COM-202 - Signal Processing

Homework 8

Please submit your answer to Exercise 6 by Apr 17, 2025

Exercise 1. IIR Filter properties

Consider a causal, stable IIR filter with impulse response $g[n]$ and transfer function $G(z)$. Which of the following statements are always true?

(a) The DTFT of $g[n]$ exists.

(b) The ROC of $G(z)$ includes the curve $|z| = 0.5$.

(c) The inverse filter $H(z) = 1/G(z)$ is FIR.

(d) The inverse filter $H(z) = 1/G(z)$ is stable.

(e) The system $D(z) = (1 - 3z^{-1})G(z)$ is stable.

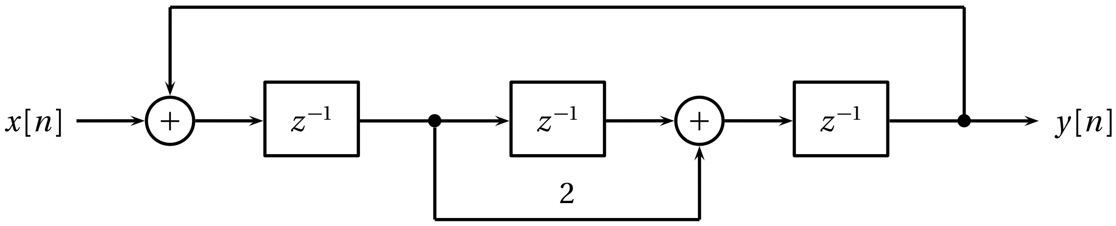
(f) The filter with transfer function $C(z) = G(z)G(z)$ is stable.

Exercise 2. FIR Filter properties

Answer the questions in the previous exercise once again, but this time assume that the filter $G(z)$ is FIR.

Exercise 3. Block diagram analysis

Consider the causal system implemented by the following block diagram:



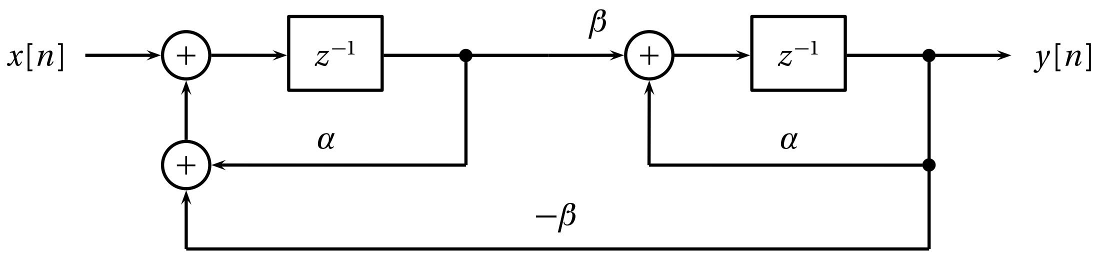
(a) Compute the system's transfer function $H(z)$.

(b) Plot the system's poles and zeros on the complex plane.

(c) Determine if the system is stable.

Exercise 4. Block diagram 1

Consider the causal system described by the following block diagram:



(a) Compute its transfer function $H(z) = Y(z)/X(z)$.

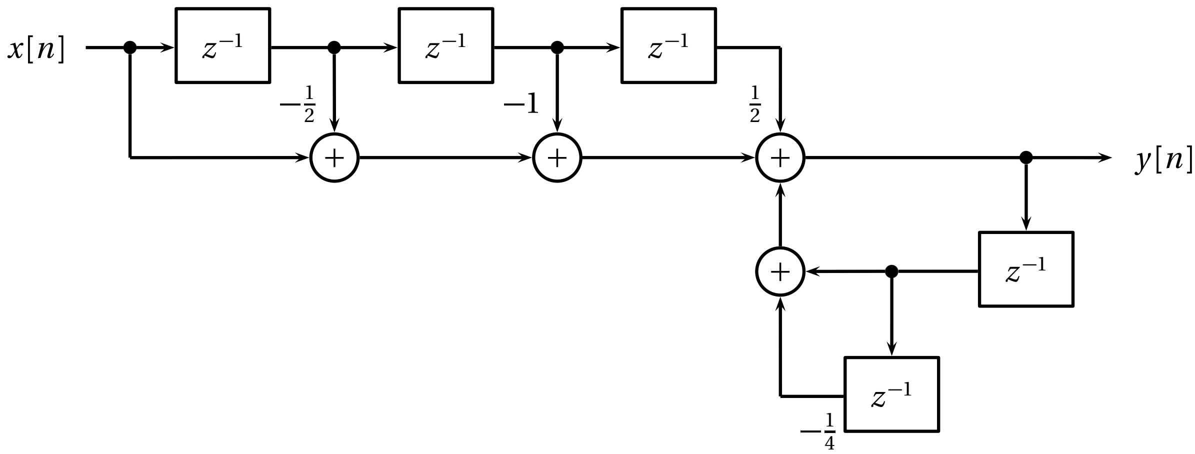
(b) Assume now that

$$\alpha = r\cos\theta$$
$$\beta = r\sin\theta$$

for $0 < r < 1$ and $0 < \theta < \pi/2$. Describe the type of filter implemented by the block diagram for this choice of coefficients and sketch its pole-zero plot.

Exercise 5. Block diagram 2

Consider the causal system described by the following block diagram:



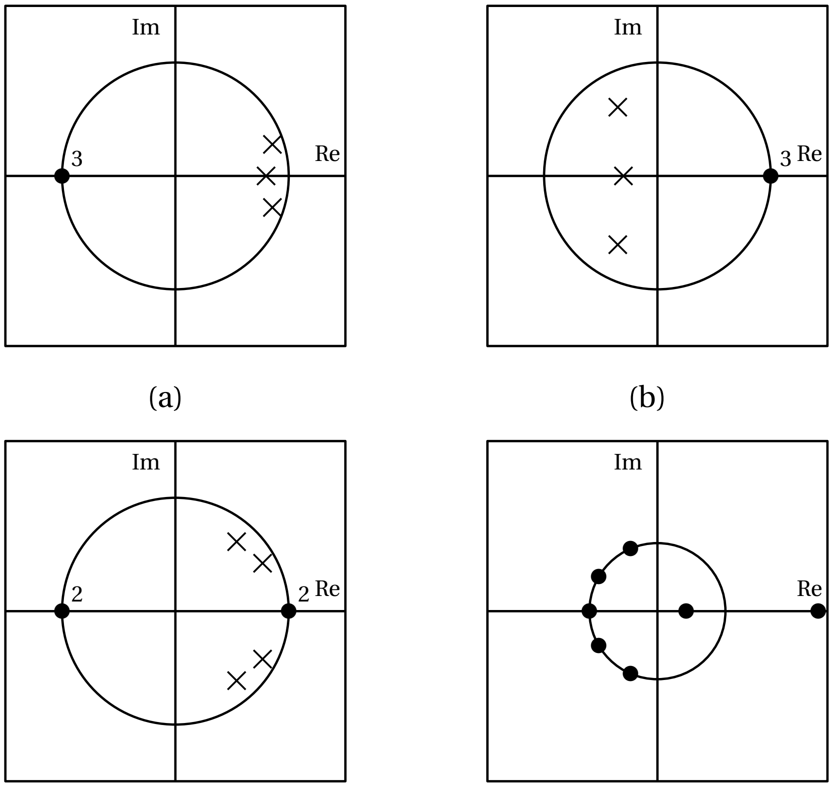
(a) Compute its transfer function $H(z) = Y(z)/X(z)$.

(b) Is the system stable?

(c) Draw a block diagram that implements the same transfer function **using only two delays** (i.e. only two z^{-1} delay blocks).

Exercise 6. Poles and zeros

For each of the following pole-zero plots, sketch the magnitude response of the corresponding filter.



(a) (b) (c) (d)

(In the plots, poles are represented by crosses and zeros by circles; if applicable, the multiplicity of each pole and zero is indicated by a number. The circle indicates the unit circle on the complex plane).

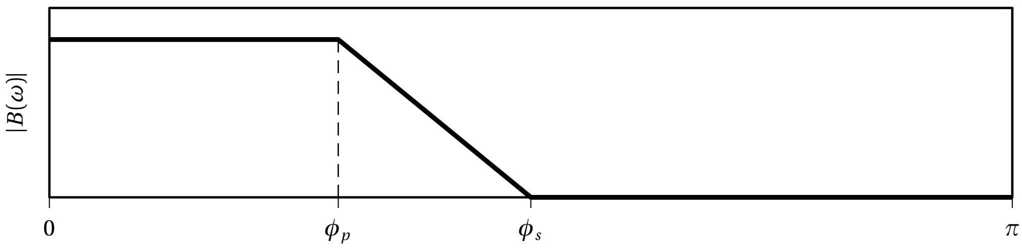
Exercise 7. Filter design

Bellanger's Approximation is an empirical formula used to estimate the length of an optimal linear-phase FIR lowpass filter based on its design specifications. For a lowpass with transition band $[\omega_p,\ \omega_s]$ and error tolerances of δ_p and δ_s in passband and stopband respectively, the required filter length will be approximately

$$N \approx \frac{-2\log_{10}(10\delta_p\delta_s)}{3(\omega_s - \omega_p)/2\pi} - 1$$

Since the order is inversely proportional to the width of the transition band, "sharp" filters (i.e., filters with a narrow transition band) will require a lot of multiplications per output sample. The following questions will ask you to analyze an alternative design strategy called *Interpolated FIR* (IFIR), used to obtain sharp filters at a lower computational cost.

To begin, assume you have designed an optimal N-tap FIR lowpass $B(z)$ with impulse response $b[n]$ and with the following magnitude response (we're showing just the positive frequencies and neglecting the ripples):



The transition band of $B(z)$ has width $\Delta_B = \phi_s - \phi_p$. We now build a derived FIR filter $B_I(z)$ with impulse response

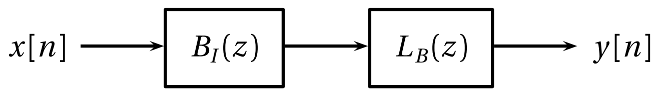
$$b_I[n] = \begin{cases} b[n/2] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

(a) Express $B_I(z)$ in terms of $B(z)$.

(b) Sketch the magnitude response $|B_I(\omega)|$; you don't need to draw the ripples but clearly show the band edges and their values.

(c) Assuming that multiplications by zero can be ignored, what is the number of multiplications per output sample required by $B_I(z)$?

Consider now the following cascade, used to implement a complete IFIR filter:



(d) Describe and sketch the frequency response of a filter $L_B(z)$ so that the cascade $B_I(z)L_B(z)$ implements a lowpass filter.

(e) Specify the passband and stopband frequencies of the lowpass implemented by the cascade $B_I(z)L_B(z)$.

(f) $L_B(z)$ will be implemented as an optimal linear-phase FIR; find the passband and stopband frequencies θ_p, θ_s that maximize the width the transition band of $L_B(z)$.

Consider now the following lowpass design specifications:

$$\omega_p = 0.3\pi$$
$$\omega_s = 0.31\pi$$
$$\delta_p = \delta_s = 0.01;$$

we want to compare a standard FIR implementation $H(z)$ with an IFIR implementation $\hat{H}(z) = B_I(z)L_B(z)$ such that $\hat{H}(z)$ has the same passband as $H(z)$.

(g) Estimate the order of a standard FIR implementation $H(z)$ using Bellanger's formula.

(h) For an IFIR implementation, determine the values of ϕ_p and ϕ_s to use for the initial filter $B(z)$ so that the final IFIR cascade $\hat{H}(z) = B_I(z)L_B(z)$ (for a suitable $L_B(z)$) has the same passband as $H(z)$.

(i) Estimate the order of an optimal linear phase FIR implementation of $B(z)$.

(j) Assume an optimal linear phase FIR implementation for $L_B(z)$, using the maximum transition band Δ_L possible and using $\delta_p = \delta_s = 0.01$; estimate the order of $L_B(z)$.

(k) Using the above estimations, determine the number of operations per output sample of the IFIR cascade $B_I(z)L_B(z)$.