COM-202 - Signal Processing

Homework 7

Please submit your answer to Exercise 5 by Apr 10, 2025

Exercise 1. A simple model for a bank

A simple discrete-time feedback loop with a single delay can be used to describe an elementary banking model where compound interest accrues yearly. Assume the following:

- you can only deposit (or withdraw) funds from your account on January 1st of each year; call the yearly deposit $x[n]$, with $n = 0$ when you open the account;
- on December 31st each year, the bank looks at your total assets and adds R percent of it to your account the next day; R is the interest rate;
- let's assume for simplicity that you never take any money out, so your balance is always positive.

With these assumptions we can model $y[n]$, the amount of money in your account in year n, via the recursive equation

$$y[n] = \alpha y[n-1] + x[n]$$

where $\alpha = 1 + R$.

Compute the closed-form expression for $y[n]$ when $x[n] = \beta u[n]$, that is, when you deposit β units of currency in your account every year.

Exercise 2. Allpass filters

An allpass filter is a filter whose magnitude response is a constant. Allpass filters are useful when we need to modify only the phase of an input signal.

(a) Consider a discrete-time LTI system with transfer function

$$H(z) = \frac{1-(1/a)z^{-1}}{1-az^{-1}}$$

where a is a real-valued constant. Show that the system is allpass, i.e., show that $|H(\omega)| = d$ for all values of the frequency ω.

(b) Determine the magnitude response of a filter with transfer function

$$G(z) = \frac{1-(1/a)z^{-1}}{1-az^{-1}} \, \frac{1-(1/b)z^{-1}}{1-bz^{-1}} \, \frac{1-(1/c)z^{-1}}{1-cz^{-1}}$$

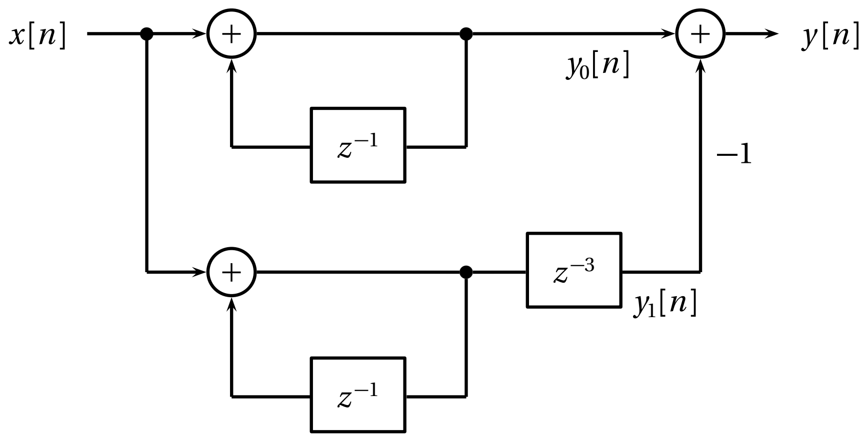
where a, b, and c are real-valued constants.

Exercise 3. Linear phase

Show that a causal, odd-length, antisymmetric FIR filter has a linear phase response. Recall that, for an FIR of length M, symmetry (or antisymmetry) is around the midpoint of the impulse response so that, in this case, the antisymmetry condition translates to $h[n] = -h[M-1-n]$ for $n = 0, \ldots, M-1$.

Exercise 4. Discrete-time system diagram

Consider the causal system described by the following block diagram. Assume a causal input ($x[n] = 0$ for $n < 0$) and zero initial conditions.



(a) Find the three constant-coefficients difference equations that describe the relationship between the input $x[n]$ and $y_0[n]$, $y_1[n]$, $y[n]$.

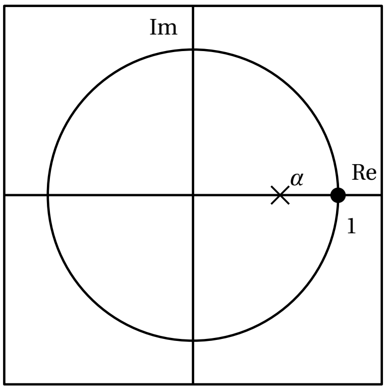
(b) Find $H_0(z)$, $H_1(z)$ and $H(z)$, the transfer functions associated to the three CCDEs you found in the previous point.

(c) Is the whole system stable?

(d) Consider the input $x[n] = u[n]$, where, as usual, $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ for $n < 0$. How do $y_0[n], y_1[n]$ and $y[n]$ evolve over time? Sketch their values.

Exercise 5. Impulse response from poles and zeros

Compute the impulse response of the causal filter with the following pole-zero plot:



Exercise 6.

A causal LTI system is described by the following difference equation, where b is a real number.

$$y[n]+(b+1)^2 y[n-1]+(2b^3+b^2)y[n-2]=x[n].$$

(a) Find the transfer function $H(z)$ and the range of b such that the system is stable.

(b) Sketch a block diagram implementing this system.
