COM-202 - Signal Processing

Solutions for Homework 10

Exercise 1. Continuous-time Fourier Transform

(a) Using the Fourier transform formula, find the Fourier transform of the following signals

- $x_1(t) = e^{-at}u(t)$, with $Re(a) > 0$
- $x_2(t) = e^{at}u(-t)$, with $Re(a) > 0$

Recall that the unit step $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$$

(b) Using the Fourier transform formula, prove the following properties of the continuous-time Fourier transform

- Scaling property: $x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right)$ where $a \neq 0$
- Shift in time property: $x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j2\pi f t_0}X(f)$

Solution:

(a) *Both signals are in $L_2(\mathbb{R})$ and so the Fourier transform formula will converge. Note that there are different conventions about what happens to $u(t)$ at zero. The value of this single point will not affect any of our results and we leave it unspecified.*

Using the transform formula we obtain

$$\begin{aligned} X_1(f) &= \int_{-\infty}^{\infty} x_1(t)e^{-j2\pi f t}\,dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j2\pi f t}\,dt \\ &= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\,dt = -\frac{1}{a+j2\pi f}e^{-(a+j2\pi f)t}\Big|_{-\infty}^{0} = \frac{1}{a+j2\pi f} \end{aligned}$$

and

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)e^{-j2\pi f t}dt = \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j2\pi f t}dt$$
$$= \int_{-\infty}^{0} e^{(a-j2\pi f)t}dt = -\frac{1}{a-j2\pi f}e^{(a-j2\pi f)t}\Big|_{-\infty}^{0} = \frac{1}{-a+j2\pi f}$$

(b) *We can prove the scaling property directly from the transform formula. Let $y(t) = x(at)$. For a<0,*

$$Y(f) = \int_{-\infty}^{\infty} x(at)e^{-j2\pi f t}dt = \int_{\infty}^{-\infty} \frac{1}{a}x(s)e^{-j2\pi f s/a}ds$$
$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(s)e^{-j2\pi f/as}ds = \frac{1}{|a|}X(f/a)$$

For $a > 0$ the result follows analogously.

We can similarly prove the shift in time property directly from the transform formula. Let $y(t) = x(t-t_0)$,

$$Y(f) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi f t}dt = \int_{-\infty}^{\infty} \frac{1}{a}x(s)e^{-j2\pi f(s+t_0)}ds$$
$$= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} \frac{1}{a}x(s)e^{-j2\pi f s}ds = e^{-j2\pi f t_0}X(f)$$

Exercise 2. Sampling Sinusoids

(a) Consider a sampler operating at a sampling frequency F_s = 500 Hz. Which of the following signals can be converted to discrete-time sequences with no loss of information by this system?

- $x_1(t) = \cos(2\pi f_1 t)$, with $f_1 = 100$ Hz
- $x_2(t) = \sin(2\pi f_2 t)$, with $f_2 = 225$ Hz
- $x_3(t) = \sin(2\pi f_3 t)$, with $f_3 = 1250$ Hz
- $x_4(t) = \cos(2\pi f_1 t) + \sin(2\pi f_4 t)$, with $f_1 = 100$ Hz and $f_4 = 400$ Hz

(b) A second sampler operates by sampling its input every $T_s = 0.5 \times 10^{-3}$ seconds. Which of the following signals can be converted to discrete-time sequences with no loss of information by this system?

- $x_5(t) = \cos(2\pi f_5 t)$, with $f_5 = 500$ Hz
- $x_6(t) = \sin(2\pi f_3 t)$, with $f_3 = 1250$ Hz

- $x_7(t) = \cos(2\pi f_6 t) + \sin(2\pi f_7 t)$, with $f_6 = 250$ Hz and $f_7 = 150$ Hz
- $x_8(t) = \sin(2\pi f_8 t)$, with $f_8 = 750$ Hz

Solution:

According to the sampling theorem, a signal can be sampled with no loss of information as long as the sampling frequency is larger than its total bandwidth.

The spectrum of a sinusoidal signal at frequency f_0 is zero for $|f| > f_0$ so the total bandwidth is $2f_0$. For each signal, therefore, we need to check if the sampling frequency is larger than twice the frequency of its fastest component.

(a) the first sampler works at a rate $F_s = 500$ Hz; therefore

- $F_s > 2f_1 = 200$*: the signal $x_1(t)$ can be sampled with no loss of information*
- $F_s > 2f_2 = 450$*: the signal $x_2(t)$ can be sampled with no loss of information*
- $F_s < 2f_3 = 2500$*: NO, the signal $x_3(t)$ will be aliased*
- $F_s > 2f_1 = 200$ *but* $F_s < 2f_4 = 800$*: NO, the signal $x_4(t)$ will be aliased*

(b) the second sampler works a rate $F_s = 1/T_s = 2000$ Hz; therefore

- $F_s > 2f_5 = 1000$*: the signal $x_5(t)$ can be sampled with no loss of information*
- $F_s < 2f_3 = 2500$*: the signal $x_6(t)$ will be aliased*
- $F_s > 2f_6 = 500$ *and* $F_s > 2f_7 = 300$*: the signal $x_7(t)$ can be sampled with no loss of information*
- $F_s > 2f_8 = 1500$*: the signal $x_8(t)$ can be sampled with no loss of information*

Exercise 3. Raw sampling

The continuous-time signal

$$x(t) = \sum_{m=1}^{4} m \cos(2\pi f_0 m\, t)$$

with $f_0 = 300$ Hz, is raw-sampled into the discrete-time signal $x[n] = x(nT_s)$ using $T_s = 5 \cdot 10^{-4}$ seconds. Sketch the DTFT of $x[n]$.

Solution:

The sampling frequency is $F_s = 1/T_s = 10^4/5 = 2000$ Hz. Since the continuous-time signal is simply a linear combination of four pure sinusoids, we can determine for each term in the sum whether the sinusoid will be aliased or not:

- *for the first term ($m = 1$) $f_0 = 300 < F_s/2 = 1000$ so there will be no aliasing*
- *for the second term ($m = 2$) $2f_0 = 600 < F_s/2 = 1000$ so there will be no aliasing*
- *for the third term ($m = 3$) $3f_0 = 900 < F_s/2 = 1000$ so there will be no aliasing*
- *for the second term ($m = 4$) $4f_0 = 1200 > F_s/2 = 1000$ so this component will be aliased*

The discrete-time signal will be

$$x[n] = x(nT_s) = \sum_{m=1}^{4} m\cos(\omega_m n)$$

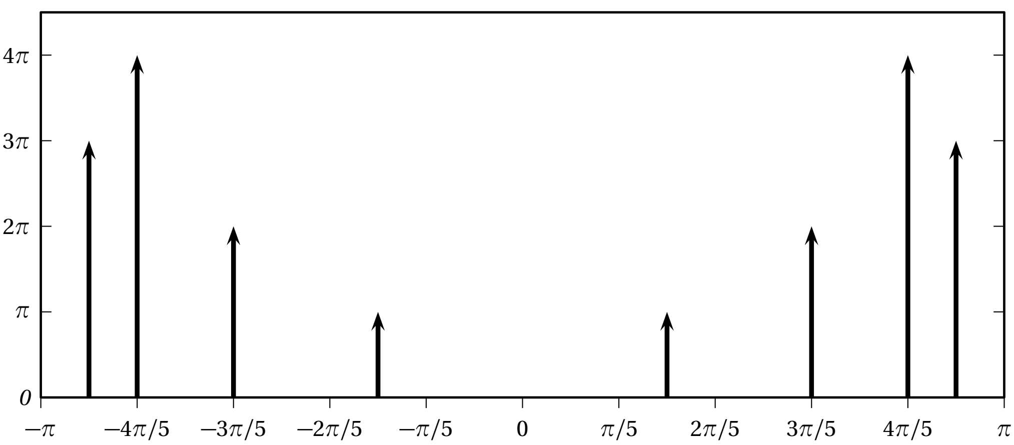
where

$$\omega_m = 2\pi\frac{mf_0}{F_s} = \frac{3m}{10}\pi, \quad m = 1,2,3$$

whereas ω_4 will be wrapped over the $[-\pi,\pi]$ interval:

$$\omega_4 = \left[\frac{6}{5}\pi\right]_{-\pi}^{+\pi} = \frac{6}{5}\pi - 2\pi = -\frac{4}{5}\pi$$

Since the generalized DTFT of each cosine component is a pair of periodized Dirac deltas at $\pm\omega_m$, the DTFT $X(\omega)$ will look like so:



Alternately, the solution can be worked out in the frequency domain starting from the generalized CTFT of the continuous-time signal, which contains 8 Dirac deltas at the frequencies $\pm mf_0$, for $m = 1,2,3,4$:

$$X_c(f) = \sum_{\substack{m=-4 \\ m\neq 0}}^{4} |m|\,\delta(f - mf_0)$$

After sampling, the periodized spectrum is

$$
\begin{aligned}
X(\omega) &= F_s \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{2\pi} F_s + k F_s\right) \\
&= F_s \sum_{\substack{m=-4 \\ m\neq 0}}^{4} |m| \sum_{k=-\infty}^{\infty} \delta\left(\frac{\omega}{2\pi} F_s + k F_s - m f_0\right) \\
&= F_s \sum_{\substack{m=-4 \\ m\neq 0}}^{4} |m| \sum_{k=-\infty}^{\infty} \delta\left(\frac{F_s}{2\pi}\left(\omega - 2\pi \frac{m f_0}{F_s} + 2\pi k\right)\right) \\
&= \sum_{\substack{m=-4 \\ m\neq 0}}^{4} |m| \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega - \frac{3m}{10}\pi + 2\pi k\right) \\
&= \sum_{\substack{m=-4 \\ m\neq 0}}^{4} |m| \tilde{\delta}\left(\omega - \frac{3m}{10}\pi\right)
\end{aligned}
$$

which yields the same plot as before. Note that we had to use the scaling property of the Dirac delta, $\delta(t/\alpha) \equiv \alpha\delta(t)$, which is perhaps not widely known. Hence the preference for the time-domain approach.

Exercise 4. Bandwidth of a signal

Consider a bandlimited continuous-time signal $x(t)$ whose total bandwidth is W Hz (in other words, the spectrum $X(f)$ is zero for $|f| > W/2$. Determine the maximum possible bandwidth for each of the following signals, assuming that $X(f) \neq 0$ over its entire bandwidth:

(a) $x_1(t) = x(t) - x(t-1)$

(b) $x_2(t) = x^2(t)$

(c) $x_3(t) = 2x(t)\cos(2\pi W t)$

(d) $x_4(t) = (x * h)(t)$ where $h(t) = \text{sinc}((W/3)t)$

Solution:

(a) $X_1(f) = (1 - e^{-j2\pi f})X(f)$; since $X(f) = 0$ for $|f| > W/2$, $X_1(f)$ will also be zero for $|f| > W/2$ so the bandwidth remains the same.

(b) since $x_2(t) = x(t)x(t)$, $X_2(f)$ is the convolution of $X(f)$ with itself:

$$
X_2(f) = \int_{-\infty}^{\infty} X(\phi)X(f-\phi)d\phi = \int_{-W/2}^{W/2} X(\phi)X(f-\phi)d\phi
$$

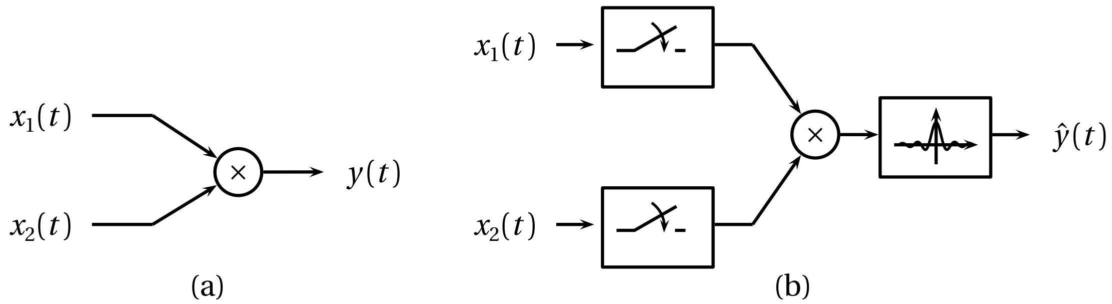
The argument of the integral will be zero if $f-\phi > W/2$ or $f-\phi < -W/2$. In the first case, this means $f > W/2+\phi$ and, since $-W/2 \leq \phi \leq W/2$, this means that $X_2(f)$ will be zero for $f > W/2+W/2 = W$. Similarly, in the second case, $X_2(f)$ will be zero for $f < -W/2-W/2 = -W$. In the end the total bandwidth for $X_2(f)$ will be doubled, that is, $2W$ Hz

(c) *$x_3(t)$ is a modulated version of $x(t)$ and $X_3(f) = X(f-W) + X(f+W)$; the support of $X(f-W)$ goes from $W-W/2$ to $W+W/2$ whereas the support of $X(f+W)$ goes from $-W-W/2$ to $-W+W/2$. In the end, the total bandwidth is $W+W/2-(-W-W/2) = 3W$*

(d) *$h(t)$ is the impulse response of a continuous-time ideal lowpass filter with cutoff frequency $W/6$ so the total bandwidth of $x_4(t)$ will be $W/3$*

Exercise 5. Discrete-time implementation of analog systems

Consider the continuous-time system shown in figure (a) below, whose output is the product of its two input signals. In order to implement a discrete-time version of this system, you build the device shown in figure (b), using two samplers and an ideal sinc interpolator, all of which work at the same rate F_s.



You know that the real-valued, continous-time input signals are bandlimited, with a maximum positive frequency $F_N = 8000$ Hz. Determine the minimum value for the rate F_s so that the discrete-time implementation produces exactly the same output as the continuous-time original system. Explain in detail your choice and, if in doubt, "test" the discrete-time system using the input signals $x_1(t) = x_2(t) = x(t) = \text{sinc}(2F_N t)$.

Solution:

Since the input signals are bandlimited to 8000 Hz, it is tempting to sample and interpolate at a rate of $F_s = 2F_N = 16000$ Hz. The multiplication of two signals is however a nonlinear operation that actually changes the bandwdith of the signals and therefore we need a higher rate. For example, consider the test signal $x(t) = \text{sinc}(2F_N t)$; in the frequency domain, $X(f) = (1/(2F_N))\,\text{rect}(f/(2F_N))$, showing that the signal is indeed 16kHz-bandlimited.

Using $x_1(t) = x_2(t) = x(t)$ as the inputs, the continuous-time system outputs the signal

$$y(t) = \mathrm{sinc}^2(2F_N t)$$

On the other hand, if the discrete-time system works at $F_s = 2F_N$, the sampling interval is $T_s = 1/(2F_N)$ and we have

$$\begin{aligned} x_{1,2}[n] &= x(nT_s) = \mathrm{sinc}(n) = \delta[n] \\ x_1[n]x_2[n] &= \delta^2[n] = \delta[n] \\ \hat{y}(t) &= \sum_n x_1[n]x_2[n]\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right) = \mathrm{sinc}(t/T_s) = \mathrm{sinc}(2F_N t) \neq y(t). \end{aligned}$$

Indeed, using the continuous-time version of the modulation theorem,

$$CTFT\{x_1 x_2\}(f) = (X_1 * X_2)(f) = \int_{-\infty}^{\infty} X_1(\phi)X_2(f-\phi)d\phi == \int_{-F_N}^{F_N} X_1(\phi)X_2(f-\phi)d\phi;$$

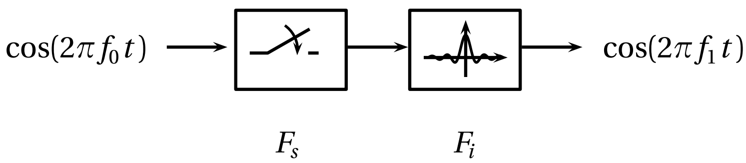
as we have seen in an earlier exercise, the argument of the integral will be zero for $f - \phi > F_N$ or for $f - \phi < -F_N$ and, since $-F_N \leq \phi \leq F_N$, the support of $(X_1 * X_2)(f)$ will be the interval $[-2F_N, 2F_N]$, which is twice the bandwidth of the input.

Therefore, the sampling rate in the discrete-time system needs to be at least $F_s = 4F_N =$ 32000 Hz.

As a side note, this fact is particularly important in digital audio processing systems that implement nonlinear effects such as distortion. Audio signals can be sampled at 48 kHz with no loss of information since the humans cannot hear frequencies above 20 kHz; however, if the internal processing is nonlinear, a digital audio workstation must use a sampling rate of at least 96 kHz to prevent aliasing artefacts.

Exercise 6. Mystery Signal

Consider the following setup, where a sinusoidal input of unknown frequency is first raw-sampled at a rate $F_s = 500$ Hz and then sinc-interpolated at a rate $F_i = 250$ Hz.



You measure the frequency of the output sinusoid and find out that $f_1 = 50$ Hz. Which of the following input frequencies would produce the measured output?

(a) $f_0 = 100$ Hz

(b) $f_0 = 150$ Hz

(c) $f_0 = 400$ Hz

(d) $f_0 = 600$ Hz

Solution: *Call* $x[n] = \cos(2\pi(f_0/F_s)n) = \cos(\omega_0 n)$ *the discrete-time signal produced by the raw sampler; the frequency of this oscillation is*

$$\omega_0 = \left[2\pi\frac{f_0}{F_s}\right]_{-\pi}^{+\pi}$$

where the notation $[\theta]_{-\pi}^{+\pi}$ *indicates that the angle* θ *has been wrapped over the* $[-\pi, +\pi]$ *interval. Mathematically, the wrapping operation can be expressed as*

$$[s]_{-a}^{+a} = s - 2a\left\lfloor \frac{s}{2a} + \frac{1}{2}\right\rfloor;$$

algorithmically, the value $[s]_{-a}^{+a}$ *can be computed by repetedly adding or subtracting* $2a$ *to* s *until the result is within* $[-a, a]$. *Note that for any* $c \in \mathbb{R}$

$$[cs]_{-a}^{+a} = c\left([s]_{-a/c}^{+a/c}\right)$$

since

$$[cs]_{-a}^{+a} = cs - 2a\left\lfloor \frac{cs}{2a} + \frac{1}{2}\right\rfloor = c\left(s - 2(a/c)\left\lfloor \frac{s}{2(a/c)} + \frac{1}{2}\right\rfloor\right) = c\left([s]_{-a/c}^{+a/c}\right)$$

and therefore

$$\omega_0 = \left[2\pi\frac{f_0}{F_s}\right]_{-\pi}^{+\pi} = 2\pi\left[\frac{f_0}{F_s}\right]_{-1/2}^{+1/2} = \frac{2\pi}{F_s}\left[f_0\right]_{-F_s/2}^{+F_s/2}$$

After the interpolator, the frequency of the output is going to be

$$f_1 = \frac{\omega_0}{2\pi}F_i = \frac{F_i}{F_s}\left[f_0\right]_{-F_s/2}^{+F_s/2} = \frac{1}{2}\left[f_0\right]_{-250}^{+250}$$

and so

(a) *if* $f_0 = 100$ *Hz, then* $f_1 = \frac{1}{2}[100]_{-250}^{+250} = 100/2 = 50$ *Hz*

(b) *if* $f_0 = 150$ *Hz, then* $f_1 = \frac{1}{2}[150]_{-250}^{+250} = 150/2 = 75$ *Hz*

(c) *if* $f_0 = 400$ *Hz, then* $f_1 = \frac{1}{2}[400]_{-250}^{+250} = (400 - 500)/2 = -50$ *Hz*

(d) *if* $f_0 = 600$ *Hz, then* $f_1 = \frac{1}{2}[600]_{-250}^{+250} = (600 - 500)/2 = 50$ *Hz*

Since $\cos(-2\pi f t) = \cos(2\pi f t)$, *the frequency of the input could be* 100, 400, *or* 600 *Hz.*

Exercise 7. Aliasing in Time?

Consider an N-periodic discrete-time signal $\tilde{\mathbf{x}}$, with N an *even* integer, and let $\tilde{\mathbf{X}}$ be its N-point DFS:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}nk} \qquad k \in \mathbb{Z}$$

Consider now a vector $\tilde{\mathbf{Y}}$ of length $N/2$ obtained by selecting the even-numbered elements of $\tilde{\mathbf{X}}$:

$$Y[m] = \tilde{X}[2m], \quad m = 0, 1, \ldots, N/2.$$

If we now compute the inverse DFS of $\tilde{\mathbf{Y}}$ we obtain the $(N/2)$-periodic signal $\tilde{\mathbf{y}}$

$$\tilde{y}[n] = \frac{2}{N} \sum_{k=0}^{N/2-1} \tilde{Y}[k] e^{j\frac{2\pi}{N/2}nk} \qquad n \in \mathbb{Z}.$$

Express $\tilde{\mathbf{y}}$ in terms of $\tilde{\mathbf{x}}$ and describe their relationship.

Solution:

$$\begin{aligned}
\tilde{y}[n] &= \frac{2}{N} \sum_{k=0}^{N/2-1} \tilde{Y}[k] e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N} \sum_{k=0}^{N/2-1} \tilde{X}[2k] e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N} \sum_{k=0}^{N/2-1} \sum_{i=0}^{N-1} \tilde{x}[i] e^{-j\frac{2\pi}{N}(2k)i}\, e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N} \sum_{i=0}^{N-1} \tilde{x}[i] \sum_{k=0}^{N/2-1} e^{j\frac{2\pi}{N/2}(n-i)k}
\end{aligned}$$

Now

$$\sum_{k=0}^{N/2-1} e^{j\frac{2\pi}{N/2}(n-i)k} = \begin{cases} N/2 & \textit{if } (n-i) \textit{ is a multiple of } (N/2) \\ 0 & \textit{otherwise} \end{cases}$$

so that the only nonzero terms in the outer sum (that for index i) are those for $i = n$ and $i = n + N/2$. In the end

$$\tilde{y}[n] = \tilde{x}[n] + \tilde{x}[n + N/2].$$

or, in compact form

$$\tilde{\mathbf{y}} = \tilde{\mathbf{x}} + \mathcal{S}^{N/2}\tilde{\mathbf{x}}.$$

Since $\tilde{\mathbf{x}}$ *is* N*-periodic,* $\tilde{\mathbf{y}}$ *is an* $(N/2)$*-periodic sequence obtained by summing two copies of* $\tilde{\mathbf{x}}$ *shifted by* $N/2$ *samples. This "time aliasing" is the dual of the frequency aliasing we incur when we sample too slowly in time; in this case, by dropping half of the DFS coefficients we are not "sampling enough" in frequency and thus the signal gets aliased in time.*