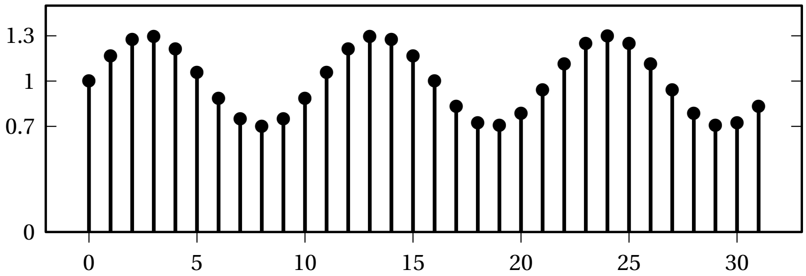
COM 202 - Signal Processing

Sample Midterm

Multiple Choice Questions: each multiple choice question is worth 4 points.

Exercise 1.

Consider the finite-length discrete-time signal $\mathbf{x} \in \mathbb{C}^{32}$ shown in this figure:



What is the DFT of this signal?

a)

$$X(k) = \begin{cases} 32, & k = 0, \\ \frac{24}{5} e^{-j\pi/2}, & k = 3, \\ \frac{24}{5} e^{j\pi/2}, & k = 29, \\ 0, & \text{otherwise} \end{cases}$$

b)

$$X(k) = \begin{cases} \frac{24}{5}, & k = 3 \text{ and } k = 29, \\ 0, & \text{otherwise} \end{cases}$$

c)

$$X(k) = \begin{cases} \frac{128}{5}, & k = 0, \\ \frac{48}{5} e^{-j\pi}, & k = 3, \\ \frac{24}{5} e^{j\pi/4}, & k = 29, \\ 0, & \text{otherwise} \end{cases}$$

d)

$$X(k) = \begin{cases} 32, & k = 1, \\ 24, & k = 7, \\ 16, & k = 30, \\ 0, & \text{otherwise} \end{cases}$$

Solution: *The signal is a periodic oscillation completing 3 cycles over 32 samples; it oscillates* *around the value 1, with a maximum positive amplitude equal to 0.3, so:*

$$x[n] = 1 + \frac{3}{10}\sin\left(2\pi\frac{3}{32}n\right).$$

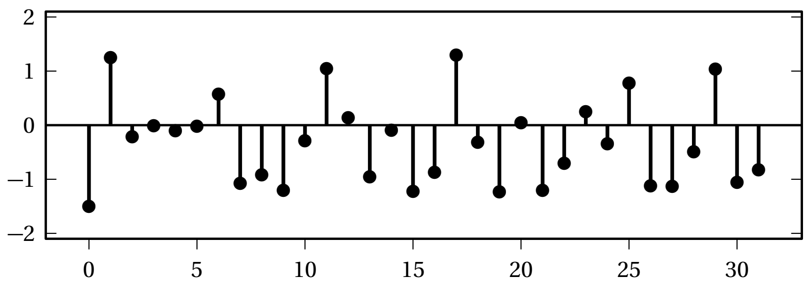
By linearity, its DFT is

$$X[k] = 32\,\delta[k] - j\frac{3}{10}\frac{32}{2}(\delta[k-3] - \delta[k-(32-3)]$$
$$= \begin{cases} 32 & k=0 \\ -(24/5)j & k=3 \\ (24/5)j & k=29 \\ 0 & \textit{otherwise} \end{cases}$$

which is given by option a).

Exercise 2.

The 32 samples of the discrete-time signal **x** shown in the figure below were obtained using a random number generator.



Say if it is possible to determine 32 complex numbers $C_0, C_1, \ldots, C_{31}$ so that the signal in the plot can be expressed as

$$x[n] = \sum_{k=0}^{31} C_k e^{j\frac{\pi}{16}nk}.$$

a) Yes, $C_k = X[k]/N$, where $\mathbf{X} = \text{DFT}\{\mathbf{x}\}$.

b) No, **x** doesn't look like a linear combination of complex exponentials.

c) No, because **x** is real-valued.

d) Yes, $C_k = X[k]$, where $\mathbf{X} = \text{DFT}\{\mathbf{x}\}$.

Solution: *The correct answer is option a). Any length-N signal can be represented exactly as a linear combination of the N DFT basis vectors; the expression for* **y** *is equivalent to an* *inverse DFT except for the leading scaling factor $1/N$ but, by setting $C_k = X[k]/N$, $\mathbf{y}$ becomes the inverse DFT of the DFT of* $\mathbf{x}$.

Exercise 3.

Given the matrix $\mathbf{H}$ below, for what values of c_1, c_2, c_3, c_4 do the rows of $\mathbf{H}$ form an *orthonormal* basis in $\mathbb{C}^4$?

$$\mathbf{H} = \begin{bmatrix} c_1 & c_1 & c_1 & c_1 \\ c_2 & c_2 & -c_2 & -c_2 \\ c_3 & -c_3 & 0 & 0 \\ 0 & 0 & c_4 & -c_4 \end{bmatrix}$$

a) $c_1 = c_2 = \frac{1}{2}$ and $c_3 = c_4 = \frac{1}{\sqrt{2}}$

b) None.

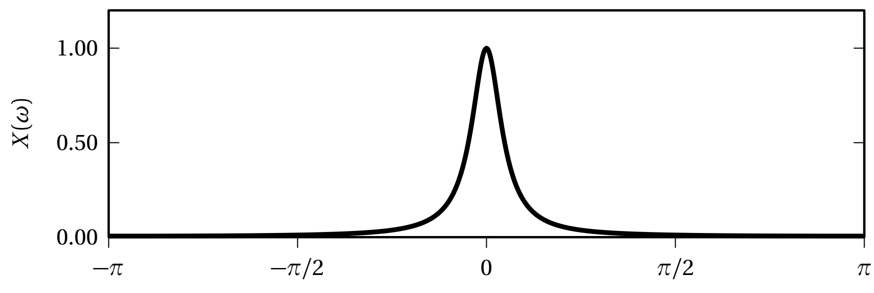
c) $c_1 = c_2 = \frac{1}{4}$ and $c_3 = c_4 = \frac{1}{2}$

d) $c_1 = c_2 = c_3 = c_4 = 1$

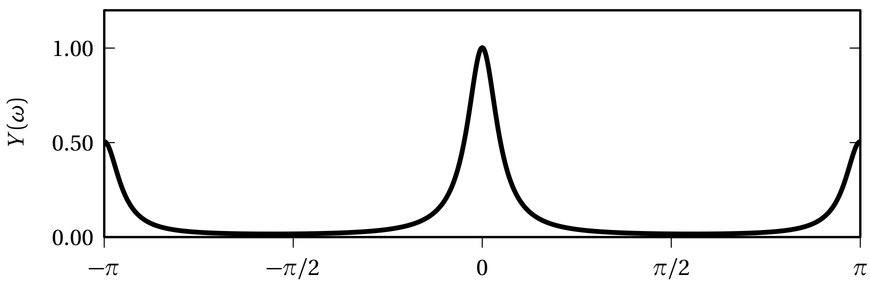
Solution: *The answer is a). Any non-zero values for c_1, c_2, c_3, c_4 would give us an orthogonal basis. But, among the provided options, these values are the only ones that would give us a basis that is orthonormal.*

Exercise 4.

Consider the infinite-length signal $\mathbf{x}$ whose DTFT $X(\omega)$ is shown in the following picture:



Consider now the following plot showing $Y(\omega)$, the DTFT of the infinite-length signal $\mathbf{y}$:



What is the correct definition of $\mathbf{y}$?

a) $y[n] = x[n] + \frac{1}{2}e^{j\pi n}x[n]$

b) $y[n] = x[n] + \frac{1}{2}x[n - \pi n]$

c) $y[n] = \frac{1}{2}e^{-j\pi n}x[n] + \frac{1}{2}e^{j\pi n}x[n]$

d) $y[n] = x[n]\left(1 + \left(\frac{1}{2}\right)^{|n|}\right)$

Solution: *The correct answer is a). Multiplication by $e^{j\pi n}$ amounts to a shift of π of the DTFT, which is what is observed here.*

Exercise 5.

Consider a causal filter with transfer function

$$H(z) = \frac{1+7z^{-4}}{4+z^{-2}}$$

Which of the following statements is correct?

a) The region of convergence of the transfer function includes the unit circle

b) The filter is FIR

c) The filter is unstable

d) The filter has four poles

Solution:

- **The region of convergence of the transfer function includes the unit circle**
- *The filter is FIR*
- *The filter is unstable*
- *The filter has four poles*

The system has two poles in $z = \pm j/2$; since it has poles, it is IIR; since the poles are less than one in magnitude they system is stable, and therefore the ROC of the transfer function includes the unit circle

---

Exercise 6.

What is the output obtained after running the following Python code?

```
import numpy as np

N = 4
w = np.pi / N
a = np.arange(N)
print( np.dot(np.exp(-2j * w * np.outer(a, a)), np.ones(N)) )
```

a) `[1.0 1.0 1.0 1.0]`

b) `[3.14159265359j 0.0 0.0 -3.14159265359j]`

c) `[4.0 0.0 0.0 0.0]`

d) `[1.0 1.0j -1.0 -1.0j]`

Solution: *The correct answer is (c):*

The code computes the DFT of a length-N signal whose samples are equal to one:

- a *is the vector* $[0\ 1\ \dots\ N-1]$
- `np.outer(a, a)` *yields a* $N \times N$ *matrix* $\mathbf{V}$ *whose entries are* $V[n,k] = nk$, $0 \le n,k < N$
- `np.exp(-2j * w * np.outer(a, a))` *yields the* $N \times N$ *Fourier matrix* $\mathbf{W}$*, with* $W[n,k] = e^{-(2\pi/N)nk}$
- `np.dot(np.exp(-2j * w * np.outer(a, a)), np.ones(N))` *returns the matrix-vector product* $\mathbf{Wx}$ *with* $\mathbf{x} = [1\ 1\ 1\ 1]^T$

Since $\mathrm{DFT}\{\mathbf{1}\} = N\boldsymbol{\delta}$*, the correct answer is (c).*

Exercise 7.

A discrete-time system is defined by an input-output relationship $y[n] = x[-n]$. Which of the following claims about the system is true?

a) The system is linear

b) The system is time-invariant

c) The system is causal

d) The system is LTI

Solution: *The answer is a). The system is linear, but not time-invariant or causal. Since it is not time-invariant, it is also not LTI.*

Exercise 8.

Which of the following claims about properties of infinite-length discrete-time signals is false?

- The signal $x[n] = \sin(\frac{n}{2}) + \cos(\frac{n}{3})$ is periodic
- If $x[n]$ has finite energy, then $\lim_{n\to\infty} |x[n]| = 0$
- A non-zero bounded periodic discrete-time signal must have finite power
- Given a real $a > 0$, the signal $x[n] = e^{-an}u[n]$ is in ℓ_2

Solution:

- *(F) The signal* $x[n] = \sin(\frac{n}{2}) + \cos(\frac{n}{3})$ *is periodic*
- *(T) If* $x[n]$ *has finite energy, then* $\lim_{n\to\infty} |x[n]| = 0$
- *(T) A non-zero bounded periodic discrete-time signal must have finite power*
- *(T) Given a real* $a > 0$*, the signal* $x[n] = e^{-an}u[n]$ *is in* ℓ_2

Exercise 9.

LTI Systems: This question is worth 20 points.

An LTI system is given by the following difference equation

$$y[n] = \lambda_1 y[n-1] + \lambda_2 x[n]$$

where $\lambda_1, \lambda_2 > 0$ are real numbers.

(a) Assume that the system satisfies the condition of initial rest. That is, if $x[n] = 0$ for all $n < n_0$, then $y[n] = 0$ for all $n < n_0$. Find the impulse response of the system. Is this system an FIR or an IIR filter?

(b) For which values of λ_1 and λ_2 is this system stable? Justify your answer.

(c) Let $\lambda_1 = \lambda_2 = \frac{1}{2}$. Given the input signal $x[n] = \left(\frac{1}{4}\right)^n u[n]$ to the system, compute the output $y[n]$.

Solution:

(a) *The impulse response of the system can be found by induction. First, we use the initial rest condition. Since $\delta[n] = 0$ for all $n < 0$, $h[n] = 0$ for all $n < 0$. Then,*

$$h[0] = \lambda_1 h[-1] + \lambda_2 \delta[0] \qquad = \lambda_2$$
$$h[1] = \lambda_1 h[0] + \lambda_2 \delta[1] \qquad = \lambda_1 \lambda_2$$

Now, assume

$$h[n] = \lambda_1^n \lambda_2 u[n]$$

Clearly, the assumption holds for all $n \leq 1$. Then, for all $n \geq 1$, by induction,

$$\begin{aligned} h[n+1] &= \lambda_1 h[n] + \lambda_2 \delta[n+1] \\ &= \lambda_1 \left(\lambda_1^n \lambda_2 u[n]\right) \\ &= \lambda_1^{n+1} \lambda_2 \end{aligned}$$

Thus, the impulse response is

$$h[n] = \lambda_2 \lambda_1^n u[n].$$

The system is an IIR filter.

(b) *The system is stable if* $|\lambda_1| < 1$. *Otherwise,* $\sum_{n=-\infty}^{\infty} |h[n]| \to \infty$.

(c) *This is similar to Homework 8 Exercise 1-(b).*

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k u[k] \left(\frac{1}{2}\right)^{n+1-k} u[n-k] \\
&= \left(\frac{1}{2}\right)^{n+1} \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k u[k] \left(\frac{1}{2}\right)^{-k} u[n-k] \\
&= \left(\frac{1}{2}\right)^{n+1} \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k \left(\frac{1}{2}\right)^{-k} u[k]u[n-k] \\
&= \left(\frac{1}{2}\right)^{n+1} \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k]u[n-k]
\end{aligned}$$

First, we notice the elements of the sum will be zero for all negative k because of $u[k]$. Then, they will be zero for all $k > n$ because of $u[n-k]$. Lastly, the sum from 0 to n is meaningful only if $n \geq 0$, so we add a factor of $u[n]$ to reflect this.

$$\begin{aligned}
y[n] &= \left(\frac{1}{2}\right)^{n+1} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k u[n-k] \\
&= \left(\frac{1}{2}\right)^{n+1} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k u[n] \\
&= \left(\frac{1}{2}\right)^{n+1} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\left(\frac{1}{2}\right)} u[n] \\
&= \left(\frac{1}{2}\right)^{n} \left(1-\left(\frac{1}{2}\right)^{n+1}\right) u[n] \\
&= \left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^{n} u[n].
\end{aligned}$$

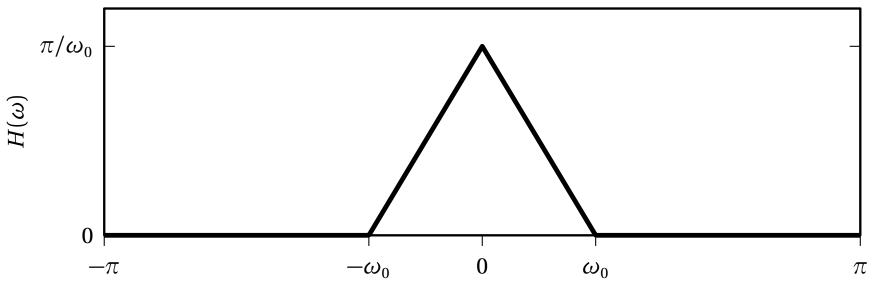
Exercise 10.

DTFT. This question is worth 22 points.

Consider an LTI system with frequency response

$$H(\omega) = \begin{cases} \frac{\pi}{\omega_0}\left(1 - \left|\frac{\omega}{\omega_0}\right|\right), & |\omega| \le \omega_0, \\ 0, & \text{otherwise,} \end{cases}$$

for some value $0 \le \omega_0 < \frac{\pi}{2}$. The real-valued frequency response has thus a triangular shape as in the following illustration:



(a) Is the system a low-pass, a high-pass, or a band-pass filter?

(b) Is the system causal?

(c) Find the system's response to the input signal $x[n] = 1 + \cos\left(\frac{\omega_0}{2} n\right) + \cos(2\omega_0 n)$.

(d) Let $\mathbf{h}$ be the impulse response of the system. Show that $\sum_{n\in\mathbb{Z}} |h[n]|^2 = \frac{\pi}{3\omega_0}$.

Solution: (a) *The system is a low-pass filter.* (b) *Since the frequency response is purely real, the impulse response must be even, that is, $h[n] = h[-n]$. Thus, the system cannot be a causal filter.* (c) *The DTFT of the input is*

$$X(\omega) = \tilde{\delta}(\omega) + \left(\tilde{\delta}(\omega - \omega_0/2) + \tilde{\delta}(\omega + \omega_0/2)\right)/2 + \left(\tilde{\delta}(\omega - 2\omega_0) + \tilde{\delta}(\omega + 2\omega_0)\right)/2$$

In the frequency domain, the DTFT of the output is $Y(\omega) = H(\omega)X(\omega)$; since $H(0) = \pi/\omega_0$, $H(\pm\omega_0/2) = \pi/(2\omega_0)$ and $H(\pm 2\omega_0) = 0$

$$Y(\omega) = (\pi/\omega_0)\tilde{\delta}(\omega) + (\pi/(2\omega_0))\left(\tilde{\delta}(\omega - \omega_0/2) + \tilde{\delta}(\omega + \omega_0/2)\right)/2$$

whose inverse Fourier transform is

$$y[n] = \frac{\pi}{\omega_0}\left(1 + \frac{1}{2}\cos(\omega_0 n/2)\right)$$

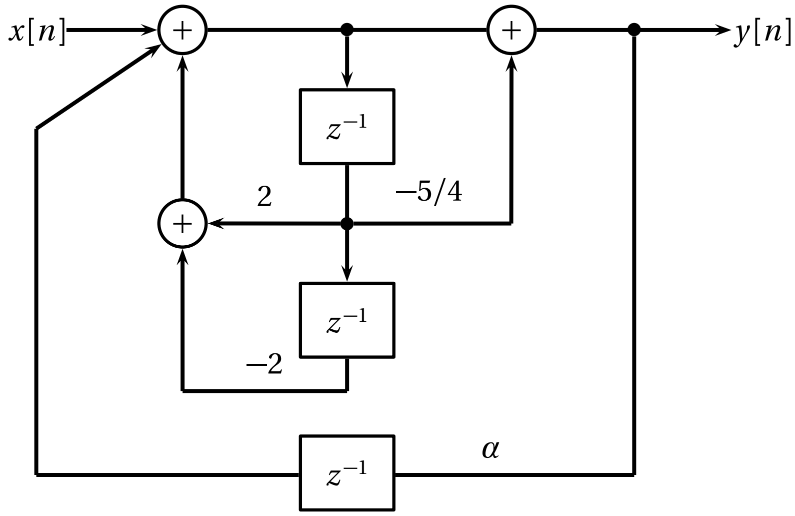
(d) *Use Parseval equality*

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} |h[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega \\
&= \frac{\pi}{2\omega_0^2}\int_{-\omega_0}^{\omega_0}\left(1-\left|\frac{\omega}{\omega_0}\right|\right)^2 d\omega \\
&= \frac{\pi}{2\omega_0^2}\left[\int_{-\omega_0}^{\omega_0} 1+\frac{\omega^2}{\omega_0^2} d\omega - \left(\frac{2}{\omega_0}\right)\int_{-\omega_0}^{\omega_0} |\omega|\, d\omega\right] \\
&= \frac{\pi}{2\omega_0^2}\left[2\omega_0+\frac{2\omega_0}{3} - \left(\frac{2}{\omega_0}\right)\int_{-\omega_0}^{\omega_0} |\omega|\, d\omega\right] \\
&= \frac{\pi}{2\omega_0^2}\left[2\omega_0+\frac{2\omega_0}{3} - \left(\frac{2}{\omega_0}\right)\omega_0^2\right] \\
&= \frac{\pi}{2\omega_0^2}\frac{2\omega_0}{3} \\
&= \frac{\pi}{3\omega_0}
\end{aligned}
$$

Exercise 11.

System analysis. This question is worth 26 points.

Consider the causal LTI system shown in the following block diagram and let $\alpha = -2$:



(a) Compute the system's transfer function $H(z)$

(b) Plot the system's poles and zeros on the complex plane

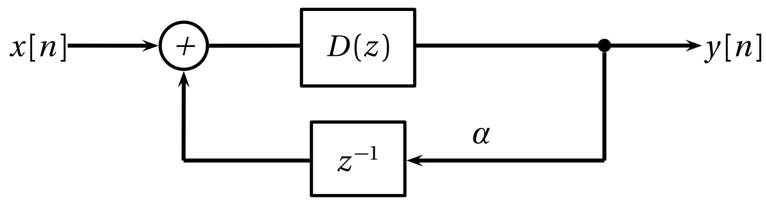
(c) Sketch the magnitude of the system's frequency response $|H(\omega)|$

(d) Find the transfer function of a *stable* filter $G(z)$ so that $|H(z)G(z)| = 1$

(e) Would the system be stable if we removed the lowest branch (i.e. if we set $\alpha = 0$)?

Solution:

The system can be simplified as



From this we can write the input/output relation in the z domain

$$Y(z) = D(z)[X(z) + \alpha z^{-1} Y(z)]$$

from which we obtain the transfer function

$$H(z) = \frac{D(z)}{1 - \alpha z^{-1} D(z)}$$

If we write $D(z)$ as a ratio of polynomials, i.e. $D(z) = B(z)/A(z)$, we finally obtain

$$H(z) = \frac{B(z)}{B(z) - \alpha z^{-1} A(z)}$$

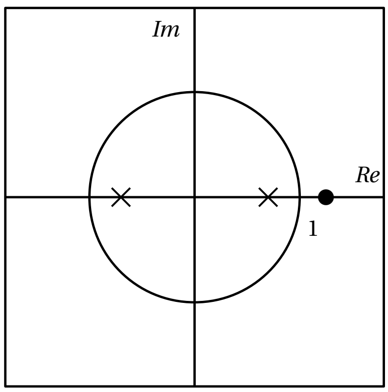
From the figure, it is immediate to see that $D(z)$ is an (incomplete) second order section in direct form II, incomplete since it has a single zero. Its transfer function is therefore

$$\begin{aligned} D(z) &= \frac{1-(5/4)z^{-1}}{1-2z^{-1}+2z^{-2}} \\ &= \frac{1-(5/4)z^{-1}}{(1-(1+j)z^{-1})(1-(1-j)z^{-1})} \end{aligned}$$

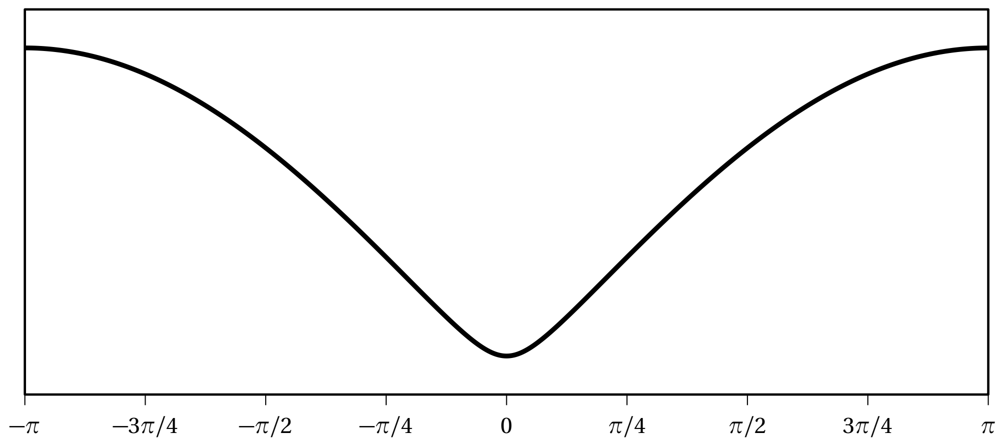
(a) *By letting $B(z) = 1-(5/4)z^{-1}$ and $A(z) = 1-2z^{-1}+2z^{-2}$, the transfer function with $\alpha = -2$ becomes*

$$H(z) = \frac{1-(5/4)z^{-1}}{1-2z^{-1}+2z^{-2}+2z^{-1}(1-(5/4)z^{-1})} = \frac{1-(5/4)z^{-1}}{1-(1/2)z^{-2}}$$

(b) *There is a zero in $z = 5/4$ and two poles in $z = \pm\sqrt{1/2}$*



(c) *since pole and zero on the positive real axis are almost equidistant from one, their effects cancel each other out; the pole in $z = -\sqrt{1/2}$ brings the magnitude of the frequency response up to create a highpass characteristic:*



(d) *The inverse transfer function is not stable because the zero of* $H(z)$ *is outside the unit circle. By choosing*

$$G(z) = \frac{1-(1/2)z^{-2}}{(5/4)-z^{-1}}$$

the product G(z)H(z) is the allpass term $(1-(5/4)z^{-1})/((5/4)-z^{-1})$ *whose frequency response magnitude is one.*

(e) *If we remove the feedback branch, the transfer function becomes* $H(z) = D(z)$. *The poles of* $D(z)$ *are larger than one in magnitude (*$|z_{1,2}| = |1 \pm j| = \sqrt{2}$*) and so the system would not be stable.*