COM-202 - Signal Processing

Homework 2

24 February 2025, Monday

Please submit your answer to Exercise 6 by 6 March 2025, Thursday, 23:59.

Exercise 1. Energy of complex-valued signals

Compute the energy of the signal defined as

$$x[n] = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^n + j\left(\frac{1}{\sqrt{3}}\right)^n & n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 2. Operators and linearity

A discrete-time signal *operator* is a transformation acting on the entire signal:

$$\mathbf{y} = \mathscr{F}\mathbf{x}$$

A *linear* operator has the following properties (where α is a complex-valued scalar):

$$\mathscr{F}(\alpha\mathbf{x}) = \alpha\mathscr{F}\mathbf{x}$$
$$\mathscr{F}(\mathbf{x}+\mathbf{y}) = \mathscr{F}\mathbf{x}+\mathscr{F}\mathbf{y}$$

(a) Show that the time-shift operator for infinite-length signals, defined by $(\mathscr{S}\mathbf{x})[n] = x[n+1]$, is a linear operator.

(b) Show that the squaring operator, defined by $(\mathscr{Q}\mathbf{x})[n] = (x[n])^2$ is *not* linear.

Exercise 3. Operators in matrix notation

Linear operators acting on finite-length signals can always be expressed as a matrix-vector product. For example, consider the shift-by-one operator in $\mathbb{C}^N$, which is defined as a right *circular* shift:

$$(\mathscr{S}\mathbf{x})[n] = x[(n-1) \mod N].$$

In vector notation we can write

$$\mathscr{S}\mathbf{x} = \mathbf{S}\mathbf{x}$$

where the matrix **S** has the following form (using $N = 4$ for convenience):

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Express in matrix form the following operators in $\mathbb{C}^4$:

(a) the first-difference operator, defined by $(\mathscr{V}\mathbf{x})[n] = x[n] - x[(n-1) \mod N]$

(b) the averaging operator, defined by $(\mathscr{A}\mathbf{x})[n] = (x[n] + x[(n+1) \mod N])/2$

(c) the time reversal operator, defined by $(\mathscr{R}\mathbf{x})[n] = x[-n \mod N]$

Exercise 4. Elementary signal operators

Using elementary signal operators

(a) express $\boldsymbol{\delta}$ in terms of $\mathbf{u}$

(b) express $\mathbf{u}$ in terms of $\boldsymbol{\delta}$

(c) express the constant signal $\mathbf{1}$, which is equal to 1 for all $n \in \mathbb{Z}$, in terms of $\mathbf{u}$ and $\boldsymbol{\delta}$

(d) express the constant signal $\mathbf{1}$ terms of $\mathbf{u}$ only

(e) express $\mathbf{x}$, with $x[n] = \cos(2n)$, in terms of the signal $\mathbf{c}$, with $c[n] = \cos(n)$, and of any of the previous signals

As a reminder

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and

$$(\mathscr{S}^{-1}\mathbf{x})[n] = x[n-1]$$
$$(\mathscr{V}\mathbf{x})[n] = x[n] - x[n-1]$$
$$(\mathscr{R}\mathbf{x})[n] = x[-n]$$
$$(\mathscr{Q}\mathbf{x})[n] = x^2[n]$$
$$(\mathscr{E}\mathbf{x})[n] = \sum_{k=-\infty}^{n} x[k]$$

Exercise 5. Vector space

For each of the definitions given below, determine whether resulting space is a vector space and, if not, explain why:

(a) the set of vectors $\begin{bmatrix} x_0 & x_1 \end{bmatrix}^T \in \mathbb{R}^2$ for which $x_1 = 3x_0 + 1$ and with the usual definitions of scalar multiplication and vector addition

(b) the set of vectors $\begin{bmatrix} x_0 & x_1 \end{bmatrix}^T \in \mathbb{R}^2$ with the standard definition for vector addition and the following definition for scalar multiplication:

$$\alpha \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \alpha x_0 \\ x_1 \end{bmatrix}$$

Exercise 6. Bases & Python

Consider the vector space $V \subset \mathbb{C}^8$ spanned by the *rows* of $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) What is an easy way to prove that the rows in $\mathbf{H}$ do indeed form a basis?

(b) Use Python to verify point (a); obviously you can use `numpy`.

The basis described by $\mathbf{H}$ is called the *Haar basis* and it is one of the most celebrated cornerstones of a branch of signal processing called wavelet analysis (which we won't study in this class). To get a feeling for its properties, however, consider the following set of Python experiments:

(c) Verify that $\mathbf{H}\mathbf{H}^H$ is a diagonal matrix, which means the vectors are orthogonal.

(d) Consider a constant signal $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ and compute its coefficients in the Haar basis.

(e) Consider an alternating signal $\mathbf{y} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$ and compute its coefficients in the Haar basis.

Exercise 7. Bases

Let $\{\mathbf{x}^{(k)}\}_{k=0,\dots,N-1}$ be a basis for a subspace S. Prove that any vector $\mathbf{z} \in S$ is *uniquely* represented in this basis.

Hint: remember that the vectors in a basis are linearly independent and use this to prove the thesis by contradiction.