COM-202 - Signal Processing

Selection of multiple choice and open questions from past exams

Question 1. DFT

Let $x[n] = \sin\left(\frac{2\pi}{N} pn\right)$ be a finite-length signal in $\mathbb{C}^N$ with $N > 2$ and p is a positive integer such that $1 < p < N/2$. Let $X[k]$ be k-th value of the N-point DFT of $x[n]$. Which one of the following statements is **true**?

- ☐ $jX[k] \in \mathbb{R}$
- ☐ $X[k] = |X[k]|$ for $k = 0, 1, \ldots, N-1$
- ☐ $|X[0]| = N/2$
- ☐ $X[k] = X[N-k]$ for $k = 0, 1, \ldots, N-1$

Question 2. DFT

Let $x[n] = \cos\left(\frac{2\pi}{N} pn\right)$ be a finite-length signal in $\mathbb{C}^N$ with $N > 2$ and p is a positive integer such that $1 < p < N/2$. Let $X[k]$ be the N-point DFT of $x[n]$. Which one of the following statements is **false**?

- ☐ $|X[N-p]| = N$
- ☐ $X[k] = |X[k]|$ for $k = 0, 1, \ldots, N-1$
- ☐ $|X[0]| = 0$
- ☐ $X[k] = X[N-k]$ for $k = 1, \ldots, N-1$

Question 3. DFT

Let $x[n]$ be a real-valued, finite-length signal in $\mathbb{C}^N$ whose N-point DFT is $X[k]$. Let $y[n]$ be a finite-length signal of length $2N$ obtained by concatenating two copies of $x[n]$:

$$y[n] = \begin{cases} x[n] & 0 \le n < N \\ x[n-N] & N \le n < 2N \end{cases}$$

Call $Y[k]$ the $2N$-point DFT of $y[n]$. Which one of the following statements is **true** for any $x[n]$?

- ☐ $Y[1] = 0$
- ☐ $Y[k] = 2X[k]$ for $k = 0, 1, \ldots, 2N-1$
- ☐ $Y[0] = X[0]$
- ☐ $Y[k] = X[k \mod 2]$ for $k = 0, 1, \ldots, 2N-1$

Question 4. DFT

Let $x[n]$ be a real-valued, finite-length signal in $\mathbb{C}^N$ whose N-point DFT is $X[k]$. Let $y[n]$ be a finite-length signal of length $2N$ obtained by appending N zeros to $x[n]$:

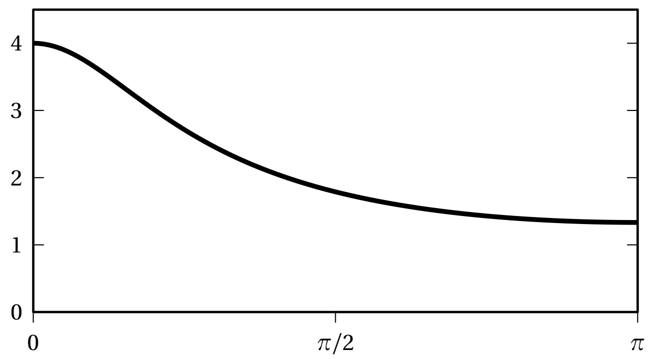
$$y[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & N \le n < 2N \end{cases}$$

Call $Y[k]$ the $2N$-point DFT of $y[n]$. Which one of the following statements is **true** for any $x[n]$?

- ☐ $Y[2k] = X[k]$
- ☐ $Y[k] = X[k]/2$ for $k = 0, 1, \ldots, 2N-1$
- ☐ $Y[k] = 0$ for $N \le k < 2N$
- ☐ $Y[k] = X[N-k]$ for $N \le k < 2N$

Question 5. DTFT

Consider the signal $x[n] = 2a^n\, u[n]$, with $a > 0$. The magnitude of its DTFT $X(\omega)$ is shown in the following plot over the $[0, \pi]$ interval. What is the value of a?



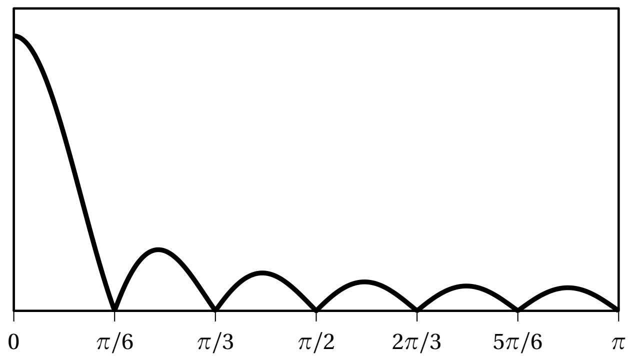
- [ ] 0.5
- [ ] 2
- [ ] It cannot be determined from this magnitude plot
- [ ] $\frac{1}{\sqrt{2}}$

Question 6. DTFT

Consider the discrete-time signal

$$x[n] = \begin{cases} 1 & 0 \le n < M \\ 0 & \text{otherwise} \end{cases}$$

The magnitude of its DTFT $X(\omega)$ is shown in the following plot over the $[0, \pi]$ interval. What is the value of the positive integer M?



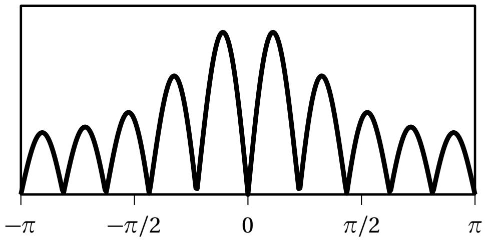
- [ ] 12
- [ ] 6
- [ ] The value cannot be determined from the plot because there are no units on the vertical axis
- [ ] 11

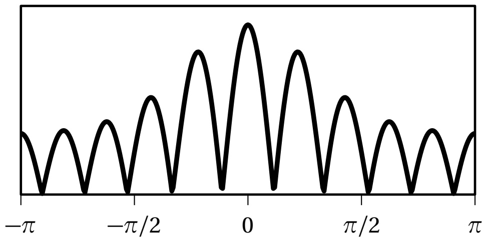
Question 7. DTFT

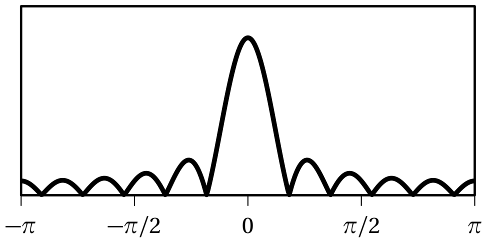
Consider the infinite-length discrete-time signal

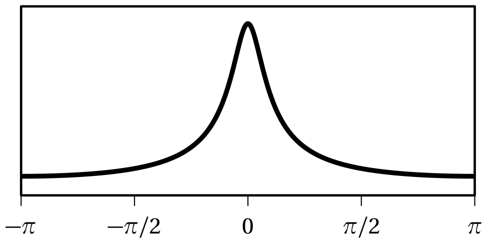
$$x[n] = \begin{cases} n^3 & |n| < M \\ 0 & \text{otherwise} \end{cases}$$

Which of the following plots could represent the magnitude of its DTFT?

☐ 

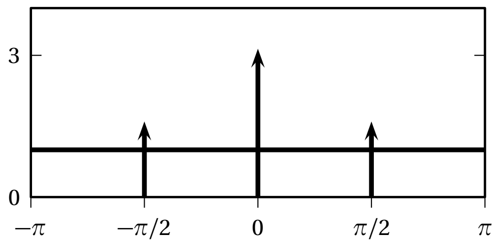
☐ 

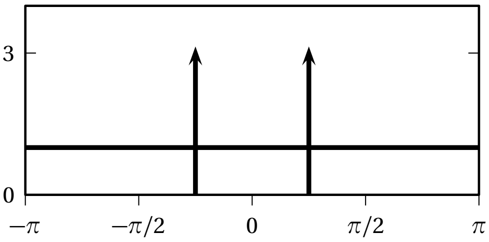
☐ 

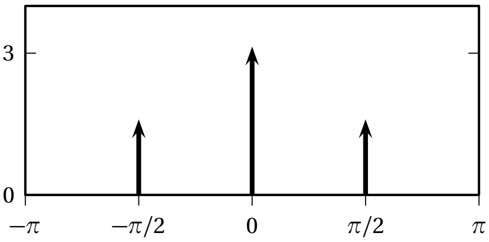
☐ 

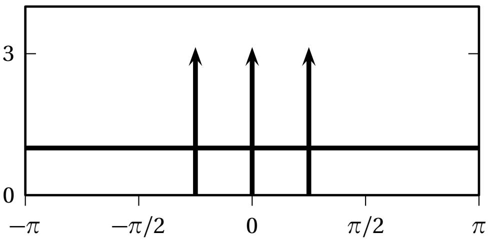
Question 8. DTFT

Consider the infinite-length discrete-time signal $x[n] = \delta[n] + \cos^2(\omega_0 n)$ with $\omega_0 = \pi/4$. Which one of the following plots represents its DTFT?

☐ 

☐ 

☐ 

☐ 

Question 9. SCQ-05

A discrete-time system is defined by an input-output relationship $y[n] = x[-n]$. Which of the following claims about the system is **true**?

- ☐ The system is linear
- ☐ The system is time-invariant
- ☐ The system is causal
- ☐ The system is memoryless

Question 10. Filters in Python

The following Python function applies a discrete-time filter to the array of values provided as an input; select which transfer function $H(z)$ is implemented by the code.

```
import numpy as np

def myfilter(data: np.ndarray) -> np.ndarray:
    w1, w2 = 0, 0
    for n in range(0, len(data)):
        w = data[n] + 1.4 * w1 - 0.6 * w2
```

```
        data[n] = 3 * w + 6 * w1 + 3 * w2
        w2 = w1
        w1 = w
    return data
```

- [ ] $H(z) = \frac{3+6z^{-1}+3z^{-2}}{1-1.4z^{-1}+0.6z^{-2}}$
- [ ] $H(z) = \frac{3+6z^{-1}+3z^{-2}}{1+1.4z^{-1}-0.6z^{-2}}$
- [ ] $H(z) = \frac{1}{3}\frac{1+1.4z^{-1}-0.6z^{-2}}{1+2z^{-1}+z^{-2}}$
- [ ] $H(z) = 4+6.4z^{-1}+2.7z^{-2}$

Question 11. Filters in Python

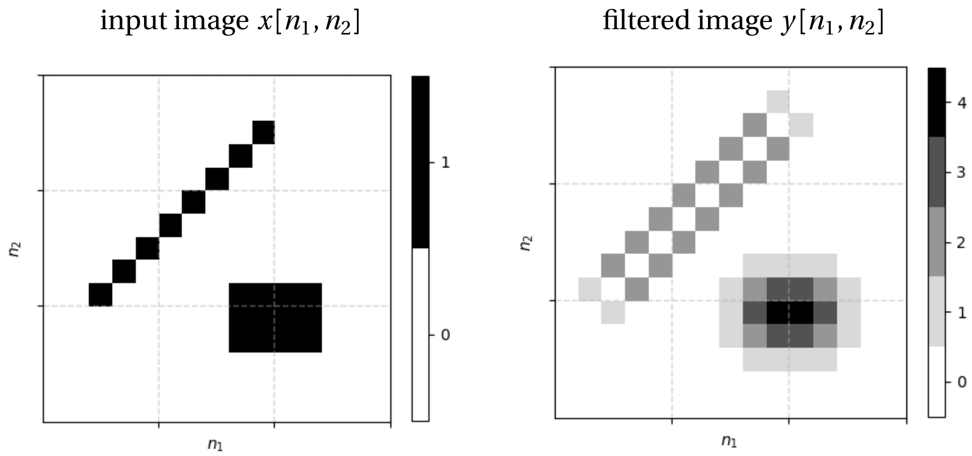
Select the output obtained by running the following Python code block:

```
import numpy as np

def myfun(data: np.ndarray) -> np.ndarray:
    y = np.zeros(len(data))
    for n in range(0, len(data)):
        y[n] = 0.5 * y[n-1] + data[n]
    return y

print(myfun([2, 0, 0, 0]))
```

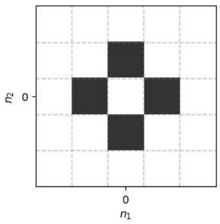
- [ ] `[2 1 0.5 0.25]`
- [ ] `[1 0 0 0]`
- [ ] the code crashes and returns an `IndexError`
- [ ] `[1 0.5 0.25 0.125]`

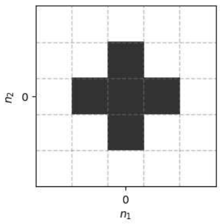
Question 12. SCQ-08

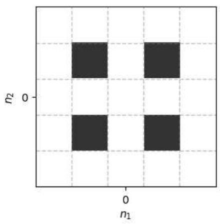
The 15×15 image $x[n_1, n_2]$ shown in the left panel of the following figure is filtered with a 2D FIR to produce the output image $y[n_1, n_2]$ shown in the right panel. The value of each pixel in both images is shown as a grasyscale level using the shades of gray shown to the right of each image.

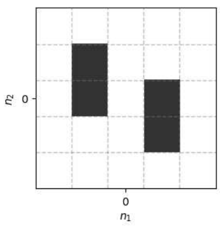


Which of the following plots shows the impulse response of the 2D FIR used to filter the image? (Note that in all of the following plots, a white pixel indicates that $h[n_1, n_2] = 0$ while a black pixel indicates that $h[n_1, n_2] = 1$)

☐ 

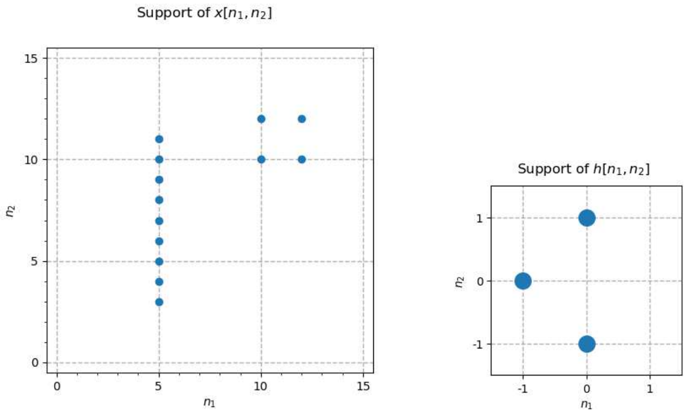
☐ 

☐ 

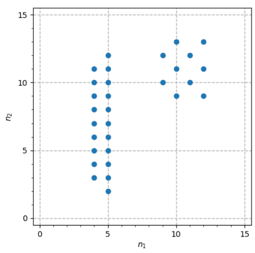
☐ 

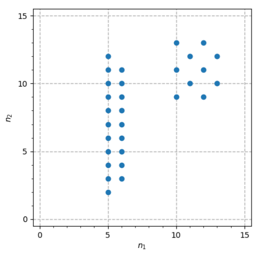
Question 13. SCQ-08

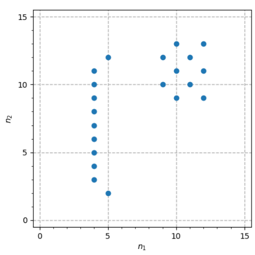
Consider a 16×16 image $x[n_1, n_2]$ and a two-dimensional FIR filter $h[n_1, n_2]$; the support of the image and the filter are shown in the following figure; remember that the support of a 2D signal is the set of coordinates $[n_1, n_2]$ for which the signal is nonzero and a support plot shows these coordinates with a dot.

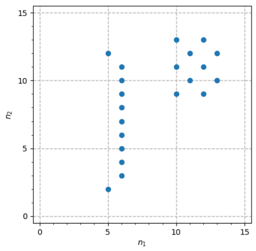


Which of the following plots shows the support of the image obtained by filtering $x[n_1, n_2]$ with $h[n_1, n_2]$?

☐ 

☐ 

☐ 

☐ 

Question 14. z-transform

Given an infinite-length signal $x[n]$ with z-transform $X(z)$, we build a signal $y[n]$ as

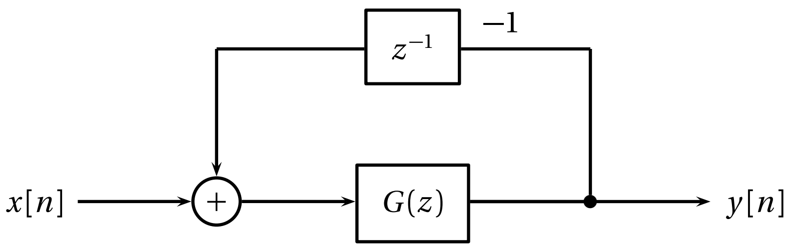
$$y[n] = \begin{cases} x[n/3] & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

What is the expression for the z-transform of $y[n]$?

- ☐ $Y(z) = X(z^{1/3})$
- ☐ $Y(z) = X((-z)^{1/3})$
- ☐ $Y(z) = X(z^3)$
- ☐ $Y(z) = X(3z)$

Question 15. z-transform

Select the transfer function implemented by the following block diagram:



- ☐ $H(z) = \dfrac{G(z)}{1 + z^{-1}G(z)}$
- ☐ $H(z) = \dfrac{1}{1 + z^{-1}G(z)}$
- ☐ $H(z) = \dfrac{1}{1 - z^{-1}G(z)}$
- ☐ $H(z) = \dfrac{G(z)}{1 - z^{-1}G(z)}$

Question 16. SCQ-14

The Leaky Integrator system,

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n], \qquad 0 < \lambda < 1$$

is an example of

- ☐ IIR lowpass filter
- ☐ FIR lowpass filter
- ☐ unstable LTI system
- ☐ memoryless LTI system

Question 17. SCQ-15

A sampling system samples continuous-time signals with sampling period $T = 0.5 \cdot 10^{-3}$. Let $x_1(t) = \cos(2\pi f_0 t) + \sin(2\pi f_1 t)$ and $x_2(t) = \cos(2\pi f_2 t)$, with

$$f_0 = 400$$
$$f_1 = 225$$
$$f_2 = 600$$

According to the sampling theorem, which of the following signals can be sampled with sampling period T_s without aliasing?

- ☐ only $x_1(t)$
- ☐ only $x_2(t)$
- ☐ both $x_1(t)$ and $x_2(t)$
- ☐ neither

Question 18. SCQ-15

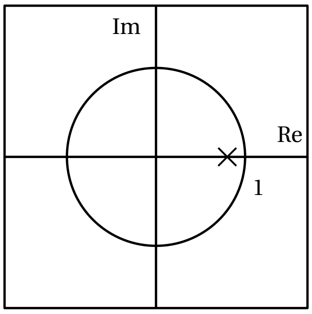
Let $x(t)$ be a band-limited signal with CTFT satisfies $|X(f)| = 0$ for $|f| \geq 400$ Hz. Which of the following statements about $x(t)$ is **true**?

- ☐ According to the sampling theorem, the signal $x[n] = x\left(\frac{n}{800}\right)$ contains all the information needed to reconstruct $x(t)$ without aliasing
- ☐ According to the sampling theorem, the sampling frequency $F_s = 400$ is sufficient to sample $x(t)$ without aliasing

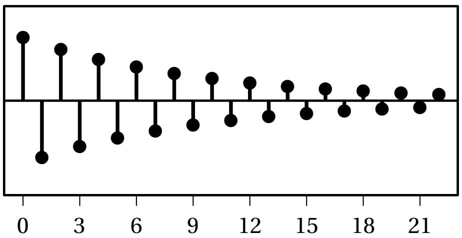
- ☐ According to the sampling theorem, it is not possible to sample $x(t)$ without aliasing
- ☐ According to the sampling theorem, to avoid aliasing we need to sample $x(t)$ with a sampling period smaller than 400.

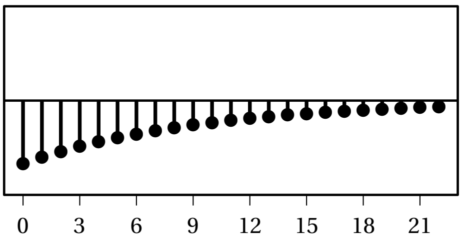
Question 19. Poles, zeros, and impulse response

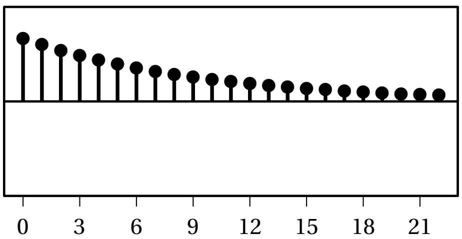
Consider a causal IIR filter with impulse response $h[n]$ and transfer function $H(z)$. The pole-zero plot for $H(z)$ is shown in the following figure and note that the multiplicity of the pole could be greater than one.

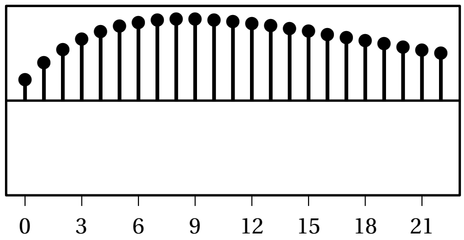


Select the impulse response that is **not** compatible with the given pole-zero plot.

- ☐ 

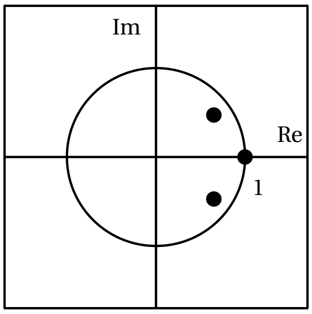
- ☐ 

- ☐ 

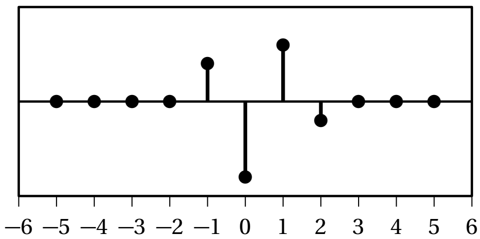
☐ 

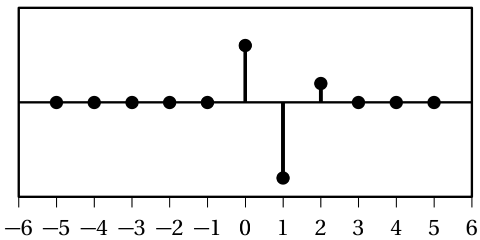
Question 20. Poles, zeros, and impulse response

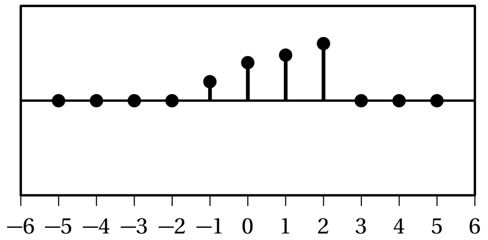
Consider an FIR filter with real-valued impulse response $h[n]$ and transfer function $H(z)$. The locations of the zeros of $H(z)$ are shown in the following figure and each zero has multiplicity one.

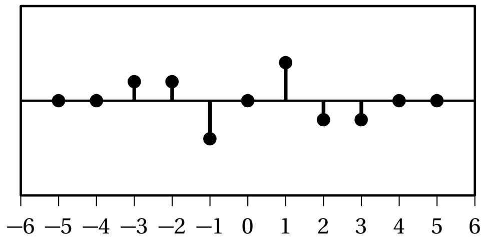


Select the only impulse response that is compatible with the given pole-zero plot. (For each impulse response, every value not shown in the plot is equal to zero.)

☐ 

☐ 

☐ 

☐ 

Question 21. *System analysis. This question is worth 10 points.*

Consider the causal system $\mathscr{H}$ defined by the following CCDE:

$$y[n] = y[n-1] + x[n] - x[n-1] - 2x[n-2] + 2x[n-3]$$

(a) (4 Pts) Compute the transfer function of the system and show that it implements an FIR filter.

(b) (2 Pts) Show that the causal inverse filter with transfer function $G(z) = 1/H(z)$ is not stable.

(c) (4 Pts) Consider now a causal FIR filter $\mathscr{W}$ with impulse response

$$w[n] = \delta[n] + (1/2)\delta[n-1] + (1/4)\delta[n-2].$$

Show that the filter $\mathscr{W}$ has a stable causal inverse.

Question 22. *LTI system. This question is worth 8 points.*

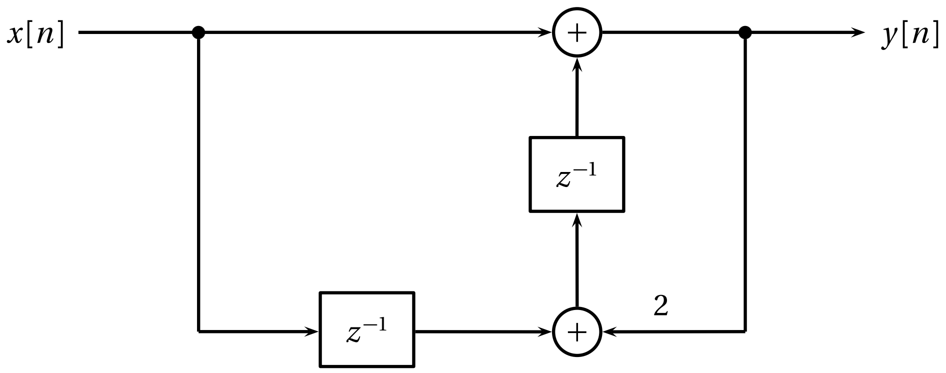
A discrete-time LTI system has an impulse response $h[n] = \delta[n] - \frac{\omega_0}{\pi}\mathrm{sinc}\left(\frac{\omega_0}{\pi}n\right)$ for some $0 < \omega_0 < \frac{\pi}{2}$.

(a) (4 Pts) Find the frequency response $H(\omega)$ for this system. Is the LTI system a low-pass, high-pass, or band-pass filter?

(b) (4 Pts) This system is applied to an input signal $x_1[n] = 1 + \sin\left(\frac{\omega_0}{2} n\right) + \sin(2\omega_0 n)$. Find the output signal $y_1[n]$.

Question 23. *System analysis. This question is worth 12 points.*

Consider the causal system $\mathscr{H}$ described by the following block diagram:



(a) (3 Pts) Compute the transfer function $H(z)$ of the system.

(b) (2 Pts) Determine if the system is stable.

We now connect $\mathscr{H}$ in cascade to the output of a causal filter $\mathscr{G}$ described by the CCDE

$$y[n] = a\,y[n-1] + b\,x[n] + c\,x[n-1]$$

where a, b, c are real-valued, non-zero coefficients.

(c) (3 Pts) Draw a block diagram that implements $\mathscr{G}$ using a single delay block (that is, a single z^{-1} element).

(d) (4 Pts) Call $W(z)$ the transfer function implemented by the cascade of $\mathscr{G}$ and $\mathscr{H}$. Assume that you know the value of b; determine the range of values for a and c so that $W(z)$ is stable.

Question 24. *Sampling and Interpolation (Sample Final): This question is worth 14 points.*

Find the CTFT for each of the following signals. Determine if it is band-limited or not. If yes, find the expression for the maximum frequency ω_N.

(a) (4 Pts) The continuous-time signal $x(t) = e^{j2\pi f_0 t} + \sin(2\pi f_1 t) + \cos(2\pi f_2 t)$, with $f_0 = 125, f_1 = 50, f_2 = 37.5$.

(b) (5 Pts) The continuous-time signal $y(t)$ obtained by passing $x(t) = \text{rect}\left(\frac{t}{2a_0}\right)$ through an LTI system with a frequency response $H(f) = 1 - \text{rect}\left(\frac{f}{2f_c}\right)$, where $a_0 > 0$ and $f_c > 0$ are both real constants.

(c) (5 Pts) The continuous-time signal $x(t)$ obtained by using the zero-order local interpolation on the discrete-time signal

$$x[n] = \sum_{k=-N}^{N} \delta[n-k].$$

That is,

$$x(t) = \sum_{n=-N}^{N} x[n]\text{rect}\left(\frac{t - nT_s}{T_s}\right)$$

for some sampling interval T_s and integer $N > 0$.

Question 25. *FIR filters. This question is worth 18 points.*

Consider a causal FIR filter $\mathscr{H}$ whose real-valued impulse response $h[n]$ is nonzero for $n = 0, 1, \ldots, N-1$; the filter is not necessarily linear phase.

The filter is applied to its own impulse response, obtaining the output signal $y[n] = (h * h)[n]$:

(a) (4 Pts) Show that the signal $y[n]$ has at most $2N-1$ nonzero samples.

(b) (2 Pts) Show that if $N = 2$, $y[n]$ will always have three nonzero samples.

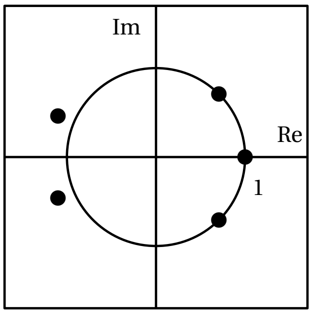
(c) (2 Pts) For $N = 3$, find an impulse response $h[n]$ so that $y[n]$ has only four nonzero samples.

We now apply the filter to its time-reversed impulse response, that is, to the noncausal input $h_r[n] = h[-n]$, and obtain the signal $w[n] = (h * h_r)[n]$.

(d) (4 Pts) Show that $w[n]$ is symmetric around zero, i.e. $w[n] = w[-n]$

(e) (2 Pts) Let $W(z)$ be the z-transform of $w[n]$ and assume $W(z_0) = 0$. Show that $W(1/z_0) = 0$

Assume that the zeros of $H(z)$, the z-transform of $h[n]$, are as shown in the following plot.

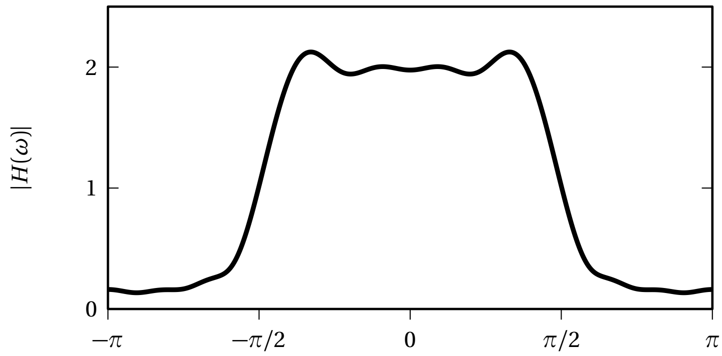


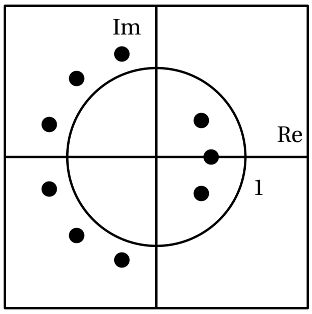
(f) [2p] Sketch the locations of the zeros of $Y(z)$

(g) [2p] Sketch the locations of the zeros of $W(z)$

Question 26. *Linear-phase filtering. This question is worth 18 points.*

Consider a causal FIR filter $\mathscr{H}$ whose real-valued impulse response $h[n]$ is nonzero for $n = 0, 1, \ldots, N-1$; its magnitude response $|H(\omega)|$ and the location of the zeros of its transfer function $H(z)$ are shown in the following figures. Note that $\mathscr{H}$ is FIR but *not* a linear phase FIR.





Consider now a new causal FIR filter $\mathscr{H}_r$ whose impulse response $h_r[n]$ is obtained by "flipping" $h[n]$, that is,

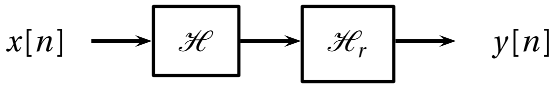
$$h_r[n] = \begin{cases} h[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Call $H_r(z)$ the transfer function of $\mathscr{H}_r$.

(a) (4 Pts) Using the information provided by the pole-zero plot displayed above, sketch as accurately as you can the locations of the zeros of $H_r(z)$.
(b) (2 Pts) Is $\mathscr{H}_r$ BIBO stable?

Hint: begin by finding the relationship between $H_r(z)$ and $H(z)$.

The two filters are now connected in cascade, as shown in the following figure; call $\mathscr{G}$ the equivalent FIR filter implemented by the cascade.



(c) (4 Pts) What is the maximum number of nonzero samples of the impulse response $g[n]$?
(d) (4 Pts) Using the information provided by the plot of the magnitude response displayed above, sketch as accurately as you can the magnitude response $|G(\omega)|$ and clearly show its value at $\omega = 0$.
(e) (4 Pts) Show that $\mathscr{G}$ is always a linear-phase FIR.