COM-202 - Signal Processing

Selection of multiple choice and open questions from past exams

Question 1. DFT

Let $x[n] = \sin\left(\frac{2\pi}{N}pn\right)$ be a finite-length signal in $\mathbb{C}^N$ with $N > 2$ and p is a positive integer such that $1 < p < N/2$. Let $X[k]$ be k-th value of the N-point DFT of $x[n]$. Which one of the following statements is **true**?

- ☐ $jX[k] \in \mathbb{R}$
- ☐ $X[k] = |X[k]|$ for $k = 0, 1, \ldots, N-1$
- ☐ $|X[0]| = N/2$
- ☐ $X[k] = X[N-k]$ for $k = 0, 1, \ldots, N-1$

Solution: $X[k]$ is the DFT of a sine whose frequency is an integer multiple of $2\pi/N$, the fundamental frequency for $\mathbb{C}^N$. Therefore $X[k] = -j(N/2)(\delta[k-p] - \delta[k-N+p])$ which is purely imaginary. Therefore $jX[k]$ is real.

Question 2. DFT

Let $x[n] = \cos\left(\frac{2\pi}{N}pn\right)$ be a finite-length signal in $\mathbb{C}^N$ with $N > 2$ and p is a positive integer such that $1 < p < N/2$. Let $X[k]$ be the N-point DFT of $x[n]$. Which one of the following statements is **false**?

- ☐ $|X[N-p]| = N$
- ☐ $X[k] = |X[k]|$ for $k = 0, 1, \ldots, N-1$
- ☐ $|X[0]| = 0$
- ☐ $X[k] = X[N-k]$ for $k = 1, \ldots, N-1$

Solution: $X[k]$ is the DFT of a cosine whose frequency is an integer multiple of $2\pi/N$, the fundamental frequency for $\mathbb{C}^N$. Therefore $X[k] = (N/2)(\delta[k-p]+\delta[k-N+p])$ which is real and symmetric ($X[k] = X[N-k]$) and whose value in $k = N-p$ is $X[N-p] = N/2$.

Question 3. DFT

Let $x[n]$ be a real-valued, finite-length signal in $\mathbb{C}^N$ whose N-point DFT is $X[k]$. Let $y[n]$ be a finite-length signal of length $2N$ obtained by concatenating two copies of $x[n]$:

$$y[n] = \begin{cases} x[n] & 0 \le n < N \\ x[n-N] & N \le n < 2N \end{cases}$$

Call $Y[k]$ the $2N$-point DFT of $y[n]$. Which one of the following statements is **true** for any $x[n]$?

- ☐ $Y[1] = 0$
- ☐ $Y[k] = 2X[k]$ for $k = 0, 1, \ldots, 2N-1$
- ☐ $Y[0] = X[0]$
- ☐ $Y[k] = X[k \mod 2]$ for $k = 0, 1, \ldots, 2N-1$

Solution:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\begin{aligned} Y[k] &= \sum_{n=0}^{2N-1} y[n] e^{-j\frac{2\pi}{2N}kn} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n=N}^{2N-1} x[n-N] e^{-j\frac{2\pi}{2N}k(n)} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}kn} + \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}kn} e^{-j\pi k} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}kn} (1 + e^{-j\pi k}) \end{aligned}$$

$$Y[k] = \begin{cases} 2X[k/2] & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \tag{1}$$

Question 4. DFT

Let $x[n]$ be a real-valued, finite-length signal in $\mathbb{C}^N$ whose N-point DFT is $X[k]$. Let $y[n]$ be a finite-length signal of length $2N$ obtained by appending N zeros to $x[n]$:

$$y[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & N \le n < 2N \end{cases}$$

Call $Y[k]$ the $2N$-point DFT of $y[n]$. Which one of the following statements is **true** for any $x[n]$?

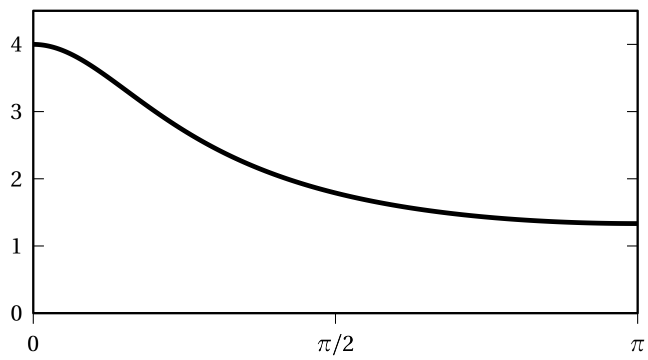
- ☐ $Y[2k] = X[k]$
- ☐ $Y[k] = X[k]/2$ for $k = 0, 1, \ldots, 2N-1$
- ☐ $Y[k] = 0$ for $N \le k < 2N$
- ☐ $Y[k] = X[N-k]$ for $N \le k < 2N$

Solution:

$$Y[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}nk}$$

Question 5. DTFT

Consider the signal $x[n] = 2a^n\, u[n]$, with $a > 0$. The magnitude of its DTFT $X(\omega)$ is shown in the following plot over the $[0, \pi]$ interval. What is the value of a?



- ☐ 0.5
- ☐ 2

☐ It cannot be determined from this magnitude plot

☐ $\frac{1}{\sqrt{2}}$

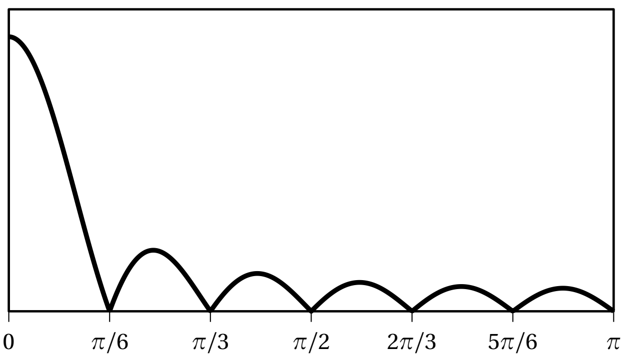
Solution: $X(\omega) = \frac{2}{1-ae^{-j\omega}}$ so that $X(\omega)|_{\omega=0} = 2/(1-a)$. From the plot, the magnitude DTFT in zero is 4, so $a = 0.5$

Question 6. DTFT

Consider the discrete-time signal

$$x[n] = \begin{cases} 1 & 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

The magnitude of its DTFT $X(\omega)$ is shown in the following plot over the $[0, \pi]$ interval. What is the value of the positive integer M?



☐ 12

☐ 6

☐ The value cannot be determined from the plot because there are no units on the vertical axis

☐ 11

Solution:

$$|X(\omega)| = \left| \frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)} \right|$$

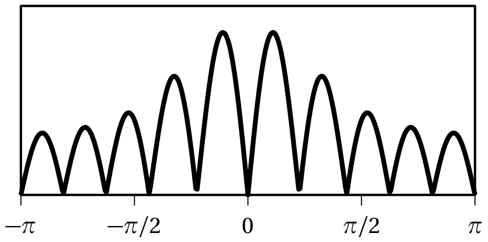
Since $X(\omega) = 0$ for $\omega = (2\pi/M)k,\ k \in \mathbb{Z}/\{0\}$, $M = 12$.

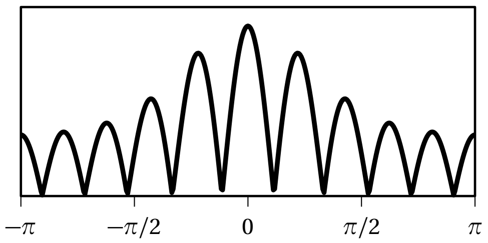
Question 7. DTFT

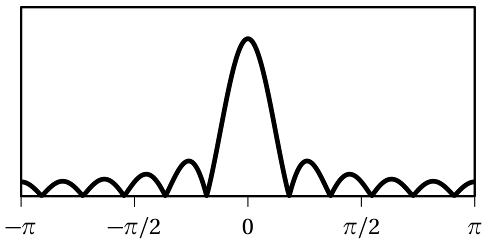
Consider the infinite-length discrete-time signal

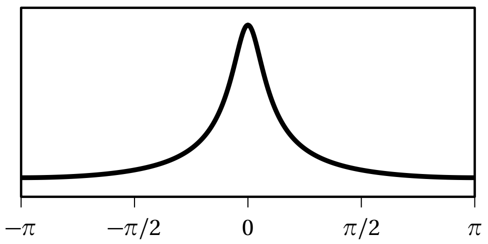
$$x[n] = \begin{cases} n^3 & |n| < M \\ 0 & \text{otherwise} \end{cases}$$

Which of the following plots could represent the magnitude of its DTFT?

☐ 

☐ 

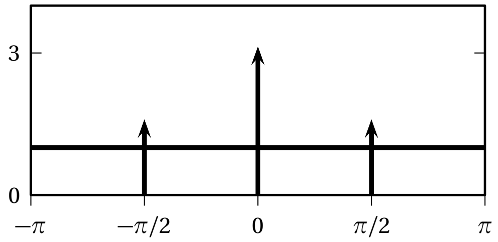
☐ 

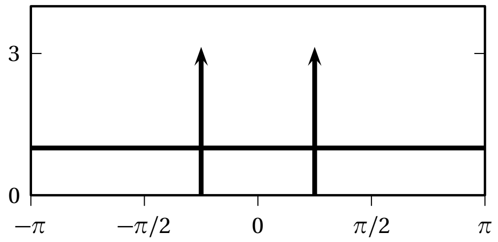
☐ 

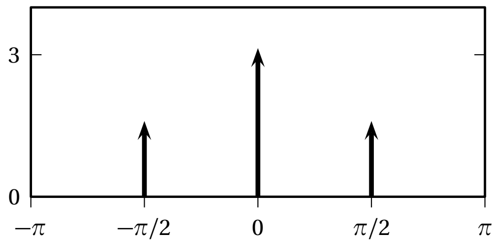
Solution: Since $\sum_n x[n] = 0$, the value of the DTFT must be zero in $\omega = 0$. Only one plot fulfills this condition.

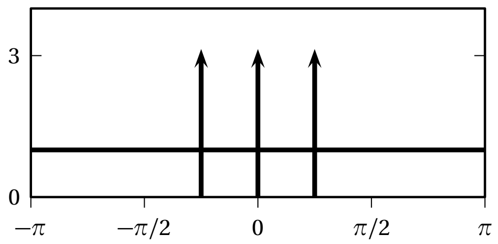
Question 8. DTFT

Consider the infinite-length discrete-time signal $x[n] = \delta[n] + \cos^2(\omega_0 n)$ with $\omega_0 = \pi/4$. Which one of the following plots represents its DTFT?

☐ 

☐ 

☐ 

☐ 

Solution: $x[n] = \delta[n] + (1/2) + (1/2)\cos(2\omega_0 n)$ so its DTFT is $X(\omega) = 1 + (1/2)\tilde{\delta}(\omega) + (1/4)\tilde{\delta}(\omega - 2\omega_0) + (1/4)\tilde{\delta}(\omega + 2\omega_0)$, where $\tilde{\delta}(\omega) = 2\pi \sum_k \delta(\omega - 2k\pi)$.

Question 9. SCQ-05

A discrete-time system is defined by an input-output relationship $y[n] = x[-n]$. Which of the following claims about the system is **true**?

- ☐ The system is linear
- ☐ The system is time-invariant
- ☐ The system is causal
- ☐ The system is memoryless

Solution:

Option (A) is correct

- (T) The system is linear since $\mathscr{H}\{a x_1[n]+b x_2[n]\} = a x_1[-n]+b x_2[-n] = a\mathscr{H}\{x_1[n]\}+ b\mathscr{H}\{x_2[n]\}$
- (F) The system is not time-invariant since $\mathscr{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0] = x[-n+n_0]$
- (F)The system is not causal since outputs at time $n < 0$ depend on future values of input.
- (F) The system is not memoryless. The outputs at time n depend on the inputs at time $-n$.

Question 10. Filters in Python

The following Python function applies a discrete-time filter to the array of values provided as an input; select which transfer function $H(z)$ is implemented by the code.

```
import numpy as np

def myfilter(data: np.ndarray) -> np.ndarray:
    w1, w2 = 0, 0
    for n in range(0, len(data)):
        w = data[n] + 1.4 * w1 - 0.6 * w2
        data[n] = 3 * w + 6 * w1 + 3 * w2
        w2 = w1
        w1 = w
    return data
```

- ☐ $H(z) = \dfrac{3+6z^{-1}+3z^{-2}}{1-1.4z^{-1}+0.6z^{-2}}$
- ☐ $H(z) = \dfrac{3+6z^{-1}+3z^{-2}}{1+1.4z^{-1}-0.6z^{-2}}$
- ☐ $H(z) = \dfrac{1}{3}\dfrac{1+1.4z^{-1}-0.6z^{-2}}{1+2z^{-1}+z^{-2}}$
- ☐ $H(z) = 4+6.4z^{-1}+2.7z^{-2}$

Solution: The function implements a second-order section in Direct Form II and therefore the correct answer is $H(z) = \dfrac{3+6z^{-1}+3z^{-2}}{1-1.4z^{-1}+0.6z^{-2}}$

Question 11. Filters in Python

Select the output obtained by running the following Python code block:

```
import numpy as np

def myfun(data: np.ndarray) -> np.ndarray:
    y = np.zeros(len(data))
    for n in range(0, len(data)):
        y[n] = 0.5 * y[n-1] + data[n]
    return y

print(myfun([2, 0, 0, 0]))
```

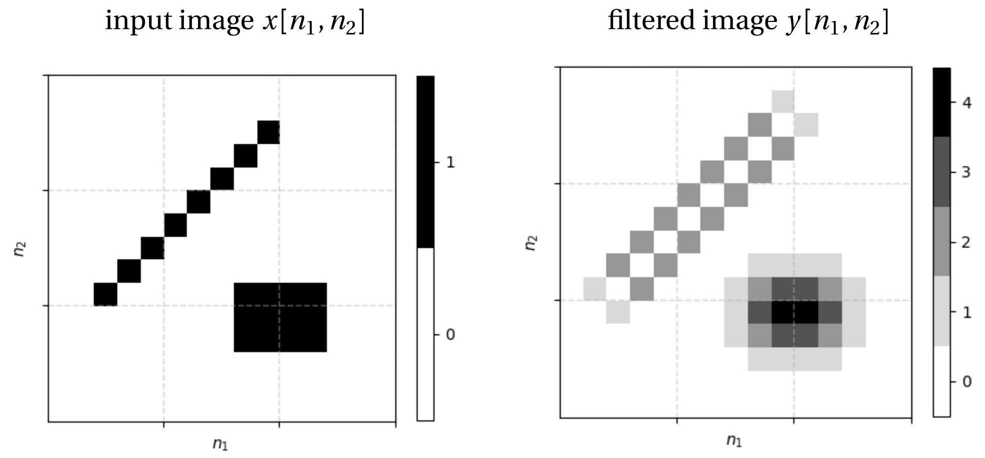
- ☐ `[2 1 0.5 0.25]`
- ☐ `[1 0 0 0]`
- ☐ the code crashes and returns an `IndexError`
- ☐ `[1 0.5 0.25 0.125]`

Solution:

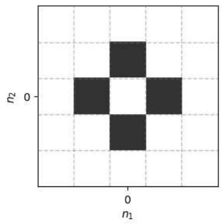
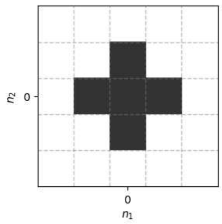
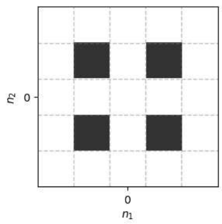
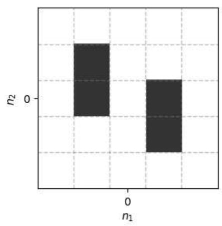
The function implements the CCDE $y[n] = 0.5\,y[n-1] + x[n]$ and it is applied to the input $x[n] = 2\delta[n]$ for $n = 0, 1, 2, 3$. The correct output is therefore `[2 1 0.5 0.25]`.

Question 12. SCQ-08

The 15×15 image $x[n_1, n_2]$ shown in the left panel of the following figure is filtered with a 2D FIR to produce the output image $y[n_1, n_2]$ shown in the right panel. The value of each pixel in both images is shown as a grasyscale level using the shades of gray shown to the right of each image.



Which of the following plots shows the impulse response of the 2D FIR used to filter the image? (Note that in all of the following plots, a white pixel indicates that $h[n_1, n_2] = 0$ while a black pixel indicates that $h[n_1, n_2] = 1$)

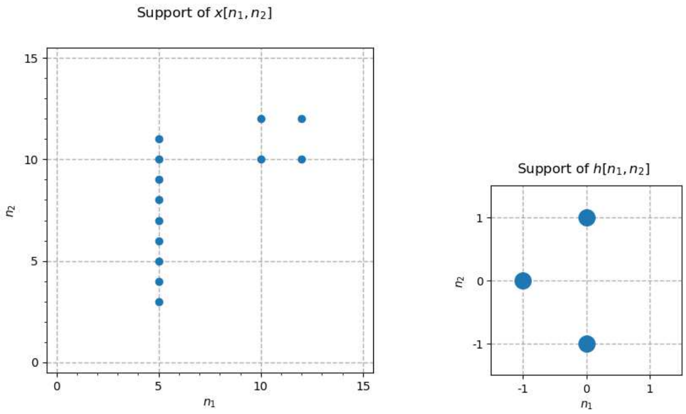
- [ ] 
- [ ] 
- [ ] 
- [ ] 

Solution: Rememeber that you can look at convolution (1D and 2D) as placing a scaled copy of the impulse response at every location, where the scaling factor is the value of the input at that location.

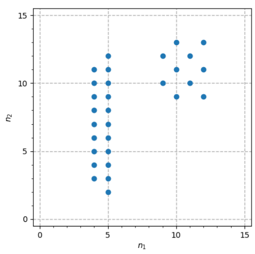
With this in mind, if you focus on one of the ends of the 45-degree line or on one of the corners of the rectangle in the input image, you can easily eliminate 2,3, and 4.

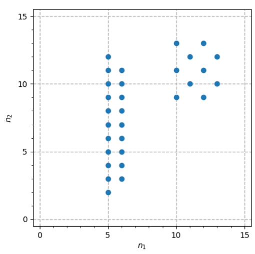
Question 13. SCQ-08

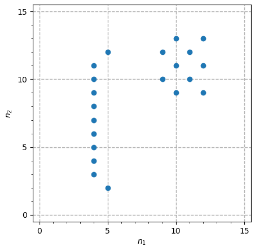
Consider a 16×16 image $x[n_1, n_2]$ and a two-dimensional FIR filter $h[n_1, n_2]$; the support of the image and the filter are shown in the following figure; remember that the support of a 2D signal is the set of coordinates $[n_1, n_2]$ for which the signal is nonzero and a support plot shows these coordinates with a dot.

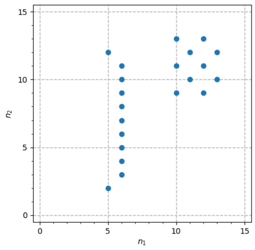


Which of the following plots shows the support of the image obtained by filtering $x[n_1, n_2]$ with $h[n_1, n_2]$?

☐ 

☐ 

☐ 

☐ 

Solution: Rememeber that you can look at convolution (1D and 2D) as placing a scaled copy of the impulse response at every location, where the scaling factor is the value of the input at that location.

With this in mind, if you focus on the vertical line you can easily eliminate 2 and 4 since the impulse response is "pointing" left but the result is "pointing" right. To decide between 1 and 3, notice that multiple overlapping copies of the impulse response would produce a line of thickness 2 pixels, so you can eliminate 3.

Question 14. z-transform

Given an infinite-length signal $x[n]$ with z-transform $X(z)$, we build a signal $y[n]$ as

$$y[n] = \begin{cases} x[n/3] & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

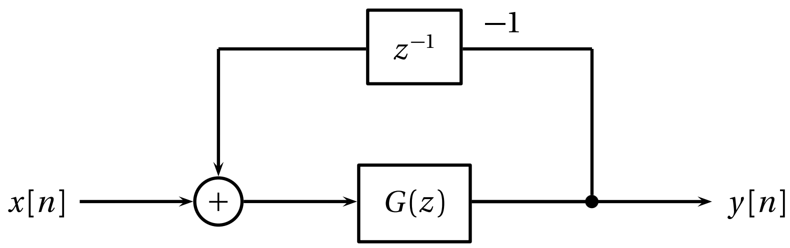
What is the expression for the z-transform of $y[n]$?

- ☐ $Y(z) = X(z^{1/3})$
- ☐ $Y(z) = X((-z)^{1/3})$
- ☐ $Y(z) = X(z^3)$
- ☐ $Y(z) = X(3z)$

Solution: We have $X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$
$Y(z) = \sum_{k=-\infty}^{\infty} y[k]z^{-k} = \sum_{l=-\infty}^{\infty} y[3l]z^{-3l} = \sum_{l=-\infty}^{\infty} x[l]z^{-3l} = X(z^3)$

Question 15. z-transform

Select the transfer function implemented by the following block diagram:



- ☐ $H(z) = \dfrac{G(z)}{1 + z^{-1}G(z)}$
- ☐ $H(z) = \dfrac{1}{1 + z^{-1}G(z)}$
- ☐ $H(z) = \dfrac{1}{1 - z^{-1}G(z)}$
- ☐ $H(z) = \dfrac{G(z)}{1 - z^{-1}G(z)}$

Solution: $H(z) = \dfrac{G(z)}{1 + z^{-1}G(z)}$

Question 16. SCQ-14

The Leaky Integrator system,

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n], \qquad 0 < \lambda < 1$$

is an example of

- ☐ IIR lowpass filter
- ☐ FIR lowpass filter
- ☐ unstable LTI system
- ☐ memoryless LTI system

Solution: Taking the Z-transform, we get $Y(z) = \lambda Y(z)z^{-1} + (1-\lambda)X(z)$. Thus the transfer function is $H(z) = \frac{Y(z)}{X(z)} = \frac{1-\lambda}{1-\lambda z^{-1}}$
Thus, $H(z) = (1-\lambda)[1+\lambda z^{-1} + (\lambda z^{-1})^2 + (\lambda z^{-1})^3 + \dots]$. Thus, it is an IIR filter.

Also $H(\omega) = \frac{1-\lambda}{1-\lambda e^{-j\omega}}$, For $\omega = 0$, the amplitude is 1 and for $\omega \to \infty$, the value tends to $1-\lambda$. Since it keeps the low frequencies and attenuates the high ones, it is a lowpass filter.

Question 17. SCQ-15

A sampling system samples continuous-time signals with sampling period $T = 0.5 \cdot 10^{-3}$. Let $x_1(t) = \cos(2\pi f_0 t) + \sin(2\pi f_1 t)$ and $x_2(t) = \cos(2\pi f_2 t)$, with

$$f_0 = 400$$
$$f_1 = 225$$
$$f_2 = 600$$

According to the sampling theorem, which of the following signals can be sampled with sampling period T_s without aliasing?

- ☐ only $x_1(t)$
- ☐ only $x_2(t)$
- ☐ both $x_1(t)$ and $x_2(t)$
- ☐ neither

Solution:

$$\frac{f_i}{1/T_s} = \frac{f_i}{2000} < 1/2 \text{ for i = 0, 1, 2}$$

All frequencies are less than half the sampling frequency.

Question 18. SCQ-15

Let $x(t)$ be a band-limited signal with CTFT satisfies $|X(f)| = 0$ for $|f| \geq 400$ Hz. Which of the following statements about $x(t)$ is **true**?

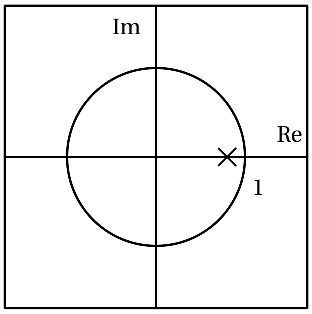
- ☐ According to the sampling theorem, the signal $x[n] = x\left(\frac{n}{800}\right)$ contains all the information needed to reconstruct $x(t)$ without aliasing
- ☐ According to the sampling theorem, the sampling frequency $F_s = 400$ is sufficient to sample $x(t)$ without aliasing

- ☐ According to the sampling theorem, it is not possible to sample $x(t)$ without aliasing
- ☐ According to the sampling theorem, to avoid aliasing we need to sample $x(t)$ with a sampling period smaller than 400.

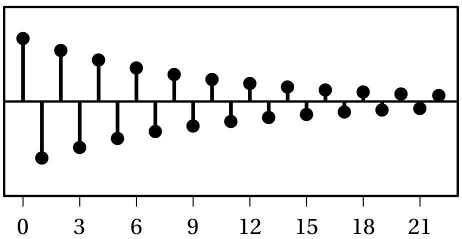
Solution: $x(t)$ is 800-BL so to avoid aliasing it must be sampled at $F_s g e 800$ Hz

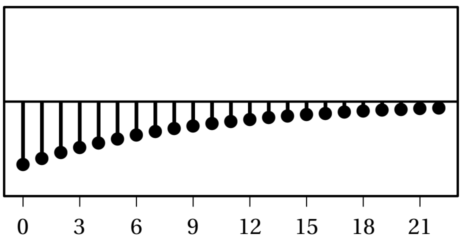
Question 19. Poles, zeros, and impulse response

Consider a causal IIR filter with impulse response $h[n]$ and transfer function $H(z)$. The pole-zero plot for $H(z)$ is shown in the following figure and note that the multiplicity of the pole could be greater than one.

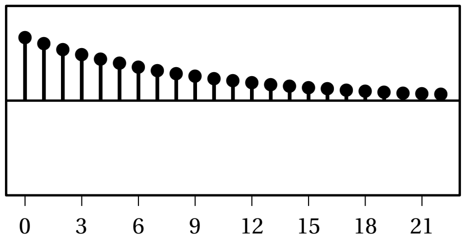


Select the impulse response that is **not** compatible with the given pole-zero plot.

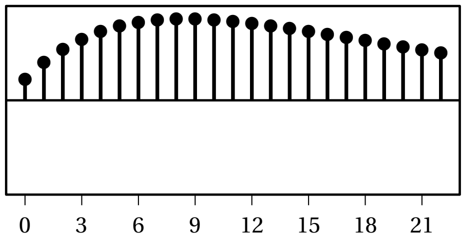
☐ 

☐ 

☐



☐

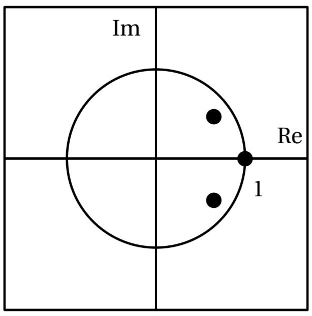


Solution:

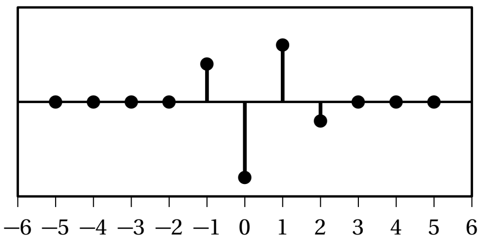
The transfer function will be of the form $H(z) = b(1 - a z^{-1})^{-M}$ where M is the multiplicity of the real-valued pole. The corresponding impulse response will thus be of the form $h[n] = b\, c[n]$ where $c[n]$ is the convolution of M copies of the sequence $a^n\, u[n]$. Since the plot clearly shows that $a > 0$, $c[n] \geq 0$ for all n and so $h[n]$ cannot be a sequence with alternating sign.

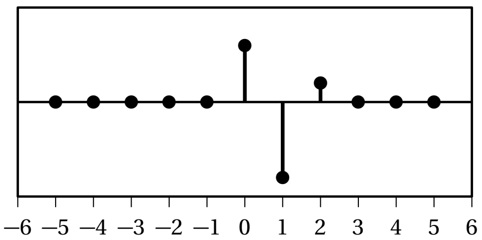
Question 20. Poles, zeros, and impulse response

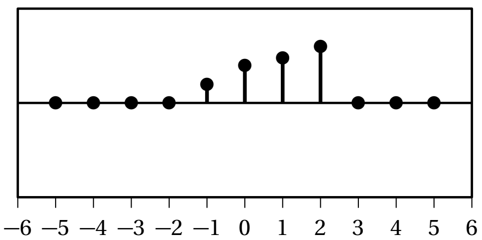
Consider an FIR filter with real-valued impulse response $h[n]$ and transfer function $H(z)$. The locations of the zeros of $H(z)$ are shown in the following figure and each zero has multiplicity one.

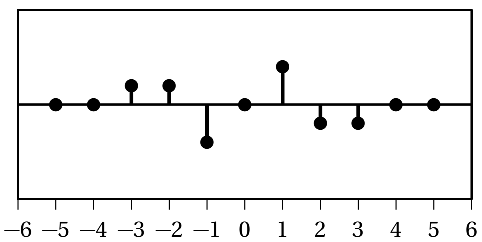


Select the only impulse response that is compatible with the given pole-zero plot. (For each impulse response, every value not shown in the plot is equal to zero.)

☐ 

☐ 

☐ 

☐ 

Solution:

The second option shows an impulse response with only three contiguous nonzero samples; the corresponding transfer function will thus have at most two zeros, but $H(z)$ has three.

The zero of $H(z)$ in $z = 1$ imply that $\sum_n h[n] = 0$, which rules out the third option where all samples are positive.

From the zero plot, we know that the transfer function will be of the form $H(z) = b(1 - z^{-1})(1-(2\rho\cos\varphi)z^{-1}+\rho^2 z^{-2})$ where $\rho\, e^{\pm j\varphi}$ are the locations of the pair of complex-conjugate zeros. Since $H(z)$ has at most four nonzero terms, the impulse response $h[n]$ will have at most four nonzero samples, which rules out the fourth option.

Question 21. *System analysis. This question is worth 10 points.*

Consider the causal system $\mathscr{H}$ defined by the following CCDE:

$$y[n] = y[n-1] + x[n] - x[n-1] - 2x[n-2] + 2x[n-3]$$

(a) (4 Pts) Compute the transfer function of the system and show that it implements an FIR filter.

(b) (2 Pts) Show that the causal inverse filter with transfer function $G(z) = 1/H(z)$ is not stable.

(c) (4 Pts) Consider now a causal FIR filter $\mathscr{W}$ with impulse response

$$w[n] = \delta[n] + (1/2)\delta[n-1] + (1/4)\delta[n-2].$$

Show that the filter $\mathscr{W}$ has a stable causal inverse.

Solution:

(a) The transfer function of the system is

$$\begin{aligned} H(z) &= \frac{1 - z^{-1} - 2z^{-2} + 2z^{-3}}{1 - z^{-1}} \\ &= \frac{(1 - z^{-1})(1 - 2z^{-2})}{1 - z^{-1}} \\ &= 1 - 2z^{-2} \end{aligned}$$

The filter is thus an FIR with impulse response $h[n] = \delta[n] - 2\delta[n-2]$.

(b) The zeros of $H(z) = 1 - 2z^{-2}$ are at $z = \pm\sqrt{2}$ and these will be the poles of the causal inverse filter $G(z) = 1/H(z)$. Since $\sqrt{2} > 1$, $G(z)$ will not be stable.

(c) We have

$$W(z) = 1 + (1/2)z^{-1} + (1/4)z^{-2}$$

The two zeros of the transfer function are at

$$z_{0,1} = \frac{-(1/2) \pm \sqrt{1/4 - 1}}{2} = (-1 \pm j\sqrt{3})/4$$

and these will be the poles of the causal inverse system $1/W(z)$. Since $|z_{0,1}| = 1/2 < 1$, the inverse system will be stable.

Question 22. *LTI system. This question is worth 8 points.*

A discrete-time LTI system has an impulse response $h[n] = \delta[n] - \frac{\omega_0}{\pi}\text{sinc}\left(\frac{\omega_0}{\pi}n\right)$ for some $0 < \omega_0 < \frac{\pi}{2}$.

(a) (4 Pts) Find the frequency response $H(\omega)$ for this system. Is the LTI system a low-pass, high-pass, or band-pass filter?

(b) (4 Pts) This system is applied to an input signal $x_1[n] = 1 + \sin\left(\frac{\omega_0}{2}n\right) + \sin(2\omega_0 n)$. Find the output signal $y_1[n]$.

Solution:

(a) The frequency response is

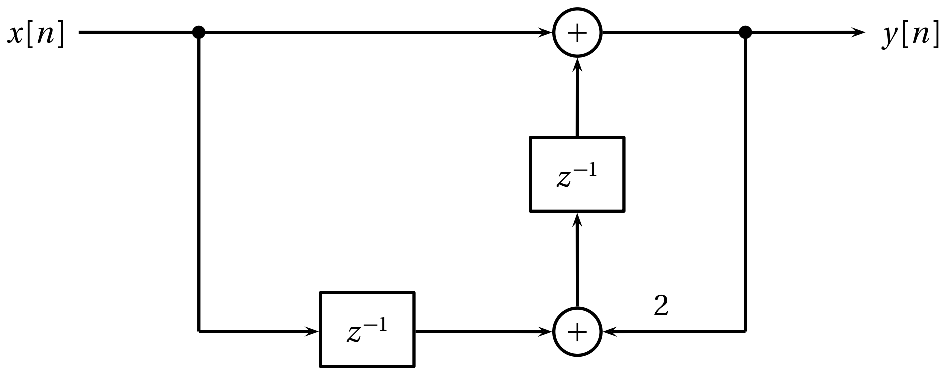
$$\begin{aligned} H(\omega) &= 1 - \text{rect}\left(\frac{\omega}{2\omega_0}\right) \\ &= \begin{cases} 0, & \omega < \omega_0 \\ 1, & \omega_0 < \omega \leq \pi \end{cases} \end{aligned}$$

This system is a high-pass filter.

(b) The input DTFT $X(\omega)$ will have non-zero components at $\omega = 0$, $|\omega| = \frac{\omega_0}{2}$, and $|\omega| = 2\omega_0$. Only the $|\omega| = 2\omega_0$ component will make it through the highpass filter, and therefore the output will be $y_1[n] = \sin(2\omega_0 n)$.

Question 23. *System analysis. This question is worth 12 points.*

Consider the causal system $\mathcal{H}$ described by the following block diagram:



(a) (3 Pts) Compute the transfer function $H(z)$ of the system.

(b) (2 Pts) Determine if the system is stable.

We now connect $\mathscr{H}$ in cascade to the output of a causal filter $\mathscr{G}$ described by the CCDE

$$y[n] = a\,y[n-1] + b\,x[n] + c\,x[n-1]$$

where a, b, c are real-valued, non-zero coefficients.

(c) (3 Pts) Draw a block diagram that implements $\mathscr{G}$ using a single delay block (that is, a single z^{-1} element).

(d) (4 Pts) Call $W(z)$ the transfer function implemented by the cascade of $\mathscr{G}$ and $\mathscr{H}$. Assume that you know the value of b; determine the range of values for a and c so that $W(z)$ is stable.

Solution:

(a) The CCDE implemented by the block diagram is

$$y[n] = x[n] + x[n-2] + 2y[n-1]$$

and thus the transfer function of the system is

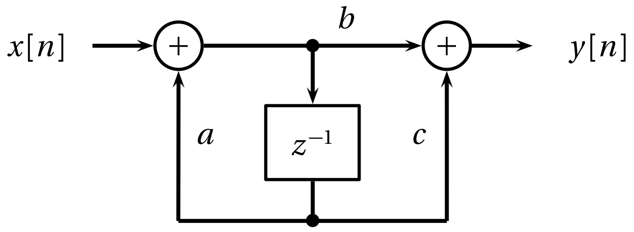
$$H(z) = \frac{1+z^{-2}}{1-2z^{-1}}$$

(b) The filter has a single pole in $z = 2$ and therefore it will not be stable.

(c) The transfer function of the pre-filter is

$$G(z) = \frac{b + cz^{-1}}{1 - az^{-1}}$$

which we can implement using a single delay as per the following block diagram



(d) The global transfer function of the cascade is

$$W(z) = G(z)H(z) = \frac{b + cz^{-1}}{1 - az^{-1}} \frac{1 + z^{-2}}{1 - 2z^{-1}}$$

In order for the cascade to be stable all poles must be inside the unit circle and therefore $G(z)$ must cancel the pole in $z = 2$ of $H(z)$; we can write

$$W(z) = b \frac{1 + (c/b)z^{-1}}{1 - 2z^{-1}} \frac{1 + z^{-2}}{1 - az^{-1}}$$

and so the pole in 2 is canceled if $c/b = -2$. The transfer function after cancellation is

$$W(z) = b \frac{1 + z^{-2}}{1 - az^{-1}}$$

and so we will also need $|a| < 1$ to ensure stability. In the end, we must have

$$\begin{cases} -1 < a < 1 \\ c = -2b. \end{cases}$$

Question 24. *Sampling and Interpolation (Sample Final): This question is worth 14 points.*

Find the CTFT for each of the following signals. Determine if it is band-limited or not. If yes, find the expression for the maximum frequency ω_N.

(a) (4 Pts) The continuous-time signal $x(t) = e^{j2\pi f_0 t} + \sin(2\pi f_1 t) + \cos(2\pi f_2 t)$, with $f_0 = 125, f_1 = 50, f_2 = 37.5$.

(b) (5 Pts) The continuous-time signal $y(t)$ obtained by passing $x(t) = \text{rect}\left(\frac{t}{2a_0}\right)$ through an LTI system with a frequency response $H(f) = 1 - \text{rect}\left(\frac{f}{2f_c}\right)$, where $a_0 > 0$ and $f_c > 0$ are both

real constants.

(c) (5 Pts) The continuous-time signal $x(t)$ obtained by using the zero-order local interpolation on the discrete-time signal

$$x[n] = \sum_{k=-N}^{N} \delta[n-k].$$

That is,

$$x(t) = \sum_{n=-N}^{N} x[n]\text{rect}\left(\frac{t-nT_s}{T_s}\right)$$

for some sampling interval T_s and integer $N > 0$.

Solution: *(a) (4 Pts)* $X(f) = \delta(f-f_0) - \frac{j}{2}(\delta(f-f_1) - \delta(f+f_1)) + (1/2)(\delta(f-f_2)) + \delta(f+f_2))$. This signal is band-limited and the highest frequency is then $F_N = f_0$.

(b) (5 Pts) since $X(f) = 2a_0\text{sinc}\left(2a_0 f\right)$ then $x(t)$ is not bandlimited. The filter is a highpass and so

$$Y(f) = H(f)X(f) = \begin{cases} 0, & f < f_c \\ 2a_0\text{sinc}\left(2a_0 f\right), & |f| > f_c \end{cases}$$

The signal $y(t)$ is not bandlimited.

(c) (5 Pts) Let $z_n(t) = \text{rect}\left(\frac{t-nT_s}{T_s}\right)$. Combining the rect/sinc pair with the shift-in-time property, we obtain $Z_n(\omega) = e^{-j\omega nT_s}\text{sinc}\left(\frac{T_s}{2\pi}\omega\right)$. Then, by linearity of the CTFT

$$X(f) = \sum_{n=-N}^{N} Z_n(\omega) = \sum_{n=-N}^{N} e^{-j\omega nT_s}\text{sinc}\left(T_s f\right)$$

This is not a band-limited signal.

Question 25. *FIR filters. This question is worth 18 points.*

Consider a causal FIR filter $\mathscr{H}$ whose real-valued impulse response $h[n]$ is nonzero for $n = 0, 1, \ldots, N-1$; the filter is not necessarily linear phase.

The filter is applied to its own impulse response, obtaining the output signal $y[n] = (h * h)[n]$:

(a) (4 Pts) Show that the signal $y[n]$ has at most $2N-1$ nonzero samples.

(b) (2 Pts) Show that if $N = 2$, $y[n]$ will always have three nonzero samples.

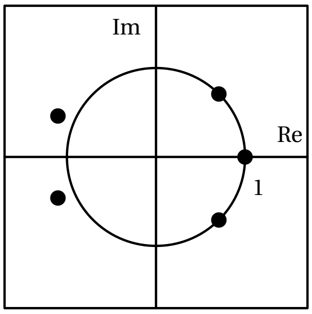
(c) (2 Pts) For $N = 3$, find an impulse response $h[n]$ so that $y[n]$ has only four nonzero samples.

We now apply the filter to its time-reversed impulse response, that is, to the noncausal input $h_r[n] = h[-n]$, and obtain the signal $w[n] = (h * h_r)[n]$.

(d) (4 Pts) Show that $w[n]$ is symmetric around zero, i.e. $w[n] = w[-n]$

(e) (2 Pts) Let $W(z)$ be the z-transform of $w[n]$ and assume $W(z_0) = 0$. Show that $W(1/z_0) = 0$

Assume that the zeros of $H(z)$, the z-transform of $h[n]$, are as shown in the following plot.



(f) [2p] Sketch the locations of the zeros of $Y(z)$

(g) [2p] Sketch the locations of the zeros of $W(z)$

Solution: Call $H(z)$ the transfer function of $\mathcal{H}$:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n};$$

note that $H(z)$ is a polynomial in z^{-1} of degree $N - 1$.

(a) From $y[n] = (h * h)[n]$, we obtain $Y(z) = H^2(z)$. Since $H(z)$ has degree $N - 1$, then $Y(z)$ has at most degree $2(N-1) = 2N-2$, which in turn implies that $Y(z)$ has at most $2N - 1$ nonzero coefficients.

(b) Let $H(z) = a + bz^{-1}$ for $a, b \in \mathbb{R}$ and $a, b \neq 0$. We have

$$Y(z) = (a + bz^{-1})^2 = a^2 + 2abz^{-1} + b^2z^{-2}$$

Since a and b are nonzero by hypothesis, all the coefficients in $Y(z)$ are nonzero as well.

(c) Proceeding as before, let $H(z) = a + bz^{-1} + cz^{-2}$; we have

$$Y(z) = (a + bz^{-1} + cz^{-2})^2 = a^2 + 2abz^{-1} + (2ac + b^2)z^{-2} + 2bcz^{-3} + c^2z^{-4}$$

Now we can choose a, b, c so that $(2ac + b^2) = 0$; for instance, with $a = b = 2$ and $c = -1$ we have

$$Y(z) = 4 + 8z^{-1} - 4z^{-3} + z^{-4}$$

which has only 4 nonzero coefficients.

(d) We have

$$w[n] = \sum_{n=-\infty}^{\infty} h_r[k]h[n-k] = \sum_{n=-\infty}^{\infty} h[-k]h[n-k]$$

and therefore

$$w[-n] = \sum_{n=-\infty}^{\infty} h[-k]h[-n-k] = \sum_{n=-\infty}^{\infty} h[-k]h[-(n+k)]$$

With the change of variable $m = n + k$ we have

$$w[-n] = \sum_{m=-\infty}^{\infty} h[-(m-n)]h[-m] = \sum_{m=-\infty}^{\infty} h[-m]h[n-m] = w[n]$$

(e) The z-transform of $w[n]$ is $W(z) = H(z)H_r(z)$ and we have

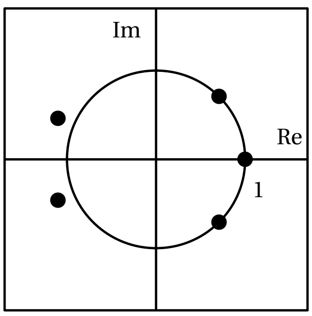
$$H_r(z) = \sum_{n=-N+1}^{0} h[-n]z^{-n} = \sum_{n=0}^{N-1} h[n]z^{n} = H(1/z).$$

With this

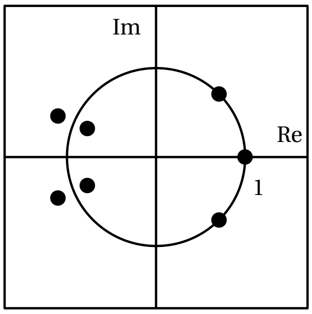
$$W(z) = H(z)H_r(z) = H(z)H(1/z)$$
$$W(1/z) = H(1/z)H_r(1/z) = H(1/z)H(z) = W(z)$$

and so, if $W(z_0) = 0$, then $W(1/z_0) = W(z_0) = 0$.

(f) Since $Y(z) = H^2(z)$, the zeros of $Y(z)$ will be in the same locations as the zeros of $H(z)$ (but each zero will have multiplicity 2):

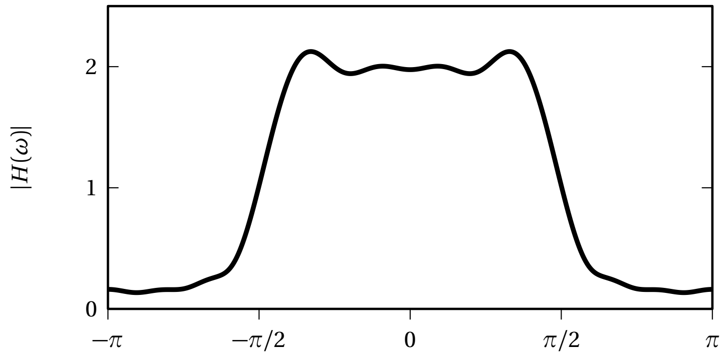


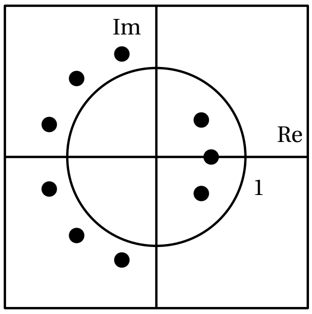
(g) Call $\{z_0, \ldots, z_{N-1}\}$ the set of zeros of $H(z)$; since $W(z) = H(z)H(1/z)$, its zeros will be the union of the zeros of $H(z)$ and of their reciprocals, $\{z_0, \ldots, z_{N-1}\} \cup \{1/z_0, \ldots, 1/z_{N-1}\}$. Zeros on the unit circle will increase their multiplicity whereas other zeros will produce an additional zero location at the reciprocal of their value



Question 26. *Linear-phase filtering. This question is worth 18 points.*

Consider a causal FIR filter $\mathscr{H}$ whose real-valued impulse response $h[n]$ is nonzero for $n = 0, 1, \ldots, N-1$; its magnitude response $|H(\omega)|$ and the location of the zeros of its transfer function $H(z)$ are shown in the following figures. Note that $\mathscr{H}$ is FIR but *not* a linear phase FIR.





Consider now a new causal FIR filter $\mathscr{H}_r$ whose impulse response $h_r[n]$ is obtained by "flipping" $h[n]$, that is,

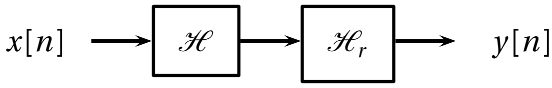
$$h_r[n] = \begin{cases} h[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Call $H_r(z)$ the transfer function of $\mathscr{H}_r$.

(a) (4 Pts) Using the information provided by the pole-zero plot displayed above, sketch as accurately as you can the locations of the zeros of $H_r(z)$.
(b) (2 Pts) Is $\mathscr{H}_r$ BIBO stable?

Hint: begin by finding the relationship between $H_r(z)$ and $H(z)$.

The two filters are now connected in cascade, as shown in the following figure; call $\mathscr{G}$ the equivalent FIR filter implemented by the cascade.



(c) (4 Pts) What is the maximum number of nonzero samples of the impulse response $g[n]$?
(d) (4 Pts) Using the information provided by the plot of the magnitude response displayed above, sketch as accurately as you can the magnitude response $|G(\omega)|$ and clearly show its value at $\omega = 0$.
(e) (4 Pts) Show that $\mathscr{G}$ is always a linear-phase FIR.

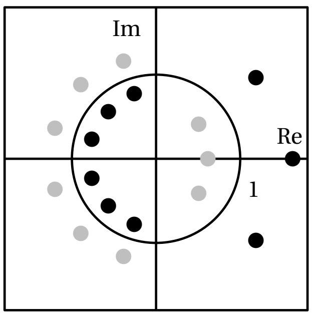
Solution: First of all, let's relate the transfer functions of $\mathscr{H}$ and $\mathscr{H}_r$; for $\mathscr{H}$ we have

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n};$$

and for $\mathscr{H}_r$ we have

$$\begin{aligned} H_r(z) &= \sum_{n=0}^{N-1} h[N-1-n]z^{-n} \qquad \leftarrow k = N-1-n \\ &= \sum_{k=0}^{N-1} h[k]z^{k-(N-1)} \\ &= z^{-(N-1)} \sum_{k=0}^{N-1} h[k]z^{k} \\ &= z^{-(N-1)} H(z^{-1}) \end{aligned}$$

(a) Call $r_0, \ldots, r_{N-1}$ the zeros of $H(z)$; since $H_r(z) = z^{-(N-1)}H(z^{-1})$ the zeros of $H_r(z)$ will be $1/r_0, \ldots, 1/r_{N-1}$; the pole-zero plot is the following (with the original zeros of $H(z)$ shown in gray).



(b) $\mathscr{H}_r$ is BIBO stable because it's FIR.

(c) Both $H(z)$ and $H_r(z)$ are simple polynomials of degree $N-1$. Since $G(z) = H(z)H_r(z)$, $G(z)$ will be a simple polynomial of degree at most $2(N-1)$. Since $G(z)$ is the z-transform of the impulse response $g[n]$, this implies that $g[n]$ has at most $2N-1$ nonzero values.

(d) Let's compute the frequency response of the cascade:

$$G(\omega) = H(\omega)H_r(\omega) = e^{-j(N-1)\omega}H(\omega)H(-\omega).$$

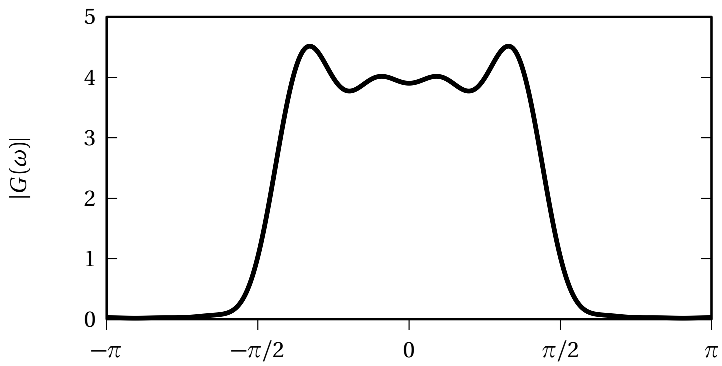
Since the filter coefficients for $\mathscr{H}$ are real-valued,

$$H(-\omega) = \sum_{n=0}^{N-1} h[n]e^{j\omega} = \sum_{n=0}^{N-1}\left(h[n]e^{j\omega}\right)^* = H^*(\omega)$$

and therefore

$$G(\omega) = |H(\omega)|^2\, e^{-j(N-1)\omega}.$$

From this, $|G(\omega)| = |H(\omega)|^2$ and, since $|H(1)| \approx 2$, then $|G(1)| \approx 4$:



(e) The simplest way to show that $\mathscr{G}$ has linear phase is to use the expression for its frequency response that we computed previously:

$$G(\omega) = |H(\omega)|^2\, e^{-j(N-1)\omega}.$$

The expression is a real-valued function multiplied by a linear-phase term with factor $-(N-1)$.

Alternatively, we can work in the time domain and show that the impulse response $g[n]$ is odd-length ($2N-1$) and symmetric and therefore $G(z)$ is a Type-I linear-phase FIR:

$$\begin{aligned}
g[n] &= \sum_{k=-\infty}^{\infty} h[k]h[k+N-1-n] \\
g[2N-2-n] &= \sum_{k=-\infty}^{\infty} h[k]h[k+N-1-(2N-2-n)] \\
&= \sum_{k=-\infty}^{\infty} h[k]h[k-(N-1)+n] \qquad \leftarrow m = k-(N-1)+n \\
&= \sum_{m=-\infty}^{\infty} h[m+(N-1)-n]h[m] = g[n]
\end{aligned}$$