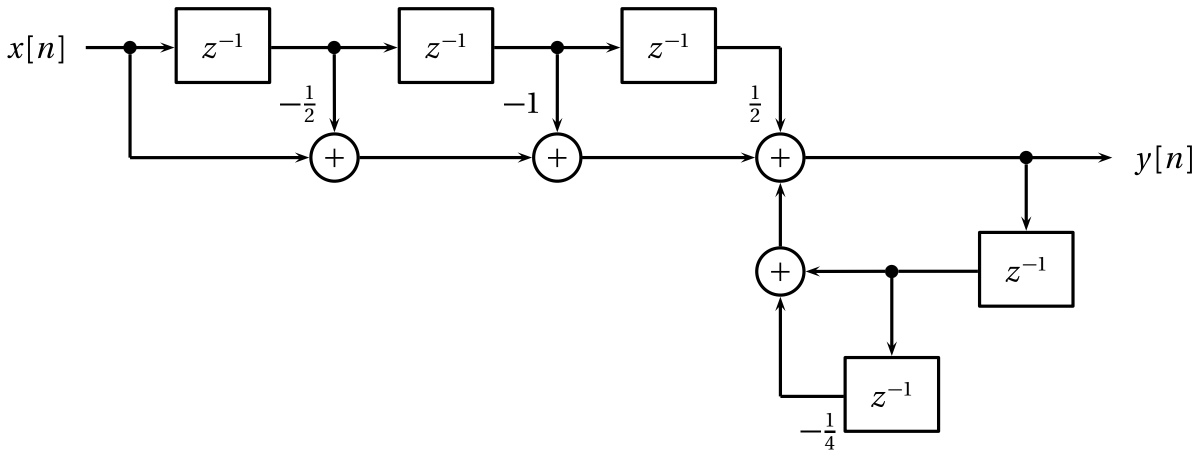
$$\begin{aligned} z_{1,2} &= (2r\cos\theta \pm \sqrt{4r^2(\cos^2\theta - 1)})/2 \\ &= r(\cos\theta \pm \sqrt{-\sin^2\theta}) \\ &= re^{\pm j\theta} \end{aligned}$$

The pole-zero plot is as follows:



Exercise 5. Block diagram 2

Consider the causal system described by the following block diagram:



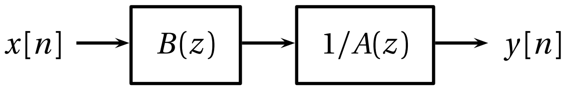
(a) Compute its transfer function $H(z) = Y(z)/X(z)$.

(b) Is the system stable?

(c) Draw a block diagram that implements the same transfer function **using only two delays** (i.e. only two z^{-1} delay blocks).

Solution:

The system can be decomposed as the cascade of an FIR and an IIR filter



where

$$B(z) = 1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{1}{2}z^{-3}$$

and

$$A(z) = 1 - z^{-1} + \frac{1}{4}z^{-2}.$$

Since we will need to determine the stability of the system later, we can already factorize $A(z)$ *by simple inspection as*

$$A(z) = (1 - \frac{1}{2}z^{-1})^2.$$

We can also try to see if the root of $A(z)$ *is also a root of* $B(z)$*: indeed* $B(1/2) = 0$. *We can now factor* $B(z)$ *either by performing polynomial division or by noticing that both* $+1$ *and* -1 *are also roots; we have*

$$B(z) = (1 - \frac{1}{2}z^{-1})(1 - z^{-2}).$$

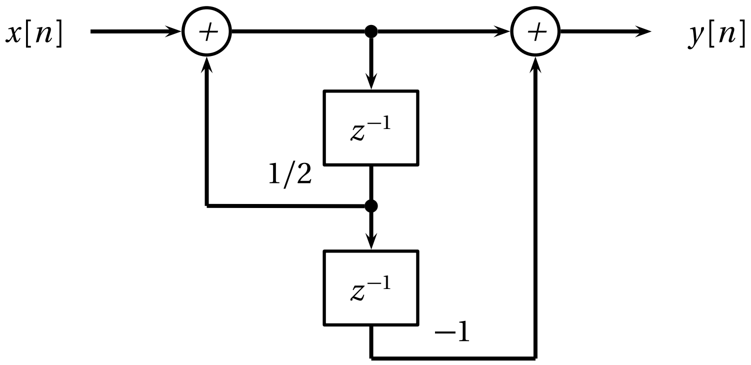
With this:

(a) *The global transfer function is*

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 - z^{-2}}{1 - \frac{1}{2}z^{-1}}.$$

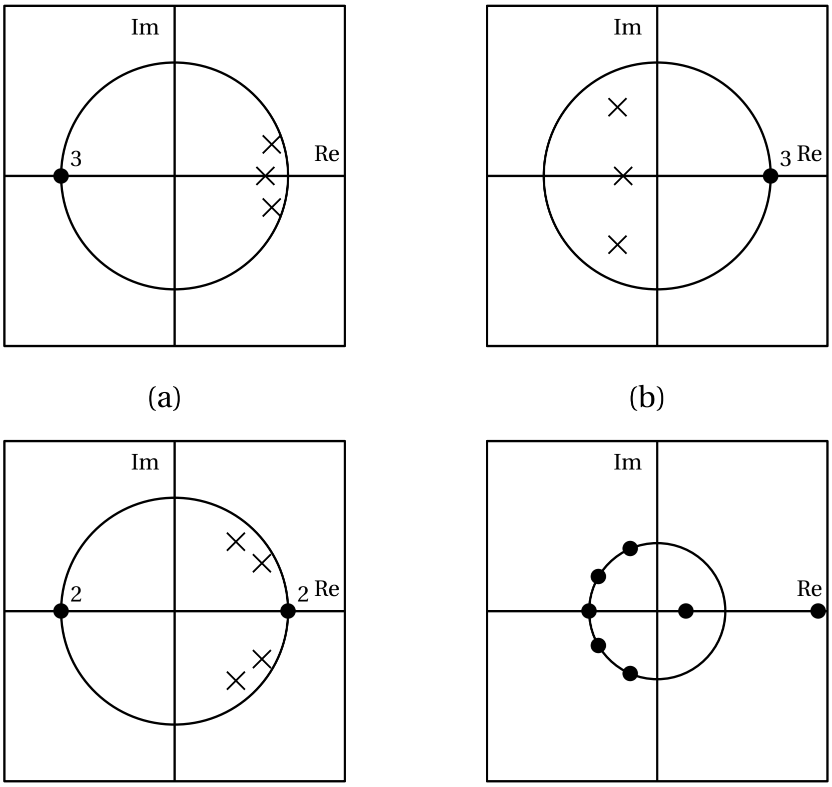
(b) *The pole of the system is in $z = \frac{1}{2}$ so the system is stable.*

(c) *The system is an incomplete second order section, so we can use the standard Direct Form II like so:*



Exercise 6. Poles and zeros

For each of the following pole-zero plots, sketch the magnitude response of the corresponding filter.

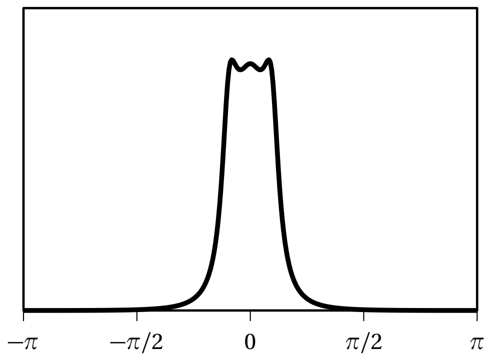


(a) (b) (c) (d)

(In the plots, poles are represented by crosses and zeros by circles; if applicable, the multiplicity of each pole and zero is indicated by a number. The circle indicates the unit circle on the complex plane).

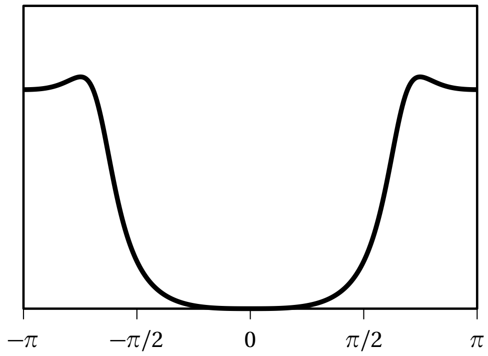
Solution:

(a) The filter has 3 zeros at $z=-1$ and poles near $z=1$, so its frequency response will be zero at $\omega=\pm\pi$ and large around $\omega=0$. This is typical of a lowpass filter.

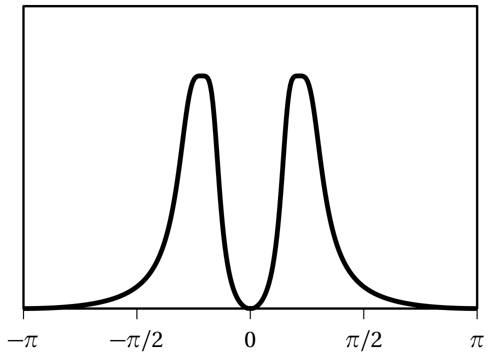


(b) Here the situation is the reverse of the previous case: there are 3 zeros at $z=1$ and poles towards $z=1$. As ω goes around the unit circle, the frequency response will start at zero for $\omega=0$ and grow large as ω becomes larger than $\pi/2$. This is typical of a

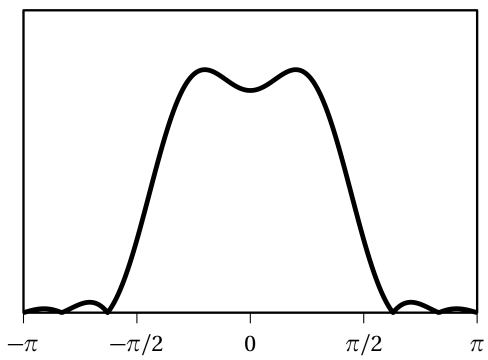
highpass filter:



(c) *Here the zeros in* $z = \pm 1$ *will set the frequency response to zero at* $\omega = 0$ *and* $\omega = \pm\pi$. *The complex-conjugate poles with phase around* $\pm\pi/4$ *will "push up" the magnitude response and create a bandpass filter:*



(d) *This filter has only zeros and it is therefore an FIR. We can use the positions of the zeros on the unit circle to approximately mark the values for* ω *where the frequency response is zero. The response will grow in amplitude as we move away from the zeros and, since the zeros on the unit circle occur for* $|\omega| > \pi/2$, *the filter is going to be a lowpass. The two zeros on the real axis will result in a ripple in the passband.*



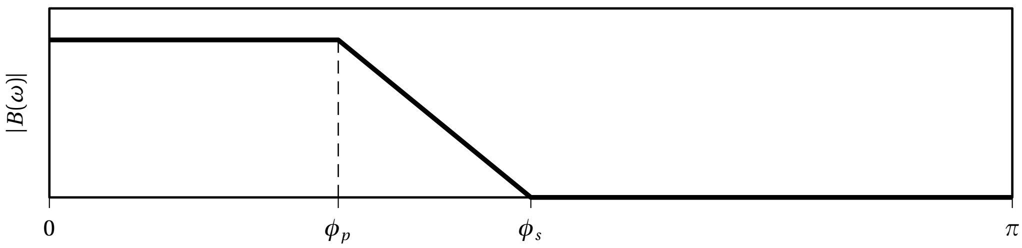
Exercise 7. Filter design

Bellanger's Approximation is an empirical formula used to estimate the length of an optimal linear-phase FIR lowpass filter based on its design specifications. For a lowpass with transition band $[\omega_p,\ \omega_s]$ and error tolerances of δ_p and δ_s in passband and stopband respectively, the required filter length will be approximately

$$N \approx \frac{-2\log_{10}(10\delta_p\delta_s)}{3(\omega_s-\omega_p)/2\pi}-1$$

Since the order is inversely proportional to the width of the transition band, "sharp" filters (i.e., filters with a narrow transition band) will require a lot of multiplications per output sample. The following questions will ask you to analyze an alternative design strategy called *Interpolated FIR* (IFIR), used to obtain sharp filters at a lower computational cost.

To begin, assume you have designed an optimal N-tap FIR lowpass $B(z)$ with impulse response $b[n]$ and with the following magnitude response (we're showing just the positive frequencies and neglecting the ripples):



The transition band of $B(z)$ has width $\Delta_B = \phi_s - \phi_p$. We now build a derived FIR filter $B_I(z)$ with impulse response

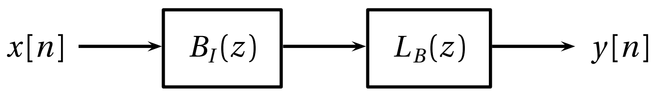
$$b_I[n] = \begin{cases} b[n/2] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

(a) Express $B_I(z)$ in terms of $B(z)$.

(b) Sketch the magnitude response $|B_I(\omega)|$; you don't need to draw the ripples but clearly show the band edges and their values.

(c) Assuming that multiplications by zero can be ignored, what is the number of multiplications per output sample required by $B_I(z)$?

Consider now the following cascade, used to implement a complete IFIR filter:



(d) Describe and sketch the frequency response of a filter $L_B(z)$ so that the cascade $B_I(z)L_B(z)$ implements a lowpass filter.

(e) Specify the passband and stopband frequencies of the lowpass implemented by the cascade $B_I(z)L_B(z)$.

(f) $L_B(z)$ will be implemented as an optimal linear-phase FIR; find the passband and stopband frequencies θ_p, θ_s that maximize the width the transition band of $L_B(z)$.

Consider now the following lowpass design specifications:

$$\omega_p = 0.3\pi$$
$$\omega_s = 0.31\pi$$
$$\delta_p = \delta_s = 0.01;$$

we want to compare a standard FIR implementation $H(z)$ with an IFIR implementation $\hat{H}(z) = B_I(z)L_B(z)$ such that $\hat{H}(z)$ has the same passband as $H(z)$.

(g) Estimate the order of a standard FIR implementation $H(z)$ using Bellanger's formula.

(h) For an IFIR implementation, determine the values of ϕ_p and ϕ_s to use for the initial filter $B(z)$ so that the final IFIR cascade $\hat{H}(z) = B_I(z)L_B(z)$ (for a suitable $L_B(z)$) has the same passband as $H(z)$.

(i) Estimate the order of an optimal linear phase FIR implementation of $B(z)$.

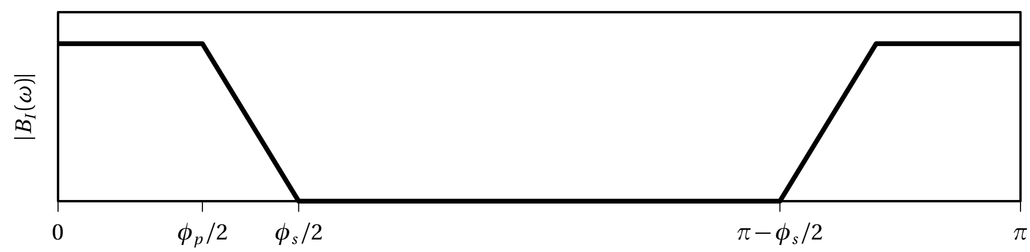
(j) Assume an optimal linear phase FIR implementation for $L_B(z)$, using the maximum transition band Δ_L possible and using $\delta_p = \delta_s = 0.01$; estimate the order of $L_B(z)$.

(k) Using the above estimations, determine the number of operations per output sample of the IFIR cascade $B_I(z)L_B(z)$.

Solution:

(a) $B_I(z) = \sum_{n=-\infty}^{\infty} b_I[n]z^{-n} = \sum_{k=-\infty}^{\infty} b[k]z^{-2k} = B(z^2)$.

(b) The frequency response is $B_I(\omega) = B(e^{j2\omega})$*, which causes a contraction of the original frequency response; the magnitude looks like so:*



(c) $B_I(z)$ *is a* $2N$*-tap filter, hence in principle we need* $2N$ *multiplications per output sample. However half those taps are zero-valued, hence* N *multiplications per output sample are sufficient.*

(d) $L_B(z)$ *should be a lowpass filter that removes the high frequency component in* $B_I(\omega)$*.*

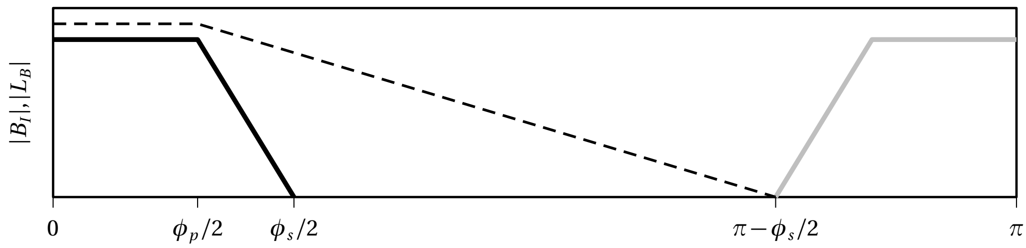
(e) *The cascade* $B_I(z)L_B(z)$ *is a lowpass with band edges:*

$$\omega_p = \phi_p/2$$
$$\omega_s = \phi_s/2$$

(f) *To minimize the computational cost of the cascade we can keep the transition band as wide as possible. We could use the following values, for instance:*

$$\theta_p = \phi_p/2$$
$$\theta_s = \pi - \phi_s/2$$

for a transition band of $\Delta_L = \pi - (\phi_s + \phi_p)/2$.



(g) *The estimated length is*

$$N \approx \frac{-2\log_{10}(10 \cdot 10^{-2} \cdot 10^{-2})}{3(0.31 - 0.3)\pi/2\pi} - 1$$
$$= \frac{6}{0.015} - 1 = 399$$

(h) *The prototype filter* $B(z)$ *has double the passband and stopband frequencies of the lowpass* $H(z)$*, i.e.*

$$\phi_p = 0.6\pi$$
$$\phi_s = 0.62\pi$$

(i) *The estimated length is* $N_B \approx \frac{6}{0.03} - 1 = 199$.

(j) $\Delta_L = \pi - (\phi_p + \phi_s)/2 = 0.39\pi$ *and so* $N_L \approx \frac{6}{3 \cdot 0.39\pi/2\pi} - 1 = \frac{6}{3 \cdot 0.39\pi/2\pi} - 1 \approx 9$.

(k) *We will need 199 multiplications for* $B_I(z)$ *and 9 for* $L_B(z)$ *for a total of 208 multiplications.*
