COM-202 - Signal Processing

Homework 5

Please submit your answer to Exercise 7 by Mar 27, 2025

Exercise 1. DTFTs

Compute the DTFTs of the following sequences:

(a) $x[n] = \frac{1}{2^n}u[n] - \frac{1}{4^n}u[n-1]$

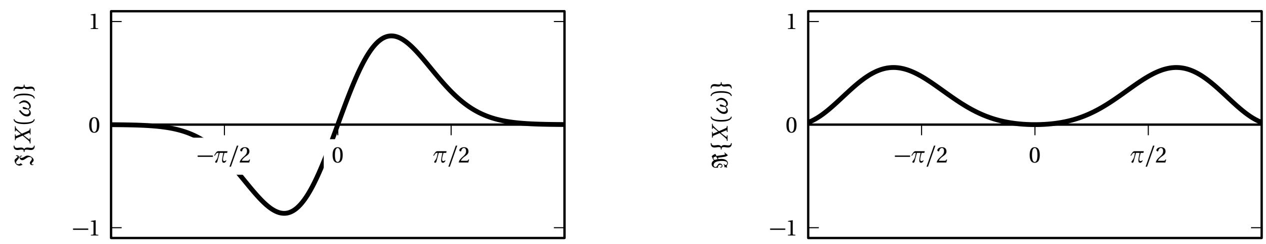
(b) $x[n] = a^n \cos(\omega_0 n)u[n], \qquad |a| < 1$

Exercise 2. DTFT visual inspection

The real and imaginary parts of $X(\omega)$ are shown in the figure below. By visual inspection of the plots, prove that:

(a) $x[n]$ is 0-mean, i.e., $\sum_{n\in\mathbb{Z}} x[n] = 0$;

(b) $x[n]$ is real valued.



Exercise 3. DTFT of a symmetric sequence

Compute the DTFT of $x[n] = a^{|n|}$ for $n \in \mathbb{Z}$ and $|a| < 1$, and sketch its magnitude for a close to 1.

Exercise 4. DTFT properties

Given a finite-energy sequence $x[n]$ and its DTFT $X(\omega)$, express the DTFT of the sequence $y[n] = (-1)^n x[n]$ in terms of $X(\omega)$.

[Hint: remember that $(-1)^n = e^{j\pi n}$*]*

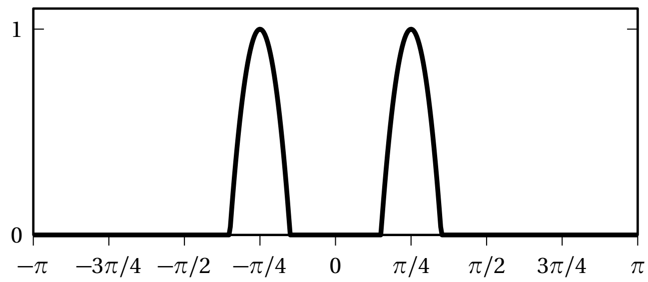
Exercise 5. Alternate notation for the DTFT

In many signal processing textbooks and publications (including the material used in previous editions of this class) DTFTs are represented using the notation

$$X(e^{j\omega}).$$

The main advantage of that choice is clarity since any expression of the form $A(e^{jb})$ is immediately interpreted as a Fourier transform; additionally, the expression automatically "encodes" the 2π-periodicity of the DTFT since obviously $X(e^{j(\omega+2k\pi)}) = X(e^{j\omega})$ with no need of knowing the actual expression for $X(e^{j\omega})$. On the other hand, the notation becomes very cumbersome when needing to apply shifts and scalings to the frequency variable, leading to hard-to-read expressions such as $X(e^{j\frac{\omega-2\pi m}{N}})$ which are very hard to read.

It's nevertheless important to become familiar with this alternate notation because it's very common. Here's a question to test your knowledge: for the DTFT $X(e^{j\omega})$ shown below, sketch the magnitude of $X(-e^{j\omega})$.



Exercise 6. DTFT of a modified sequence

Consider a causal sequence $x[n] \in \ell_2(\mathbb{Z})$ (i.e. an integer valued energy signal) with DTFT

$X(\omega)$ where $x[n] = 0 for n < 0$. A new sequence $y[n]$ is defined as

$$y[n] = \begin{cases} 0 & \text{for } n < 0 \\ x[n] & \text{for } n \text{ even} \\ a^n & \text{for } n \text{ odd} \end{cases}$$

with $|a| < 1$. Derive the expression for $Y(\omega)$ in terms of $X(\omega)$.

[*Hint: the sequence* $(1 + e^{j\pi n})/2$ *may prove useful.*]

Exercise 7. DTFT, DFT, and numerical computations

Consider the following finite-support signal, where $M \in \mathbb{N}$:

$$x[n] = \begin{cases} 1 & 0 \le n < M, \\ 0 & \text{otherwise.} \end{cases}$$

In this exercise you will need to write some Python/NumPy code to compare the theoretical value of the DTFT $X(\omega)$ to its numerical approximation.

(a) Derive the closed-form expression for $X(\omega)$.

(b) Using Python, plot the values of $|X(\omega)|$ over a set of uniformly spaced frequency values between $-\pi$ and π; set $M = 20$ and use 10,000 frequency values.

(c) Now generate an N-point finite-length sequence $x_N[n]$ where $x_N[n] = x[n]$ for $n = 0, 1, \ldots, N-1$. Compute its DFT using NumPy's FFT package and plot the magnitude of the coefficients for $N = 31, 51, 101$. Align the DFT plots so that you can visually compare the DFT coefficients to the analytical values of the DTFT obtained in the previous step.
