COM-202 - Signal Processing

Mathematical review and self-test

In this course it will be assumed that you are familiar and comfortable with a number of mathematical ideas and techniques that you have studied in your previous classes. The questions in the remainder of this homework are designed to help you identify potential "trouble spots" in your mathematical background so that, if needed, you can revise the relevant topics ahead of time. Please attempt to solve the quiz without external help (that includes seraching online); since this is a self-test review, solutions to the questions will not be provided and you are encouraged to work on the problems until you are confident with your answers. Please let the instructors know if some of the questions look completely unfamiliar to you so that we can organize a review session if needed.

Complex Numbers

A complex number z admits two canonical representations:

- cartesian form: $z = a + jb$
- polar form: $z = \rho e^{j\theta}$

Please note that, according to usual engineering practice, we use the symbol j to denote the imaginary unit.

Exercise 1. Conversion Between Representations.

Given $z = \rho e^{j\theta}$, express z in cartesian form $a + jb$:

$$\begin{aligned} a &= \\ b &= \end{aligned}$$

Conversely, given $z = a + jb$, express z in polar form $\rho e^{j\theta}$:

$$\begin{aligned} \rho &= \\ \theta &= \end{aligned}$$

Exercise 2. Elementary Operations.

Assume $z = a + jb = \rho e^{j\theta}$ and $w = c + jd = \sigma e^{j\varphi}$. Write the results of the following operations in canonical form (either cartesian or polar) using the one which is most convenient for each case (see the first example, where the cartesian form is the most practical):

$$\begin{aligned} z + w &= (a + jb) + (c + jd) = (a + c) + j(b + d) \\ z - w &= \\ z/w &= \\ |z| &= \\ z^* &= \\ z^5 &= \\ \sqrt{z} &= \end{aligned}$$

Exercise 3. The Complex Plane.

Plot the following complex numbers on the complex plane (mark the points on the plane with the corresponding letter):

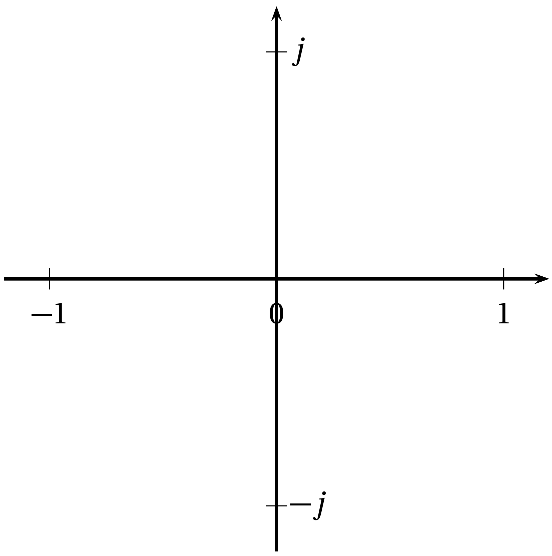
a) 1

b) $1 + j$

c) $\sqrt{-1}$

d) $e^{-j\pi/4}$

e) $(1/3)e^{j\pi}$



Exercise 4. Curious, isn't it?

Compute the numerical value of the imaginary unit raised to the power of the imaginary unit:

$$j^j =$$

Linear Algebra

N-dimensional complex vectors are by default *column* vectors and are indicated by the notation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = [x_1\ x_2\ \ldots\ x_N]^T$$

where the superscript T denotes transposition. Similarly, an $N \times M$ matrix is represented by

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1M} \\ x_{21} & x_{22} & \ldots & x_{2M} \\ & \ldots & \ldots & \\ x_{N1} & x_{N2} & \ldots & x_{NM} \end{bmatrix}$$

Exercise 5. Vectors and Matrices

Assume $\mathbf{x} = [1 \quad 2 \quad 3]^T$ and consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. Compute the following quantities:

$$\mathbf{x}^T\mathbf{x} =$$

$$\mathbf{x}\mathbf{x}^T =$$

$$|\mathbf{x}| =$$

$$\mathbf{A}\mathbf{x} =$$

$$\mathbf{x}\mathbf{A} =$$

$$\mathbf{A}\mathbf{A}^T =$$

Exercise 6. Matrix inversion

Compute (if it exists), the inverse of each of the following matrices

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}^{-1} =$$

Exercise 7. Inner product

Given a pair of real-valued N-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$, their inner product (or dot product) is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x_n \, y_n.$$

Two vectors are called orthogonal if their inner product is zero.

Find three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ with the following properties:

- the vectors have three dimensions, e.g. $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix}^T$
- none of the elements in any of the three vectors is zero
- the vectors are mutually orthogonal, i.e. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{y} \rangle = 0$

Exercise 8. Unitary matrices

Given a matrix $\mathbf{A}$ with complex-valued entries, its Hermitian transpose $\mathbf{A}^H$ is a matrix obtained by transposing $\mathbf{A}$ and taking the complex-conjugate of each element:

$$\mathbf{A}^H_{n,m} = \overline{\mathbf{A}_{m,n}}$$

A square matrix is called *unitary* if its Hermitian transpose is also its inverse, that is, if $\mathbf{A}^H \mathbf{A} = \mathbf{I}$.

Show that the following matrix is unitary:

$$\mathbf{A} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix}$$

Calculus

Exercise 9. Polynomials

Find the roots of the following polynomials (remember that, in the complex field, a polynomial of degree N has N roots!):

a) $p(x) = 2x^2 - 5x + 2$;

b) $p(x) = x^3 + 1$;

c) $p(x) = x^4 + 3x^2 + 2$;

Exercise 10. Series

Compute the following sums:

$$\sum_{n=0}^{17} n =$$

$$\sum_{n=0}^{N} a^n =$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} =$$

$$\sum_{n=0}^{\infty} \frac{1}{n} =$$

Exercise 11. Limits

Compute the following limits:

$$\lim_{x\to\infty} \frac{\log x}{x} =$$

$$\lim_{x\to 0} \frac{\sin x}{x} =$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x =$$

Exercise 12. Integrals

Compute the following integrals:

$$\int_{-\pi}^{\pi} \sin x \, dx =$$

$$\int e^{\alpha x} \, dx =$$

$$\int x \cos nx \, dx =$$

Probability theory

Exercise 13. Mean and variance

Assume a is a discrete random variable assuming values in $\{-1, 2\}$ with probabilities $P[a = -1] = 0.3$ and $P[a = 2] = 0.7$.

- compute the mean of a
- compute the variance of a

Exercise 14. Probability distribution function and expectation

Assume x is a uniformly distributed random variable over the interval $[1, 3]$. Compute the following values

$$E[x] =$$

$$E[(x-2)^2] =$$

$$E[1/x] =$$