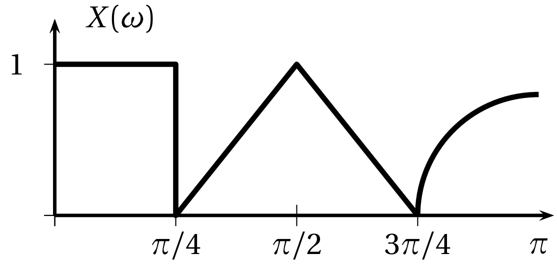
COM-202 - Signal Processing

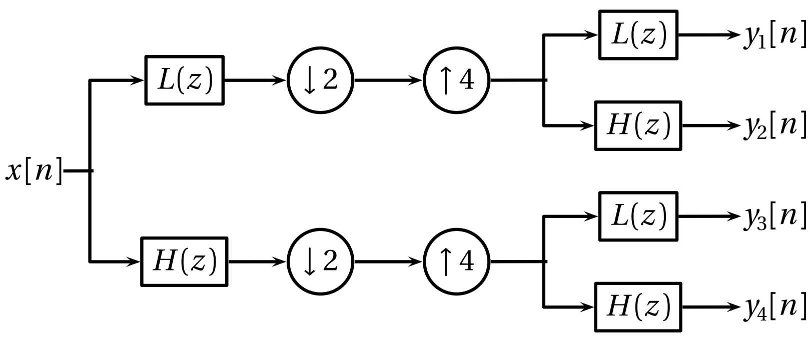
Solutions for Homework 11

Exercise 1. Multirate Signal Processing

Consider a real-valued discrete-time signal $x[n]$ with the following spectrum (shown here between 0 and π and, because the signal is real, symmetric for negative frequencies):

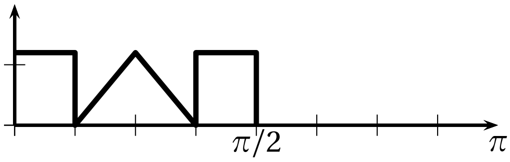


Now consider the following multirate processing scheme in which $L(z)$ is an ideal *lowpass* filter with cutoff frequency $\pi/2$ and $H(z)$ is an ideal *highpass* filter with cutoff frequency $\pi/2$:

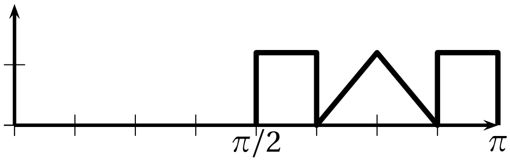


Plot the four spectra $Y_1(\omega)$, $Y_2(\omega)$, $Y_3(\omega)$, $Y_4(\omega)$.

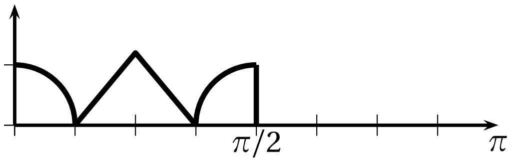
Solution:



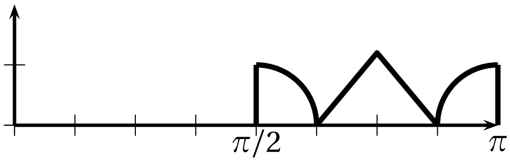
$Y_1(\omega)$



$Y_2(\omega)$



$Y_3(\omega)$



$Y_4(\omega)$

Exercise 2. Digital processing of continuous-time signals

In your grandmother's attic you just found a treasure: a collection of super-rare 78rpm vinyl jazz records. The first thing you want to do is to transfer the recordings to compact discs, so you can listen to them without wearing out the originals. Your idea is obviously to play the record on a turntable and use an A/D converter to convert the line-out signal into a discrete-time sequence, which you can then burn onto a CD. The problem is, you only have a "modern" turntable, which plays records at 33rpm. Since you're a DSP wizard, you know you can just go ahead, play the 78rpm record at 33rpm and sample the output of the turntable at 44.1 KHz. You can then manipulate the signal in the discrete-time domain so that, when the signal is recorded on a CD and played back, it will sound right.

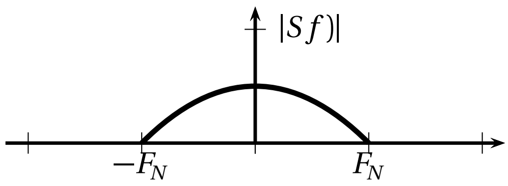
Design a system which performs the above conversion. If you need to get on the right track, consider the following:

- Call $s(t)$ the continuous-time signal encoded on the 78rpm vinyl (the jazz music)
- Call $x(t)$ the continuous-time signal you obtain when you play the record at 33rpm on the modern turntable
- Let $x[n] = x(nT_s)$, with $T_s = 1/44100$.

and answer the following questions:

(a) Express $x(t)$ in terms of $s(t)$.

(b) Sketch the Fourier transform $X(f)$ when $S(f)$ is as in the following figure. The highest positive frequency of $S(f)$ is $F_N = 16000$ Hz (old records have a smaller bandwidth than modern ones).



(c) Design a system to convert $x[n]$ into a sequence $y[n]$ so that, when you interpolate $y[n]$ to a continuous-time signal $y(t)$ with interpolation period T_s, you obtain $Y(f) = S(f)$.

(d) What if you had a turntable which plays records at 45rpm? Would your system be different? Would it be better?

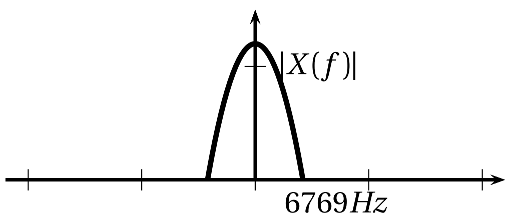
Solution:

(a) Playing the record at lower rpm slows the signal down by a factor $33/78$. *Therefore*

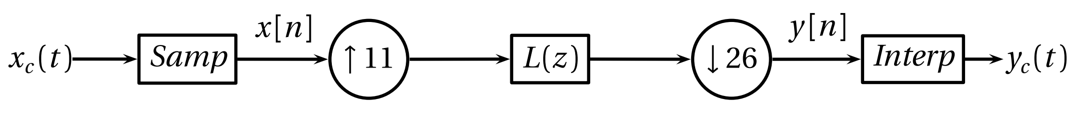
$$x(t) = s\left(\frac{33}{78}\, t\right) = s\left(\frac{11}{26}\, t\right)$$

(b) From the rescaling property of the Fourier transform

$$X(f) = \frac{26}{11}\, S\left(\frac{26}{11} f\right)$$



(c) We need to change the sampling rate so that, when $y[n]$ *is interpolated at 44.1 KHz its spectrum is equal to* $S(f)$. *The rational sampling rate change factor is clearly* $33/78$ *which is simply* $11/26$ *after factoring. The processing scheme is as follows:*



where $L(z)$ *is a lowpass filter with cutoff frequency* $\pi/26$ *and gain* $L_0 = 11$*; both the* *sampler and interpolator work at $F_s = 1/T_s = 44100$ Hz. We have:*

$$
\begin{aligned}
X_c(f) &= \frac{26}{11} S\left(\frac{26}{11} f\right) \\
X(\omega) &= F_s X_c(\frac{\omega}{2\pi} F_s) \\
Y(\omega) &= L_0 \frac{1}{26} X(\frac{11}{26}\omega) \\
&= \frac{11}{26} F_s X_c\left(\frac{11}{26}\frac{\omega}{2\pi} F_s\right) \\
&= F_s S\left(\frac{\omega}{2\pi} F_s\right) \\
Y_c(f) &= (1/F_s)\, Y(2\pi f/F_s) \\
&= S(f)
\end{aligned}
$$

(d) *The sampling rate change scheme stays the same except that now $45/78 = 15/26$. Therefore, the final upsampler has to compute more samples than in the previous scheme. The computational load of the sampling rate change is entirely dependent on the filter $L(z)$. If we upsample more before the output, we need to compute more filtered samples and therefore at 45rpm the scheme is less efficient.*

Exercise 3. Oversampled sequences

Consider a real-valued infinite sequence $x[n]$ for which:

$$X(\omega) = 0 \qquad \frac{\pi}{3} \le |\omega| \le \pi$$

A single sample of $x[n]$ may have been corrupted and we would like to approximately or exactly recover it. We denote n_0 the time index of the corrupted sample and $\hat{x}[n]$ the corresponding corrupted sequence.

(a) Specify a strategy for exactly recovering $x[n]$ from $\hat{x}[n]$ if n_0 is known.

(b) What would you do if the value of n_0 is not known?

(c) Now suppose we have $k-1$ corrupted samples: what is the condition that $X(e^{j\omega})$ must satisfy to be able to exactly recover $x[n]$?

Solution:

Given that $X(\omega) = 0$ for $\frac{\pi}{3} \le |\omega| \le \pi$, $x[n]$ can be thought of as a signal that has been sampled at 3 times the minimal sampling frequency. Therefore, we can downsample the signal at least by a factor of 3 without loss of information.

(a) *Assume n_0 is odd; we can then downsample $x[n]$ by 2, without loss of information and the corrupted sample will be discarded in the downsampling operation. We can then upsample by 2 and recover the original signal eliminating the error. If n_0 is even, simply shift the signal by 1 and perform the same operation.*

(b) *If the value of n_0 is not known, we need to determine whether n_0 is odd or even. We can write*

$$\hat{x}[n] = x[n] - \epsilon\delta[n - n_0]$$

and therefore

$$\hat{X}(\omega) = X(\omega) - \epsilon e^{-j\omega n_0}$$

Now, if we compute the DTFT at $\omega = \frac{\pi}{2}$ we have:

$$\hat{X}(\frac{\pi}{2}) = X(\frac{\pi}{2}) + \epsilon(-j)^{n_0} = \epsilon(-j)^{n_0}$$

since, by hypothesis, $X(\frac{\pi}{2}) = 0$. Therefore, If $\hat{X}(\frac{\pi}{2})$ is real, n_0 is even and if it is imaginary, n_0 is odd.

(c) *If there are $k-1$ corrupted samples, the worst case is when the corrupted samples are consecutive. In that case we need to downsample $\hat{x}[n]$ by a factor of k and then upsample it back. To do so without loss of information it must be:*

$$X(\omega) = 0 \quad \textit{for} \quad \frac{\pi}{k} \le |\omega| \le \pi.$$
